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Generalized space-time supersymmetries and octonionic M-theory*

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Abstract

It is shown that an octonionic version of the M-superalgebra can be defined. It involves only 52 real bosonic generators instead of the 528 of the standard M-superalgebra and presents a novel and surprising feature, its octonionic $M5$ (super-5-brane) sector coincides with the $M1$ and $M2$ sectors.

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1 Introduction.

The generalized supersymmetries going beyond the standard HŁS scheme [1] admit the presence of bosonic abelian tensorial central charges associated with the dynamics of extended objects (branes). It is widely known since the work of [2] that supersymmetries are related to division algebras. Indeed, even for generalized supersymmetries, classification schemes based on the associative division algebras (**R**, **C**, **H**) are now available, see [3]. For what concerns the remaining division algebra, the octonions, much less is known due to the complications arising from non-associativity. Octonionic structures were, nevertheless, investigated in [4, 5], in application to the superstring theory.

I discuss here the investigations in [6] concerning the realization of general supersymmetries in terms of the non-associative division algebra of the octonions. It was shown there that, besides the standard realization of the M-algebra (which supposedly underlines the M-theory) and involves real spinors, an alternative formulation, requiring the introduction of the octonionic structure, is viable. This is indeed possible due to the existence of an octonionic description for the Clifford algebra defining the 11-dimensional Minkowskian spacetime and its related spinors. The features of this octonionic M-superalgebra are puzzling. It is not at all surprising that it contains fewer bosonic generators, 52, w.r.t. the 528 of the standard M-algebra (this is expected, after all the imposition of an extra structure puts a constraint on a theory). What is really unexpected is the fact that new conditions, not present in the standard M-theory, are now found. These conditions imply that the different brane-sectors are no longer independent. The octonionic 5-brane contains the same degrees of freedom and is equivalent to the $M1$ and the $M2$ sectors. We can write this equivalence, symbolically, as $M5 \equiv M1 + M2$.

2 Generalized supersymmetries and octonions.

In the $D = 11$ Minkowskian spacetime, where the M-theory should be found, the spinors are real and have 32 components. Since the most general symmetric 32×32 matrix admits 528 components, one can easily prove that the most general supersymmetry algebra in $D = 11$ can be presented as

$$\{Q_a, Q_b\} = (C\Gamma_\mu)_{ab}P^\mu + (C\Gamma_{[\mu\nu]})_{ab}Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{ab}Z^{[\mu_1\ldots\mu_5]} \qquad (1)$$

(where C is the charge conjugation matrix), while $Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ are totally antisymmetric tensorial central charges, of rank 2 and 5 respectively, which correspond to extended objects [7, 8], the p-branes. Please notice that the total number of 528 is obtained in the r.h.s as the sum of the three distinct sectors, i.e.

$$528 \;=\; 11 + 66 + 462. \qquad (2)$$

The algebra (1) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra, recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1\ldots\mu_5]} \equiv 0$.

On the other hand, in particular spcae-time dimensions, it is possible to realize Clifford algebras in terms of the octonions. For instance one can realize the 7-dimensional

Euclidean Clifford algebra $C(0,7)$ in terms of the seven imaginary octonions τ_i satisfying the algebraic relation

$$\tau_i \cdot \tau_j \;=\; -\delta_{ij} + C_{ijk}\tau_k, \tag{3}$$

for $i,j,k = 1,\cdots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{4}$$

and vanishing otherwise. Similarly, the Minkowskian Clifford algebra in 11 dimensions can be realized in terms of 4×4 matrices with octonionic entries.

One should be aware of the properties of the non-associative realizations of Clifford algebras. In the octonionic case the commutators $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$ are no longer the generators of the Lorentz group. They correspond instead to the generators of the coset $SO(p,q)/G_2$, being G_2 the 14-dimensional exceptional Lie algebra of automorphisms of the octonions. As an example, in the Euclidean 7-dimensional case, these commutators give rise to $7 = 21 - 14$ generators, isomorphic to the imaginary octonions. Indeed

$$[\tau_i, \tau_j] \;=\; 2C_{ijk}\tau_k. \tag{5}$$

The algebra (5) is not a Lie algebra, but a Malcev algebra (due to the alternativity property satisfied by the octonions, a weaker condition w.r.t. associativity, see [9]).

3 The octonionic M-superalgebra.

The octonionic M-superalgebra is introduced by assuming an octonionic structure for the spinors which, in the $D=11$ Minkowskian spacetime, are octonionic-valued 4-component vectors. The algebra replacing (1) is given by

$$\{Q_a, Q_b\} = \{{Q^*}_a, {Q^*}_b\} = 0, \qquad \{Q_a, {Q^*}_b\} = Z_{ab}, \tag{6}$$

where $*$ denotes the principal conjugation in the octonionic division algebra and, as a result, the bosonic abelian algebra on the r.h.s. is constrained to be hermitian

$$Z_{ab} \;=\; {Z_{ba}}^*, \tag{7}$$

leaving only 52 independent components.

The Z_{ab} matrix can be represented either as the $11+41$ bosonic generators entering

$$Z_{ab} = P^\mu (C\Gamma_\mu)_{ab} + Z_{\mathbf{O}}^{\mu\nu}(C\Gamma_{\mu\nu})_{ab}, \tag{8}$$

or as the 52 bosonic generators entering

$$Z_{ab} = Z_{\mathbf{O}}^{[\mu_1\ldots\mu_5]}(C\Gamma_{\mu_1\ldots\mu_5})_{ab}\,. \tag{9}$$

Due to the non-associativity of the octonions, unlike the real case, the sectors individuated by (8) and (9) are not independent. Furthermore, as we have already seen for $k=2$, in the antisymmetric products of k octonionic-valued matrices, a certain number of them are redundant. In the general case [10] a table can be produced expressing the number of independent components in D odd-dimensional spacetime octonionic realizations of Clifford algebras, by taking into account that out of the D Gamma matrices, 7 of them are octonionic-valued, while the remaining $D-7$ are purely real. We get the following table, with the columns labeled by k, the number of antisymmetrized Gamma matrices and the rows by D (up to $D=13$)

$D \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
7	1	7	7	1	1	7	7	1						
9	1	9	22	22	10	10	22	22	9	1				
11	1	11	41	75	76	52	52	76	75	41	11	1		
13	1	13	64	168	267	279	232	232	279	267	168	64	13	1

(10)

For what concerns the octonionic equivalence of the different sectors, it can be symbolically expressed, in different odd space-time dimensions, according to the table

$D=7$	$M0 \equiv M3$
$D=9$	$M1+M2 \equiv M4$
$D=11$	$M1+M2 \equiv M5$
$D=13$	$M2+M3 \equiv M6$
$D=15$	$M3+M4 \equiv M0+M7$

(11)

In $D=11$ dimensions the relation between $M1+M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into the 4 real indices, labeled by $a,b,c,\ldots$ and the 7 octonionic indices labeled by $i,j,k,\ldots$. The 52 independent components are recovered from $52 = 4+2\times 7+6+28$, according to

4	$M1_a$	$M5_{[aijkl]} \equiv M5_a$
7	$M1_i$, $M2_{[ij]} \equiv M2_i$	$M5_{[abcdi]} \equiv M5_i$,$M5_{[ijklm]} \equiv \widetilde{M5}_i$
6	$M1_{[ab]}$	$M5_{[abijk]} \equiv M5_{[ab]}$
$4\times 7=28$	$M2_{[ai]}$	$M5_{[abcij]} \equiv M5_{[ai]}$

(12)

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