CBPF-NF-046/94

A Quantum String Representation for Four Dimensional Einstein Quantum Gravity

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ABSTRACT

We propose a quantum string functional for the Wheeler-De Witt quantum Einstein gravity constraint in the Ashtekar-Sen field coordinates.

Key-words: Ashtekar-Sen quantization; Quantum gravity; String representation.

1 Introduction

The Ashtekar-Sen propose of a new set of complexified SU(2) coordinates for Einstein action ([1]) has became very promising at the quantum level by allowing explicit classical loop space solution for the Wheeler-De Witt equation without cosmological term ([2]).

In this rapid communication we follow our previous studies ([3],[4],[6], [10],[11]) by proposing to replace the classical closed loop in the Smolin-Jacobson wave functional by a random surface (quantum string) possessing SU(2) color degrees ([3]). We show, thus that this proposed quantum string wave functional satisfies the Wheeler- De Witt equation and the diffeomorphism constraint.

2 The Random Surface Wave Functional

Let us start our analysis by considering the problem of associating a wave functional for an arbitrary self-intersecting random surface S with boundary $C_{xx} = \{C_\mu(\sigma), 0 \leq \sigma \leq 2\pi, C_\mu(0) = C_\mu(2\pi) = x\}$ and possessing SU(2) color degrees of freedom interacting with an external SU(2) connection $A^i_\mu(x)\lambda_i$. Here λ_i denote the SU(2) generators in the fundamental representation.

The surface S is characterized by two fields: firstly, by the usual (bosonic) vector position $X_\mu(\xi)$, $\xi \in D$ and D is the appropriate parameter associated to the surface $X_\mu(D) = S$. The surface SU(2) color variable $g(\xi)$ belongs to the fundamental SU(2) group. The intrinsic metric properties of S are represented by $2D$ metric fields $h_{ab}(\xi)$ ([4]).

The classical action for this color SU(2) surface is given in the Polyakov's formalism ([4])

$$S = S_0 + S_1^{(B)} \tag{1}$$

with

$$S_0 = \frac{1}{2}\int_D d^2\xi(\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu)(\xi) + \mu^2 \int_D d^2\xi\sqrt{h} \tag{1.a}$$

$$S_1^{(B)} = \frac{1}{4\pi m}\int_D d^2\xi(\sqrt{h}.Tr^{(c)}(g^{-1}\partial_a g)^2) + 4\pi i\Gamma_{wz}[g] \tag{1.b}$$

where $\Gamma_{wz}[g]$ denotes the two-dimensional Wess-Zumino functional. Its existence, together with the integer m in the written SU(2) σ-model afford us to consider the more suitable fermionic equivalent action for $S_1^{(B)}$

$$S_1^{(F)} = \int_D (\sqrt[4]{h}\,\bar{\psi}(i\gamma_a\nabla^a)\psi)(\xi)d^2\xi \tag{2}$$

where the two-dimensional Dirac field $\psi(\xi)$ belongs to the fermionic fundamental SU(2) representation.

At this point, the simplest action taking into account the interaction with the external Ashtekar-Sen connection is given by

$$S^{int}[\psi(\xi), \bar{\psi}(\xi); A_\mu(x)] = e\int_D (\sqrt{h}\,\bar{\psi}[\gamma_a\partial^a X_\mu.A^\ell_\mu(x).\lambda_\ell]\psi)(\xi)d^2\xi \tag{3}$$

It is instructive point out the interaction eq. (3) written in terms of the bosonic SU(2) variable $g(\xi)$ [5].

The complete classical interacting action eq. (1), eq. (2) and eq. (3) is invariant under the gauge transformations

$$\begin{aligned} A_\mu(X^\alpha(\xi)) &\rightarrow (\ell^{-1}A_\mu\ell+\ell^{-1}\partial_\mu\ell)(X^\alpha) \\ \psi(\xi) &\rightarrow \ell(X^\alpha(\xi))\psi(\xi) \\ \bar\psi(\xi) &\rightarrow \bar\psi(\xi)\ell^{-1}(X^\alpha(\xi)) \end{aligned} \tag{4}$$

where $\ell_{(x)} \in$ SU(2)

We shall now use eq. (1) and eq. (3) to propose the following random surface fermionic functional integral as a surface Wilson Loop ([6])

$$\begin{aligned} W_{ab}[S,C_{xx},A] &= Tr^{color}\left\{\int D^c[\psi(\xi)]D^c[\bar\psi(\xi)]\psi_a(0,0)\bar\psi_b(2\pi,0)\right. \\ &\qquad \left. exp\{-(S_0+S_1^{(F)}+S^{int})\}\right\} \end{aligned} \tag{5}$$

Notice that our above proposed random surface phase factor is a 2×2 matrix in the flat domain $D(a,b=1,2)$.

The covariant fermion functional integral is defined by the functional element of volume associated to the following functional Riemann metric with the fermions fields satisfying the Neumann boundary condition

$$||\delta\psi||^2 = \int_D [\sqrt[4]{h}(\delta\psi\delta\psi)](\xi)d^2\xi \tag{6}$$

The quantum surface functional will be defined by the Nambu-Goto functional integral over the $X_\mu(\xi)$ variables as written in ref. [7] - eq. (23) with the Dirichlet boundary condition $\partial S = C_{xx}$ and in the orthonormal surface coordinates

$$\sum_{C_{xx}}\left\{\sum_{\{S\};\partial S=C_{xx}}\{W_{ab}[S,C_{xx},A]\}\right\} = \langle W_{ab}[S,C_{xx},A]\rangle = \Phi[A] \tag{7}$$

where

$$\sum_{C_{xx}} = \int_{C_\mu(0)=C_\mu(2\pi)=x} D[C_\mu(\sigma)]e^{-\frac{1}{2}\int_0^{2\pi}(\dot C^\mu(\sigma))^2d\sigma} \tag{7.a}$$

is the x-dependent loop average and

$$\begin{aligned} \sum_{\{S\};\partial S=C_{xx}} &= \int_{X_\mu(\sigma,0)=C^\mu(\sigma)}(\Pi_{\xi,\mu}dX^\mu(\xi)\left(-\frac{1}{2}\int_D d\xi^+d\xi^-[(\partial_+X^\mu\partial_-X_\mu)](\xi^+,\xi^-)\right) \\ & exp\left\{-\frac{26-(4+3)}{48\pi}\int_D d\xi^+d\xi^-\left(\frac{(\partial_+^2X^\mu)(\partial_-X^\mu)}{[(\partial_+X^\mu)(\partial_-X^\mu]^2}(\partial_-^2X^\mu)(\partial_+X^\mu)\right)(\xi^+,\xi^-)\right\} \end{aligned} \tag{7.b}$$

denotes the correct way to sum random surfaces with wheight given by the Nambu-Goto action.

Let us show that eq. (7) satisfies formally the Wheller-De Witt constraint ([2]) integrated over the three-dimensional manifold $M \subset R^4$ (([8]) - chapter 3).

$$\int_M d^3x\, \varepsilon^{ijk}\, F^i_{\mu\nu}(x) \frac{\delta}{\delta A^i_\mu(x)} \frac{\delta}{\delta A^j_\nu(x)} \{\langle W_{ab}[S, C_{xx}, A]\rangle\} \equiv 0 \tag{8}$$

It is a straightforward calculation to show that

$$\begin{aligned}
&\int_M d^3x\, \varepsilon_{ijk} F^i_{\mu\nu}(x) \frac{\delta^2}{\delta A^j_\mu(x) \delta A^k_\nu(k)} W_{ab}[S, C_{xx}, A] = \\
&= \int_D d^2\xi \sqrt[4]{h(\xi')} \int_D d^2\xi \sqrt[4]{(\xi)} \left(\partial^a X^\mu(x) \delta^{(3)}(X^\alpha(\xi) - X^{\alpha'}(\xi')), \partial^b X^\nu(\xi')\right) \\
&F^i_{\mu\nu}(X^\alpha(\xi)).(\varepsilon_{ijk}).Tr^{color} \left\{ \int_D D^c[\psi(\xi)] D^c[\bar{psi}(\xi)] \right. \\
&\left. \psi_a(0,0)[\bar\psi(\xi)\gamma_a \lambda^j \psi(\xi) \psi \xi') \gamma_b \lambda^k \psi(\xi')] \bar\psi_b(2\pi, 0) exp\{-(S_0 + S_1^{(F)} + S^{int})\} \right\}
\end{aligned} \tag{9}$$

In the context of random surfaces sum eq. (8) we have that the $X_\mu(\xi)$ functional integral lead us to the following condition in the e-perturbative expansion for eq. (7) ([9]).

$$\begin{aligned}
&\langle (\partial^a X^\mu(\xi)) \delta^{(3)}(X^\alpha(\xi) - X^{\alpha'}(\xi'))(\partial^b X^\nu)(\xi') \theta_{\mu\nu}[X(\xi)]\rangle \\
&= \langle \delta_{\mu\nu} (\partial^a X^\mu(\xi)) \delta^{(3)}(X^\alpha(\xi) - X^{\alpha'}(\xi'))(\partial^b X^\nu)(\xi') \theta_{\mu\nu}[X(\xi)]\rangle
\end{aligned} \tag{10}$$

As a consequence of $F_{\mu\nu}(X(\xi))$ being anti-symmetric in the (μ, ν)-indexes we get the result eq. (8).

It is interesting point out that only in the condition of non self-intersection surfaces $X_\mu(\xi) = X_\mu(\xi') \to \xi = \xi'$ one obtains that eq. (5) is solution of the integrated Wheeler-De Witt equation ([6]-Appendix C).

In order to satisfy automatically general coordinate invariance on the 3D-manifold M

$$\delta x^i = \varepsilon^i(X^j) =^\varepsilon (x^i) \tag{11}$$

with $\varepsilon^i(x)$ being the vector fields generator of a element of $G_{DIFF}(M)$, one could consider formally the functional integral over $G_{DIFF}(M)$ of the action piece of our proposed solution eq. (5) involving the random surface S coordinates namelly

$$\begin{aligned}
\bar S[X^\mu(\xi), \varepsilon^\mu(X)] = &\frac{1}{2} \int_D d^2\xi (\sqrt{h}\, h^{ab} \partial_a [^\varepsilon X^\mu(\xi) \partial_b [^\varepsilon X^\mu(\xi) \\
&+ e \int_D d^2\xi \sqrt{h}(\xi) \bar\psi(\xi) [\gamma_a \partial^{a\varepsilon} X^\mu(\xi) A^\ell_\mu(^\varepsilon X(\xi)] \psi(\xi)
\end{aligned} \tag{12}$$

we have, thus, the $G_{DIFF}(M)$ invariant solution

$$\tilde\Phi[\vec A, x] = \sum_{\varepsilon(x) \varepsilon G_{DIFF}(M)} \left\{ \sum_{C_{xx}} \sum_{\{S; \partial S = C_{xx}\}} W_{ab}[S, \varepsilon, C_{xx}, A] \right\} \tag{13}$$

where the sum over the fields generators $\varepsilon^i(x)$ on M must be weighted with the non-compact formal Haar measure associated to $G_{DIFF}(M)$.

Another $G_{DIFF}(M)$ invariant solution of the Wheeler-De Witt equation (8) is to consider the domain parameter D contained on M in such way the $G_{DIFF}(M)$ when restricted to the two-dimensional sub-manifold DcM coincides with $G_{DIFF}(D)$ which is automatically satisfied by eq. (5) since we have imposed the Neven-Schwarz condition on the fermion fields ([6]).

$$\begin{aligned}\psi(0,0) &= \psi(2\pi,0)\\ \bar{\psi}(0,0) &= \bar{\psi}(2\pi,0)\end{aligned} \tag{14}$$

Another point worth to remark is that the supersymmetric random surface generalization of eq. (5)-eq. (7) $X^F_\mu(\xi,\theta) = X_\mu(\xi) + i\theta\psi^\mu(\xi)$ ([10]) still satisfies the integrated Wheeler-De Witt equation without cosmological constant.

Finally we comment on the geometrical-physical observables associated to our proposed random surface solution for Einstein quantum gravity. We propose that the simplest operator which measures metrical information in this quantum geometrical approach is the formal average of the desitised inverse object ([2])

$$\langle (q\, q^{\mu\nu})(z)\rangle = \frac{1}{2}\int d\mu[A]\tilde{\Phi}[A,z]\left(\sum_{1=1}^{3}\frac{\delta^2}{\delta A^i_\mu(z)\delta A^i_\nu(z)}\right)\tilde{\Phi}[A,z] \tag{15}$$

where $\{q_{\mu\nu}(z)\}$ is a possible outcome of a classical observable detect the topological manifold M possesses the metric $ds^2 = q_{\mu\nu}(z)dz^\mu dz^\nu$ at Planck scale. The average in eq. (5) $d\mu[A]$ should be defined by the Chern-Simon theory on M in order to preserve diffeomorphism and gauge invariance of eq. (15).

$$d\mu[A] = exp\{-\int Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\} \tag{16}$$

In the case of our random surface degenerates to its boundary $X_\mu(\xi) \to C_\mu(\sigma)$, the object eq. (15) will be given by Chern-Simon average of the Wilson loop defined by $C_\mu(\sigma)$ with the marked point z and leading to the generalized Jones polynomial associated to our fermionic topological string by integrating out the fermion degrees of freedom ([11])

A extended paper on the results presented on this rapid communication will appear elsewhere.

This research was supported by CNPq – Brasil.

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