CBPF-NF-046/01

The Wigner representation of classical mechanics, quantization and classical limit*

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Abstract

Starting from the Liouvillian formulation of classical physics it is possible by means of a Fourier transform to introduce the Wigner representation and to derive an operator structure to classical mechanics. The importance of this new representation lies on the fact that it turns out to be the suitable route to establish a general method of quantization directly from the equations of motion without alluding to the existence of Hamiltonian and Lagrangian functions. Following this approach we quantize only the motion of a Brownian particle with non-linear friction in the Markovian approximation – the thermal bath may be quantum or classical –, thus when the bath is classically described we obtain a master equation which reduces to the Caldeira-Leggett equation (*vide*, A. O. Caldeira and A. J. Leggett, *Physica A* **121** (1983) 587) for the linear friction case, and when the reservoir is quantum we get an equation reducing to the one found by Caldeira *et al.* (*Phys. Rev. A* **40** (1989) 3438). By neglecting the environmental influence we show that the system can be approximately described by equations of motion in terms of wave function, such as the Schrödinger-Langevin equation and equations of the Caldirola-Kanai type. Finally to make the present study self-consistent we evaluate the classical limit of these dynamical equations employing a new classical limiting method $\hbar \to 0$.

Key-words: quantization; classical limit; the Wigner representation of classical mechanics; Fokker-Planck equation.
PACS *numbers*: 03.65.-w, 05.30.-d, 05.40.+j, 52.65.Ff;

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*To appear in Physica A (2001).

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I Introduction: The problem of quantization

Let us begin with a brief historical overview in order to fix what we mean by quantization and note that this process of transition from classical to quantum mechanics does bring about problems of mathematical, physical and epistemological nature. The first glimpses of a quantum world were catched by Planck (1900) on *postulating* that, in the emission and/or absorption of radiation in a black-body, the *energy* is to be conceived as a *discrete* quantity [1]. On the basis of Planck's hypothesis Bohr (1913) also *postulated* that the *angular momentum* associated with the electron motion around the nucleus should be quantized, that is, an integer multiple of $\hbar$ (Planck's constant divided by 2π) [2]. Wilson [3], Planck [4], Sommerfeld [5], Schwarzschild [6], Epstein [7] summed up the postulates of Bohr and Planck in the following *ad hoc* quantization rule [8,9]: Given a Hamiltonian $H(q^i, p_i)$ describing a bound physical system with N degrees of freedom, one *postulates* that the constants of the motion I_i, e.g., the angular momentum, obey the condition

$$I_i = \frac{1}{2\pi}\oint p_i dq^i = \nu_i \hbar, \qquad (i = 1, 2, ..., N; \nu_i = 0, 1, 2, ...) \tag{1}$$

where each integral is taken along the trajectory which the system *actually* runs during a cycle of the motion of the coordinate q^i. The quantized values of the energy are obtained when $E = H(I_i)$. Assumption (1) is neither mathematically well-defined, since is not invariant under canonical transformations [10], nor does entirely agree with the experimental data: half-integer quantum numbers are not taken into account.

Unsatisfied with rule (1) Heisenberg [11] proposed a quantization method by starting with the Newtonian equation of motion $dp^i/dt = f(q_i, t)$, where $p^i = m dq_i/dt$ and *postulating* that in the quantum domain the variables q^i and p_i are replaced respectively by the operators $\hat{q}^i$ and $\hat{p}_i$ which obey the Born-Jordan-Dirac relations [12,13]:

$$\begin{array}{l} [\ \hat{q}^i, \hat{q}^j] = 0 = [\ \hat{p}_i, \hat{p}_j] \\ [\ \hat{q}^i, \hat{p}_j] = \imath\hbar\delta_{i,j}, \end{array} \tag{2}$$

δ being the Kronecker delta, so Eq.(2) replaces conditions (1) and allows to calculate the energy levels. This quantization method of Heisenberg has the advantage to be based on Newton's equations, so that it can be applied to stochastic process described by the Langevin equations [14]. Nevertheless, the main (theoretical) difficulty of this method is related to the ambiguities arising from the operator ordering in the calculation of physical quantities $A(\hat{q}^i, \hat{p}_i)$ [14].

In order to disclose the true nature of the quantum conditions (1) Schrödinger [15] proposed a different method of quantization: One starts with the Hamilton-Jacobi equation for the function $S(q_i, t)$ and one introduces the relation $S(q_i, t) = -\imath\ell \ln\psi(q_i, t)$, ℓ having dimensions of action and ψ being a complex value function. Inserting $S = -\imath\ell\ln\psi$ into the Hamilton-Jacobi equation one can build a functional $\mathcal{F}(\psi, \psi^\dagger)$ and *assume* that the integral $\mathcal{I}(\psi, \psi^\dagger) = \int \psi\psi^\dagger \mathcal{F} dq^i$ be an extreme: $\delta\mathcal{I} = 0$. From this variational problem, which replaces the *ad hoc* Bohr-Sommerfeld-Wilson rules, and making $\ell \rightarrow \hbar$ one arrives at the Schrödinger equation. Using this method Schönberg [16] has obtained non-linear

Schrödinger and Dirac equations from the generalization of the Hamilton-Jacobi equation for a charged particle subjected to arbitrary potentials; while Razavy [17], Herrera *et al.* [18] and Pal [19] have introduced dissipation in quantum mechanics from dissipative Hamilton-Jacobi equations. However, this procedure of quantization is severely criticized by Schrödinger himself [15,20]:

i) The relation between S and ψ given above is incompreensible (*unverständlich*). Indeed, Schrödinger originally had considered $S = \ell \ln\psi$, ψ being a *real* function;

ii) The formulation of the integral $\mathcal{I}(\psi, \psi^\dagger) = \int \psi\psi^\dagger \mathcal{F} dq^i$ is not entirely devoid of ambiguities (*nicht ganz eindeutig*);

iii) Finally, it is incompreensible (*unverständlich*) the existence of the variational principle: $\delta\mathcal{I} = 0$.

To turn around these difficulties Schrödinger left out his quantization method and tried to establish a more secure foundation for his equation making use of the Hamiltonian *analogy* between mechanics and optics [20,8]. However, it is worth noticing that these two procedures do not constitute a logical derivation of the Schrödinger equation from deeper physical principles: This equation should be taken as fundamental *postulate* of quantum mechanics. What matters is the agreement of its consequences with the experiments [21]. Hence, any search for a suitable process of quantization, i.e., a method for obtaining quantum mechanical equations of motion from the equations of classical mechanics, turns out to be considered only as a pseudo-problem. Unfortunately, this is the point of view dominant in many textbooks.

On the basis of the works by Heisenberg and Schrödinger, Dirac [13,22] introduced the following mnemonical rules of quantization: A classical system is to be described by a Hamiltonian $H(q^i, p_i)$, so that after replacing q^i by $\hat{q}^i$ and p_i by $\hat{p}_i = (\hbar/\imath)\partial/\partial q^i$, one obtains the Schrödinger equation $H(\hat{q}^i, \hat{p}_i)\psi = \imath\hbar\partial\psi/\partial t$. Besides privileging the Cartesian coordinates [23], as remarked by Dirac himself [24] this procedure of quantization is not mathematically well-defined due to the problem of operator ordering in the transition from the c-numbers to q-numbers [25,26]. The Dirac method is also too restrictive because it only works when, from an infinite set of Hamiltonian functions providing the same Newtonian equations of motion, one chooses a Hamiltonian identical with the total energy of the system. Hence, the Dirac rules are just physically inconsistent when applied to the quantization of dissipative systems [27-37] and to different classically equivalent Hamiltonians [38]. In addition, the wave function ψ upon which act the operators $\hat{q}^i, \hat{p}_i$ arises *ex nihilo* in the theory, so no light is risen to its physical interpretation.

Rather than seeking Hamiltonians, Feynman [39] started from the Lagrangian formalism of classical mechanics in order to quantize a given physical system. To this end he *postulated* that the propagator $K(q_2^i, t_2; q_1^i, t_1)$ in the equation

$$\Omega(q_2^i, t_2) = \int K(q_2^i, t_2; q_1^i, t_1)\Omega(q_1^i, t_1)d[q_1^i] \tag{3}$$

is proportional to $e^{\imath S/\hbar}$, $S = \int_{t_1}^{t_2} L dt$ being the classical action and L the Lagrangian of the system. In Eq.(3) $d[q_1^i]$ is the volume element at q_1^i. Following a determined limiting process in the calculation of $K(q_2^i, t_2; q_1^i, t_1)$, Feynman was able to obtain the Schrödinger equation for the wave function $\psi \equiv \Omega$. In spite of working out as a good computational tool [40] the Feynman formalism suffers from some drawbacks that make it unsuitable for quantization:

α) Mathematical difficulties arise in defining the Feynman integral in general [39,41];

β) There exists arbitrariness in the choise of the kernel K: The Feynman method does not provide the correct quantization of a double pendulum [42];

γ) Other source of ambiguity lies in the choice of the point at which the kernel is evaluated for infinitesimal time differences [43,44];

δ) Finally, the Feynman method does not clarify the problem of the quantization of a linearly damped harmonic oscillator [45].

In summary, the arbitrarinesses subjacent to the quantization methods of Schrödinger, Heisenberg, Dirac and Feynman are associated with the problem of transition from a c-number theory to a q-number theory, and to the fact that they are quantization methods fundamentally based on the existence of a Lagrangian or Hamiltonian from which one can arrive at the equations of motion.

In a recent work [46] we have investigated an alternative process of quantization starting from Newtonian equations of motion within the Liouvillian formulation of classical mechanics for open systems

$$\frac{\partial F}{\partial t} + \frac{p_i}{m}\frac{\partial F}{\partial q^i} + f\frac{\partial F}{\partial p_i} + \Theta(p_i, q^i, t; \alpha)F = 0, \tag{4}$$

where we have used the definition of physical momentum $mdq^i/dt = p_i$ and the Newton dynamical equations $dp_i/dt = f$, f being a force of any nature. $F = F(p_i, q^i, t) \geq 0$ is the probability density function and $\Theta(p_i, q^i, t; \alpha)F$ denotes a set of terms specifying the non-isolatedness character of the system through the parameter α. Considering the particular case in which the force f is derived from a potencial function $V(q^i, t)$, Eq.(4) turns out to be the non-Hamiltonian Liouville equation

$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial F}{\partial p_i} + \Theta(p_i, q^i, t; \alpha)F = 0, \tag{5}$$

provided the Hamiltonian has the form

$$H(p_i, q^i, t) = \frac{p_i^2}{2m} + V(q^i, t). \tag{6}$$

Thus α in Eq.(5) is called a parameter of non-Hamiltonianity of the system. In Ref. [46] we also introduced the Wigner representation of classical mechanics by means of the

Fourier transform

$$\chi(q_1^i, q_2^i, t) = \int F(p_i, q^i, t)\exp(\imath p_i \eta^i) dp_i, \tag{7}$$

where $q_1^i = q^i + \eta^i \ell/2$ and $q_2^i = q^i - \eta^i \ell/2$, ℓ being a variable parameter with dimensions of action and η^i a quantity with dimension of (linear momentum)$^{-1}$. The quantization occurs when we simultaneously consider $(q_1^i - q_2^i)^3 \ll 1$ and make $\ell \to \hbar$ in the evolution equation for $\chi(q_1^i, q_2^i, t)$. Consequently, we obtain the non-Hamiltonian quantal von Neumann equation. For the case $\alpha = 0$ in Eq.(5) the von Neumann equation, obtained by the quantization, is reducible to the Schrödinger equation.

Historically, the idea of using the Fourier transform (7) for derivation of the Schrödinger equation is atributed to Prof. S. Olbert in Hayakawa's paper [47]. Such procedure was followed by Surdin [48] within the stochastic electrodynamics. In these works ℓ (in Eq.(7)) is considered as a constant to be identified with the Planck constant. Bohm and Hiley [49] in passing use Eq.(7) in an attempt of algebrization of quantum and classical theories. More recently, in the context of the stochastic electrodynamics and using the Wigner representation of classical mechanics, Dechoun and França [50] have approximately resolved the Liouville equation for a oscillator immersed in a stochastic vacuum field; while Dechoun, Malta and França [51] studied particles following well-defined and continuous trajectory, and also continuous orientation of the spin vector by means of an approximate equation of the Pauli-Schrödinger type. Olavo [52] in turn, on deriving the Schrödinger equation, emploies Eq.(7) in its infinitesimal form about the Liouville equation (Eq.(5) above for $\alpha = 0$) attempting to establish an axiomatization of the quantum theory. Moreover he is able to obtain the operator structure of quantum mechanics and the quantization in generalized coordinates. Nevertheless his approach does ignore the existence of the Wigner representation of classical mechanics. These several procedures of quantization based on Eq.(7) [46-52] we call *dynamical quantization*.

Our main purposes in this article are:

(i) to provide an operator framework for classical mechanics thus avoiding the mathematical drawbacks subjacent to the usual methods of quantization: classical mechanics (c-number)$\mapsto$ quantum mechanics (q-number);

(ii) to explore the generality and fertility of *dynamical quantization* by quantizing the Brownian-type motion of a particle with non-linear friction and shed some light on the physical interpretation of quantal equations of motion for nonconservative systems usually found in the literature: the Caldeira-Leggett master equation, the Schrödinger-Langevin equation and equations of the Caldirola-Kanai type; and

(iii) to verify the logical consistency of our quantization scheme by evaluating the classical limit ($\hbar \to 0$) of these dynamical equations using a new classical limiting process. Thus we point out that quantization and classical limit generate a circular structure in the relation between classical and quantum mechanics.

The present paper is organized as follows. In Section II we present the Liouvillian formulation of classical mechanics. We introduce then the Wigner representation (Section III) from which we define a quantization process in Section IV. As an example, we quantize in Section V a particle in a thermal bath and subjected to a non-linear friction force and in Section VI we evaluate the classical limit of the quantum-mechanical equations of motion obtained in the previous section using a new method which we define in Appendix. Finally, we make our final remarks in Section VII.

II Liouvillian Formalism of Classical Mechanics

In the Liouville formulation of classical mechanics the state of a system is specified by a probability density function $F(z_1, z_2, ..., z_N, t)$ in terms of the N variables z_i and of time t. The evolution equation for F, generated by the (deterministic) dynamical system

$$\dot{z}_i \equiv \frac{dz_i}{dt} = K_i(z_1, ..., z_N, t), (i = 1, ..., N) \tag{8}$$

is given by the generalized Liouville equation [53]

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N} K_i \frac{\partial F}{\partial z_i} = -F \sum_{i=1}^{N} \frac{\partial K_i}{\partial z_i}. \tag{9}$$

Here we should note that the system is conservative as the divergence associated with (8), div$\mathbf{K} = \sum_{i=1}^{N} \partial K_i/\partial z_i$, is null and nonconservative as div$\mathbf{K}$ is different from zero. Using F we can calculate the mean value of any observable $A(z_1, ..., z_N, t)$: $< A >= \int AF dz_i ... dz_N$.

III The Wigner Representation of Classical Mechanics

Let us consider a Newtonian dynamical system

$$\dot{p} = f(p, q, t) \tag{10}$$

$$\dot{q} = \frac{p}{m}, \tag{11}$$

where q is the position, p the linear momentum, m the mass and f an arbitrary force. The Lioville equation is

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial F}{\partial p} = -\frac{\partial}{\partial p}(\mathcal{F} F). \tag{12}$$

where $f(p, q, t)$ was split into a conservative force $-\partial V(q, t)/\partial q$ and a nonconservative force $\mathcal{F}(q, p, t)$.

We now introduce the Wigner representation of classical mechanics using the following Fourier transform [46]

$$\chi(q + \frac{\ell\eta}{2}, q - \frac{\ell\eta}{2}, t) = \int F e^{\imath p\eta} dp. \tag{13}$$

Because the Wigner factor $e^{\imath p\eta}$ is an adimensional term and $\ell\eta$ has dimension of length, it follows that ℓ should have dimension of action. Inserting the classical Wigner function [1] (13) into Eq.(12), we obtain

$$\imath\ell\frac{\partial\chi}{\partial t}+\frac{\ell^2}{2m}\left[\frac{\partial^2\chi}{\partial q_1^2}-\frac{\partial^2\chi}{\partial q_2^2}\right]-\left[\,V(q_1,t)-V(q_2,t)-O(q_1,q_2,t)\right]\chi=-\imath\ell\Omega\chi \tag{14}$$

where

$$O(q_1,q_2,t)=-\sum_{n=3,5,7,\ldots}^{\infty}\frac{2}{n!}\left(\frac{q_1-q_2}{2}\right)^n\left(\frac{\partial}{\partial q_1}+\frac{\partial}{\partial q_2}\right)^n V(q_1,q_2,t), \tag{15}$$

$$\Omega\chi=\int\frac{\partial}{\partial p}(\mathcal{F}F)e^{\imath p\eta}dp \tag{16}$$

and

$$q_1=q+\frac{\ell\eta}{2}\quad;\quad q_2=q-\frac{\ell\eta}{2}. \tag{17}$$

From this Eq.(14) we note that the Wigner function χ is a complex valued function due the presence of $\imath=(-1)^{1/2}$. Using the inverse of Eq.(13) the average value of $A(p,q,t)$ is now given by

$$\langle A\rangle=\frac{1}{2\pi}\int\chi(q+\frac{\ell\eta}{2},q-\frac{\ell\eta}{2},t)A(p,q,t)e^{-\imath p\eta}d\eta dp dq. \tag{18}$$

It follows then the properties of χ

a) *Hermiticity*: $\langle\chi\rangle=\langle\chi^\dagger\rangle\Rightarrow\chi=\chi^\dagger$.

b) *Non-positivity*: even though $F\geq 0$, χ is not definite positive.

c) *Non-normalization*: $\int F(p,q,t)dqdp=1\Rightarrow\int\chi(q,q,t)dq=1$. However, $\chi(q_1,q_2,t)$ is not normalizable.

Now let us derive an operator structure for classical mechanics. Using (18) we write down

$$\langle p\rangle=\frac{1}{2\pi}\int(p\chi)e^{-\imath p\eta}d\eta dp dq=\frac{1}{2\pi}\int(-\imath\frac{\partial\chi}{\partial\eta})e^{-\imath p\eta}d\eta dp dq. \tag{19}$$

This result allows that we can associate with the momentum p the Hermitian operator

$$\tilde{p}=-\imath\frac{\partial}{\partial\eta}. \tag{20}$$

An analogous procedure for the position q yields the multiplicative operator

$$\tilde{q}=q. \tag{21}$$

[1] It is worth remarking that the Wigner function $\chi(q,\eta,t)$ in the classical domain (Eq.(13)) is not obtained as classical limit from the quantum Wigner function $W(q,p,t)$ (Eq.(71)).

Hence, with the general function $G(p,q)$ we associate the operator $\tilde{G} = G(\tilde{p},\tilde{q})$. The operators (20) and (21) act upon the classical Wigner function $\chi(q+\eta\ell/2, q-\eta\ell/2, t)$ defined at two points in space (q,η). By evaluating

$$[\,\tilde{q},\tilde{p}]\,\chi = (\tilde{q}\tilde{p} - \tilde{p}\tilde{q})\chi \tag{22}$$

we find

$$[\,\tilde{q},\tilde{p}] = 0 \tag{23}$$

which in turn leads to

$$\triangle q \triangle p \geq 0, \tag{24}$$

where $\triangle x = \sqrt{< x^2 > - < x >^2}$. It remains to investigate for which cases we have the expression $\triangle q \triangle p > 0$ or $\triangle q \triangle p = 0$. To this end we suppose that χ may be factorized [2], that is,

$$\chi(q+\eta\ell/2, q-\eta\ell/2, t) = \phi^\dagger(q-\eta\ell/2, t)\phi(q+\eta\ell/2, t), \tag{25}$$

and from Eq.(19) we get

$$\langle p\rangle = \int \phi^\dagger \left(\frac{\ell}{\imath}\frac{\partial}{\partial q}\right)\phi dq \tag{26}$$

after expanding $\phi^\dagger$ and ϕ according to

$$u(q \pm \frac{\eta\ell}{2}) = u(q) \pm \frac{\eta\ell}{2}\frac{\partial u}{\partial q} + \frac{1}{2!}\left(\frac{\eta\ell}{2}\right)^2\frac{\partial^2 u}{\partial q^2} \pm \frac{1}{3!}\left(\frac{\eta\ell}{2}\right)^3\frac{\partial^3 u}{\partial q^3} + O(\eta^4) \tag{27}$$

and after using the property $\int g(x)\delta(x-y)dx = g(y)$. So we can associate with p the operator

$$\hat{p} = \frac{\ell}{\imath}\frac{\partial}{\partial q} \tag{28}$$

and with q the operator

$$\hat{q} = q. \tag{29}$$

These operators act upon the function ϕ. It follows immediately that

$$[\,\hat{q},\hat{p}] = \imath\ell \tag{30}$$

and

$$\triangle q \triangle p \geq \frac{\ell}{2}. \tag{31}$$

The relation $\triangle q \triangle p = \ell/2$ occurs for Gaussian functions (See Eq.(41) or (42), below). Making $\ell = 0$ in this Eq.(31) we recover Eq.(24) and the Gaussian turns a delta function. Therefore, on the basis of the operator structure only, derived above, we show the existence of dispersion relations for all conservative and nonconservative systems *described* in the Wigner representation of classical mechanics with $\ell > 0$ playing the role of a variable geometric parameter.

[2] In general the factorization of χ occurs for conservative systems ($\Omega\chi = 0$ in Eq.(14)). For the case of the harmonic oscillator, for example, the factorization is obtained exactly, while for other potentials we can get the factorizability of χ only approximately, since the condition $(\partial/\partial q_1 - \partial/\partial q_2)^3 \ll 1$ is inserted in Eq.(14).

IV Dynamical Quantization

In order to quantize a given classical system described by Eq.(14) or (12) we impose the condition [46]

$$(q_1 - q_2)^4 \ll (q_1 - q_2) \tag{32}$$

and take the quantum limit

$$\ell \to \hbar = \frac{h}{2\pi}. \tag{33}$$

Thus, we arrive at the nonconservative von Neumann equation

$$\imath\hbar\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right] - [\, V(q_1,t) - V(q_2,t)]\,\rho = -\imath\hbar\tilde{\Omega}\rho \tag{34}$$

with $\tilde{\Omega}$ being obtained from Ω after using (32) and (33). $\rho = \rho(q - \eta\hbar/2, q + \eta\hbar/2, t)$, known also as "density matrix", is a quantum function defined in space (q, η); it inherits all properties of the classical function χ. In the case $\eta \to 0$, ρ turns to have the usual properties:

$$\int \rho(q,t)dq = 1$$

$$\rho(q,t) \geq 0.$$

(Let us note that for conservative systems Eq.(34) is the usual von Neumann equation giving rise to a Schrödinger equation at point q_1 and its complex-conjugate at point q_2). Using the Wigner transformation (see Eq.(71) below) this Eq.(34) turns to act on phase space (q, p). This phase-space representation of quantum mechanics is fundamental to evaluate the classical limit of non-consevative systems (see Sec.VI).

Conditions (32) and (33) are mathematically admissible; physically (33) establishes the quantum nature of the phenomena, while (32) is responsible for the consistence of the quantization process, i.e., here we can justify assumption (32) only operationally. Together (32) and (33) have to imply $\ell > \hbar$, so that the quantum domain is characterized by the smallness of the Planck constant with respect to the classical actions.

In brief, the method of dynamical quantization presents the following advantages:

(a) it is not based on the Lagrangian and Hamiltonian formalisms of classical mechanics; it starts from the Newtonian equations of motion, hence it may applied without physical ambiguities to deterministic and stochastic systems;

(b) it is mathematically well-defined due to the fact that classical mechanics, in the Wigner representation, is expressed as a q-number theory; thus before quantizing the ordering rule is pre-fixed as being the symmetrical or Weyl rule. One avoids therefore the known drawbacks inherent in the transition: classical physics (c-number) $\to$ quantum physics (q-number).

(c) conceptually, the trajectory concept of a particle in the classical domain does not dissapear by the quantization process (32) and (33).

V Example: Brownian type motion

Let us suppose that the theoretical construct of the physical reality lies on the fact that a system is always in interaction with the environment surrounding it. Our paradigmatic model being therefore the Brownian type motion of a physical system. An isolated system is just a particular and highly idealistic case.

Let m be the mass of a particle characterized by a position q and momentum $p = mdq/dt = m\dot{q}$ in one dimension. Acting upon this particle there exist four forces:

a) a force derived from an external scalar potencial $V(q,t)$: $f_1 = -\partial V/\partial q$;

b) a force proportional to powers of the velocity $\dot{q}$ and to a normal force $N = \partial Z(q,t)/\partial q$, where $Z(q,t)$ is a potential function between the surface of our particle in contact with the surfaces of the environment particles [54]: $f_2 = -2\gamma m\dot{q}^k - (\beta/m)\partial Z/\partial q$, with $k = 1, 2, 3, ...$, and 2γ and β being the friction coefficients; the term $-2\gamma m\dot{q}^k$ is a Markovian approximation of the non-Markovian expression

$$-\int_{-t}^{t'} K(t-s)\dot{q}^k ds$$

when $K(t-s) \approx 2\gamma m\delta(t-s)$.

c) a stochastic force derived from a general random potencial $V_R(q,t)$: $f_3 = -\partial V_R/\partial q$;

d) and, finally, a fluctuating force $f(t)$ with average value $< f(t) >= 0$ and correlation function $< f(t)f(t') >= 2D\delta(t-t')$, where δ is the Dirac delta-function and D a diffusion coefficient which could be

$$D = 2m\gamma\omega\hbar\coth\left(\frac{\omega\hbar}{k_B T}\right)$$

for a quantum thermal bath whose particles have a natural frequency ω, or

$$D = 2\gamma m k_B T$$

when the reservoir is classically described, i.e., when its quantum effects are neglected. There, k_B is the Boltzmann constant and T the temperature of the environment at thermal equilibrium.

The dynamics of our particle may be described by the following Markovian Fokker-Planck equation

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - \left[\frac{2\gamma}{M}p^k + \frac{\partial U}{\partial q}\right]\frac{\partial F}{\partial p} - \frac{2\gamma}{M}kp^{k-1}F - 2\gamma m k_B T\frac{\partial^2 F}{\partial p^2} = 0, \qquad (35)$$

where $M = m^{k-1}$, $U = V + V_R + (\beta/m)Z$, and $F = F(p,q,t) \geq 0$ is the probability density. We set the initial conditions as being $F(p,q,t=0) = \delta(p-p')\delta(q-q')$, where $p' = p(t=0)$ and $q' = q(t=0)$. Inserting into Eq.(35) the Fourier transform (13), we obtain the evolution equation in the Wigner representation of classical mechanics:

$$\frac{\partial \chi}{\partial t} - \frac{\imath}{m}\frac{\partial^2 \chi}{\partial \eta \partial q} + \imath\eta\frac{\partial U}{\partial q}\chi + \imath^{1-k}\frac{2\gamma}{M}\eta\frac{\partial^k \chi}{\partial \eta^k} + D\eta^2\chi = 0. \tag{36}$$

After changing the variables

$$q_1 = q + \frac{\ell\eta}{2} \quad ; \quad q_2 = q - \frac{\ell\eta}{2} \tag{37}$$

Eq.(36) reads

$$\imath\ell\frac{\partial \chi}{\partial t} + \frac{\ell^2}{2m}\left[\frac{\partial^2 \chi}{\partial q_1^2} - \frac{\partial^2 \chi}{\partial q_2^2}\right] - \left[\, U(q_1,t) - U(q_2,t) + O(q_1,q_2,t)\right]\chi + I\chi = 0, \tag{38}$$

with

$$O(q_1,q_2,t) = -\sum_{n=3,5,7,\ldots}^{\infty} \frac{2}{n!}\left(\frac{q_1-q_2}{2}\right)^n \left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)^n U(q_1,q_2,t), \tag{39}$$

$$I = \frac{2\gamma}{M}\imath^{2-k}\,(\,q_1-q_2)\left(\frac{\ell}{2}\right)^k\left(\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}\right)^k + \frac{\imath D}{\ell}\,(\,q_1-q_2)^2\,. \tag{40}$$

Equation (36) or (38) sets up the dynamics of classical mechanics in the Wigner space (q,η) entirely tantamount to the Fokker-Planck equation (35) in phase space (q,p).

As we are working in the Wigner representation we replace the initial condition $F = \delta(q-q')\delta(p-p')$ assumed for Eq.(35) with the Gaussian function

$$F(p,q,t=0) = \frac{1}{\pi\ell}e^{-(\epsilon/\ell)(q-q')^2}e^{-(\lambda/\ell)(p-p')^2}, \quad (\epsilon\lambda = 1) \tag{41}$$

or, equivalently,

$$\chi(q_1,q_2,t=0) = \left(\frac{\epsilon}{\pi\ell}\right)^{1/2} e^{-(\epsilon/2\ell)(q_1-q_1')^2}e^{-(\epsilon/2\ell)(q_2-q_2')^2} \tag{42}$$

for Eq.(38).

V.1- *Description in terms of the von Neumann function ρ*

We want now to quantize the Brownian system described by (38) obtained above. To this end we take simultaneously into account the "approximation" (32) and the quantum limit (33). So Eq.(38) becomes the generalized quantum von Neumann equation

$$\imath\hbar\frac{\partial \rho}{\partial t} + \frac{\hbar^2}{2m}\left[\frac{\partial^2 \rho}{\partial q_1^2} - \frac{\partial^2 \rho}{\partial q_2^2}\right] - \left[\, U(q_1,t) - U(q_2,t)\right]\rho +$$

$$\frac{2\gamma}{M}\imath^{2-k}\,(\,q_1-q_2)\left(\frac{\hbar}{2}\right)^k\left(\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}\right)^k\rho + \frac{\imath D}{\hbar}\,(\,q_1-q_2)^2\rho = 0 \tag{43}$$

with the initial condition

$$\rho(q_1, q_2, t=0) = \left(\frac{\epsilon}{\pi\hbar} \right)^{1/2} e^{-(2\epsilon/\hbar)\left(\frac{q_1+q_2}{2} - \frac{q_1'+q_2'}{2} \right)^2} e^{(\epsilon/\hbar)(q_1-q_1')(q_2-q_2')} \tag{44}$$

obtained from (42) by replacing ℓ for $\hbar$.

For $k=1$, i.e., when the friction force is linear, $U \equiv V(q,t)$ and considering as classical the reservoir ($D = 2\gamma m k_B T$) we get from (43) the equation

$$\begin{gathered} \imath\hbar \frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m} \left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2} \right] - \left[V(q_1,t) - V(q_2,t) \right] \rho + \\ \gamma\imath\hbar \left(q_1 - q_2 \right) \left(\frac{\partial\rho}{\partial q_1} - \frac{\partial\rho}{\partial q_2} \right) + \frac{2\gamma\imath m k_B T}{\hbar} \left(q_1 - q_2 \right)^2 \rho = 0 \end{gathered} \tag{45}$$

which is formally similar to the Caldeira-Leggett master equation [55] obtained by following the Feynman path integral method. Equation (45) presents the following features:

i) It is easy to verify that Eq.(45) is a particular case of the non-Markovian von Neumann equation

$$\imath\hbar \frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m} \left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2} \right] - \left[V(q_1,t) - V(q_2,t) \right] \rho + \mathcal{H}(q_1,q_2,t)\rho + \frac{\imath D}{\hbar} \left(q_1 - q_2 \right)^2 \rho = 0,$$

when one uses the (Markovian) approximation $K(t-s) \approx 2\gamma m \delta(t-s)$ into the expression for the friction force

$$v(p,t) = \frac{1}{m} \int_{-t}^{t'} K(t-s) p(s) ds.$$

present in the term

$$\mathcal{H}(q_1,q_2,t)\rho = \int \frac{\partial(vF)}{\partial p} e^{\imath\eta p} dp.$$

So, besides being Markovian the master equation (45) has not the Lindblad form [56], consequently, $\rho(q_1,q_2,t)|_{q_1=q_2=q}$ can have negative values giving rise to physically undesirable results [57-62]. It is worth remarking that the mere mathematical consistency, in turn, is not enough to assure that the master equation of the Lindblad type be physically acceptable [59,61,63-68]; this occurs because the effects of non-Markovicity are not taken into account.

ii) It should be noted that the (dynamical) quantization of non-Markovian open systems is independent of initial conditions. Our assumed initial condition (44) is a Gaussian function coupling particle and environment, provided the reservoir coordinates be given by $Q'_{eq} = (q_1' + q_2')/2$ in the thermal equilibrium at $t=0$ [61,69]; furthermore, (44) cannot be factorized, i.e., $\rho(q_1,q_2,t=0;Q'_{eq}) \neq X(q_1,q_2)Y(Q'_{eq})$. On the other hand, due the Feynman quantization method in the Caldeira-Leggett theory the derivation of (45) is only obtained by assuming that particle and environment are initially uncorrelated and by supposing the factorization of the density matrix in terms of variables of the particle and the bath. This factorization assumption is non-realistic and leads to non-physical results [59,61,64,69-72].

iii) In our quantization process we do not need to distinguish regimes of high and low temperatures, whereas after the procedure of Caldeira and Leggett the thermal reservoir is treated quantally and their master equation is *only* valid in a quasiclassical realm of quantum mechanics characterized by high temperatures [55,73]. (After Polykronakos and Tzani [74] the quantal nature of the bath may be responsible by the divergence of the quantity $< q^4 >$. Our results show that this assertion is incorrect: it is the approximate character of Eq.(45) the responsible for the divergence of this quantity and not the physical nature of the reservoir).

iv) To illustrate the application of Eq.(45) let us consider a single model for a free particle where the dominant term is the diffusion one, so that we obtain the following solution

$$\rho(q_1, q_2, t) = \rho(q_1, q_2, t = 0)e^{-(Dt/\hbar^2)(q_1-q_2)^2}, \tag{46}$$

where $D = 2\gamma m k_B T$ and $\rho(q_1, q_2, t = 0)$ is given by (44) with $q_1' = q_2' = 0$. Equation (46) yields

$$(\triangle q)^2 = \frac{\hbar}{2\epsilon} \tag{47}$$

$$(\triangle p)^2 = 2Dt + \frac{\epsilon\hbar}{2} \tag{48}$$

which reduce to the classical results as $\hbar \to 0$.

When the thermal bath is *quantally* treated we obtain rather than Eq.(45) the equation

$$\imath\hbar\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right] - [\,V(q_1,t) - V(q_2,t)]\,\rho +$$

$$\gamma\imath\hbar\,(\,q_1 - q_2)\left(\frac{\partial\rho}{\partial q_1} - \frac{\partial\rho}{\partial q_2}\right) + \frac{2m\gamma\omega\hbar\coth\left(\frac{\omega\hbar}{k_BT}\right)}{\hbar}(\,q_1 - q_2)^2\,\rho = 0$$

which also is Markovian. This equation was firstly found by Caldeira *et al.* [85] following the Feynmann quantization and making hypotheses about the weakness of the damping.

V.2- *Description in terms of wave function* ψ

It is worth noticing that Eq.(45) is irreducible to any Schrödinger equation, that is, it is impossible a description of a Brownian particle in terms of wave function. Nevertheless, if we neglect the environmental influence upon the particle, we arrive at the following approximate quantum equation ($q_1 \approx q_2$)

$$\imath\hbar\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right] - [\,U(q_1,t) - U(q_2,t)]\,\rho = 0 \tag{49}$$

which may be reduced, making $\rho = \psi^\dagger(q_2,t)\psi(q_1,t)$, to

$$\imath\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q_1^2} - [\,V(q_1,t) + V_R(q_1,t)]\,\psi + \frac{\imath\hbar\beta}{2m}\ln\left(\frac{\psi}{\psi^\dagger}\right)\psi = 0 \tag{50}$$

with initial condition $\psi(q_1, t=0) = (\alpha/\pi\hbar)^{1/4}e^{-(\alpha/2\hbar)(q_1-Q'_{eq})^2}$. We have used $U = V + (\beta/m)Z + V_R$ and $\psi = (\rho)^{1/2}\exp(\imath Z/\hbar)$. Although its correspondent master equation (49) is linear, Eq.(50) is non-linear due the presence of the potential function $Z(q,t)$.

In order to elucidate the physical meaning of Eq.(50) we present the various manners of deriving it. Employing the *Heisenberg quantization* method [11] directly on the classical Langevin equation describing a Brownian particle in contact with a thermal reservoir, Ford *et al.*[14] obtained the so-called Heisenberg-Langevin equation. Kostin [75], *assuming* that this Brownian particle has a wave function, was thus able to associate to it the Schrödinger-Langevin equation (50). Recently Razavy [76] showed that the Kostin's procedure is only valid if the nondissipative forces are linear. That implies that it is not always guaranteed that the equation originally obtained by Kostin describes actually a Brownian particle. Skagerstam [77] and Yasue[78] also derived Eq.(50) using now the assumptions of the *Nelson stochastic quantization* [79]. However, it is not clear in this approach to what extent the wave function related to the Brownian particle is a mere artifact of the considered hypothesis. Razavy [17], modifying arbitrarily the Hamilton-Jacobi equation for dissipative systems and using the *Schrödinger quantization method*, arrived at Eq.(50) as well. Herrera *et al.* [18] has shown that the Razavy-Hamilton-Jacobi equation may be derived from a variational principle. Nevertheless, Srivastava *et al.* [45] based on this variational principle and quantizing via the *Feynman method* arrived at unphysical results for the linearly damped harmonic oscillator. In *our approach* on the other hand the non-linear Schrödinger-Langevin equation (50) is indeed an approximate description of systems where the force depends only on the normal force $\partial Z/\partial q$ between the surfaces in contact. Such a force does not generate dissipation. Therefore, it is not odd for us that solutions of Eq.(38) seem to have several drawbacks [34].

Now performing the area-nonpreserving transformation in phase space

$$P = e^{2\gamma t}p \quad ; \quad Q = q \tag{51}$$

the Fokker-Planck equation (35) turns out to be for the case $k=1$:

$$\frac{\partial f}{\partial t} + \frac{P}{m}e^{-2\gamma t}\frac{\partial f}{\partial q} - e^{2\gamma t}\frac{\partial U}{\partial q}\frac{\partial f}{\partial P} - 4\gamma f - 2\gamma P\frac{\partial f}{\partial P} - 2\gamma m k_B T e^{4\gamma t}\frac{\partial^2 f}{\partial P^2} = 0, \tag{52}$$

where $f \equiv f(P,q,t) = e^{2\gamma t}F$; replacing the classical Wigner function

$$\chi(q_1,q_2,t) = \int f(P,q,t)e^{\imath P\xi}dP, \tag{53}$$

with $q_1 = q + \ell\xi/2$ and $q_2 = q - \ell\xi/2$, into Eq.(52) and quantizing via Eqs.(32),(33) we arrive at the following approximate equation

$$\imath\hbar\frac{\partial\rho}{\partial t}+\frac{\hbar^2}{2m}e^{-2\gamma t}\left[\frac{\partial^2\rho}{\partial q_1^2}-\frac{\partial^2\rho}{\partial q_2^2}\right]-e^{2\gamma t}\left[\,U(q_1,t)-U(q_2,t)\right]\rho+2\gamma\imath\hbar\rho=0 \tag{54}$$

which is reducible to

$$\imath\hbar\frac{\partial\psi}{\partial t}+e^{-2\gamma t}\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q_1^2}-e^{2\gamma t}U(q_1,t)\psi+\gamma\imath\hbar\psi=0 \tag{55}$$

and its complex conjugate at point q_2. If we had introduced the area-preserving transformation

$$P=e^{2\gamma t}p\quad;\quad Q=e^{-2\gamma t}q \tag{56}$$

rather than (51), we would have obtained the Fokker-Planck equation

$$\frac{\partial f'}{\partial t}+\frac{P}{m}e^{-4\gamma t}\frac{\partial f'}{\partial Q}-2\gamma\frac{\partial U'}{\partial Q}\frac{\partial f'}{\partial P}-2\gamma f'-2\gamma P\frac{\partial f'}{\partial P}-2\gamma mk_BTe^{4\gamma t}\frac{\partial^2 f'}{\partial P^2}=0, \tag{57}$$

where $f'\equiv f'(P,Q,t)=F(p,q,t)$ and $U'=U'(Q,t)$. Using

$$\chi'(Q+\frac{\ell\xi}{2},Q-\frac{\ell\xi}{2},t)=\int f'(P,Q,t)e^{\imath P\xi}dP, \tag{58}$$

defining new variables $Q_1=Q+\ell\xi/2$ and $Q_2=Q-\ell\xi/2$, quantizing and next considering $Q_1-Q_2\ll 1$ we obtain

$$\imath\hbar\frac{\partial\rho'}{\partial t}+\frac{\hbar^2}{2m}e^{-4\gamma t}\left[\frac{\partial^2\rho'}{\partial Q_1^2}-\frac{\partial^2\rho'}{\partial Q_2^2}\right]-\left[\,U'(Q_1,t)-U'(Q_2,t)\right]\rho'=0 \tag{59}$$

which reduces to

$$\imath\hbar\frac{\partial\psi'}{\partial t}+e^{-4\gamma t}\frac{\hbar^2}{2m}\frac{\partial^2\psi'}{\partial Q_j^2}-U'(Q_j,t)\psi'=0.(j=1,2) \tag{60}$$

Equations (55) and (60) are equations of the Caldirola-Kanai type [80,81] describing *approximately* dissipation in quantum mechanics. Here dissipation is introduced kinematically through the transformations (51) and (56).

Let us evaluate Eqs.(55) and (60) for the case of a harmonic potential $U\equiv V=m\omega^2q^2/2$ with $V_R,Z=0$. It follows then that Eq.(55) at a generic point q provides the solutions

$$\psi_n=\left(\frac{\alpha}{\pi^{1/2}2^n n!}\right)^{1/2}e^{-\left(\frac{\gamma}{2}+\frac{\imath\epsilon_n}{\hbar}\right)t}e^{-\frac{1}{2}\left(\alpha^2+\frac{\imath m\gamma}{\hbar}\right)\xi^2}H_n(\alpha\xi), \tag{61}$$

where $\xi=qe^{\gamma t}$, $\epsilon_n=(n+(1/2))\hbar\Omega$, $\alpha=(m\Omega/\hbar)^{1/2}$, $\Omega^2=(\omega^2-\gamma^2)$ is the damped frequency, and $H_n(\alpha\xi)$ denotes the Hermite polinomials. Thus we find

$$\langle E(p,q)\rangle=\left(n+\frac{1}{2}\right)\frac{\hbar\omega^2}{2\Omega}\left(1+e^{-4\gamma t}\right) \tag{62}$$

and

$$\triangle q \triangle p = \left(n + \frac{1}{2} \right) \frac{\hbar\omega}{2\Omega} e^{-2\gamma t} \tag{63}$$

as being the means values of the energy $E = p^2/2m + m\omega^2 q^2/2$ and the uncertainty relation, respectively. Whereas in terms of Q, P we have

$$\langle E(P,Q) \rangle = \left(n + \frac{1}{2} \right) \frac{\hbar\omega^2}{2\Omega} \left(1 + e^{-4\gamma t} \right) \tag{64}$$

and

$$\triangle Q \triangle P = \left(n + \frac{1}{2} \right) \frac{\hbar\omega}{2\Omega}. \tag{65}$$

On the other hand, the solutions of Eq.(60) are given by

$$\psi_n' = \left(\frac{\alpha}{\pi^{1/2} 2^n n!} \right)^{1/2} e^{\left(\gamma + \frac{\imath \epsilon_n}{\hbar} \right) t} e^{-\frac{1}{2}\left(\alpha^2 + \frac{2\imath m \gamma}{\hbar} \right) Q^2 e^{4\gamma t}} H_n(\alpha Q e^{2\gamma t}), \tag{66}$$

yielding

$$\langle E(p,q) \rangle = \langle E(P,Q) \rangle = \left(n + \frac{1}{2} \right) \frac{\hbar\omega^2}{\tilde{\Omega}} \tag{67}$$

and

$$\triangle q \triangle p = \triangle Q \triangle P = \left(n + \frac{1}{2} \right) \frac{\hbar\omega}{\tilde{\Omega}} \tag{68}$$

with $\tilde{\Omega}^2 = \omega^2 - 4\gamma^2$.

To end this section we emphasize that the dynamical quantization method does reveal that the von Neumann function (or density matrix) is the fundamental object of quantum mechanics. The usual Schrödinger equation is *derived* from von Neumann equation only for systems without dissipation and fluctuation (isolated systems).

VI Classical limit

In order to verify the logical consistency of our quantization procedure our purpose in this section is to investigate the classical limit of the von Neumann equation (43) in quantum phase space, the non-linear Schrödinger-Langevin equation (50), and the equations of the Caldirola-Kanai type (55) and (60), and also its respective expressions in phase space. To this end we employ the method of classical limit defined in Appendix. Let us begin applying this method to the quantal Brownian motion with non-linear friction given by Eq.(43) in phase space

$$\hbar\frac{\partial W}{\partial t} + \hbar\frac{p}{m}\frac{\partial W}{\partial q} - \left[\hbar\frac{2\gamma}{M}p^k + \hbar\frac{\partial U}{\partial q} \right] \frac{\partial W}{\partial p} - \hbar\frac{2\gamma}{M}kp^{k-1}W - 2\hbar\gamma m k_B T \frac{\partial^2 W}{\partial p^2} + GW = 0, \tag{69}$$

with

$$GW = -\frac{2}{\imath 3!}\left(\frac{-\hbar}{2\imath} \right)^3 \frac{\partial^3 U}{\partial q^3}\frac{\partial^3 W}{\partial p^3} - \frac{2}{\imath 5!}\left(\frac{-\hbar}{2\imath} \right)^5 \frac{\partial^5 U}{\partial q^5}\frac{\partial^5 W}{\partial p^5} - ..., \tag{70}$$

obtained after using the quantum Wigner function [82]:

$$W(p,q,t)=\frac{1}{2\pi}\int\rho(q+\frac{\hbar\eta}{2},q-\frac{\hbar\eta}{2},t)e^{-\imath p\eta}d\eta. \tag{71}$$

The transformation (92) (see Appendix) given in the form $W'=e^{-\alpha\xi/\hbar}W$, α being a infinitesimal parameter so that $\alpha^2\approx 0$, and the limit $\hbar\to 0$ provide the equation

$$\frac{\partial\xi}{\partial t}+\frac{p}{m}\frac{\partial\xi}{\partial q}-\left[\frac{2\gamma}{M}p^k+\frac{\partial U}{\partial q}\right]\frac{\partial\xi}{\partial p}-\frac{2\gamma}{M}kp^{k-1}\xi-2\gamma mk_BT\frac{\partial^2\xi}{\partial p^2}=0, \tag{72}$$

which becomes indeed Eq.(35) when $\xi\equiv F\geq 0$. This Eq.(72) is obtained since the following asymptotic conditions be obeyed:

$$\lim_{\hbar\to 0}W'\sim W''\neq 0 \tag{73}$$
$$\lim_{\hbar\to 0}\hbar W'\sim\alpha\xi W'' \tag{74}$$
$$\lim_{\hbar\to 0}\hbar^n W'\sim 0,\quad (n=2,4,6,...) \tag{75}$$
$$\lim_{\hbar\to 0}\hbar\frac{\partial W'}{\partial x}\sim 0,\quad (x=q,t) \tag{76}$$
$$\lim_{\hbar\to 0}\frac{\partial W'}{\partial p}\sim 0 \tag{77}$$
$$\lim_{\hbar\to 0}\hbar^j\frac{\partial^n W'}{\partial p^n}\sim 0,\quad (j,n=1,2,3,...). \tag{78}$$

In the expression (78), $n\leq j$ for j even and $n=j$ for j odd. Eqs.(73-78) are valid in a semiclassical domain of quantum mechanics not necessarily specified by high temperature as claimed in the Caldeira-Leggett approach [55] for the linear case: $k=1$.

Applying Eq.(92) on the Schrödinger-Langevin equation (50) we get

$$\left[\frac{\partial\xi}{\partial t}-\frac{1}{2m}\left(\frac{\partial\xi}{\partial q}\right)^2-V-V_R-\frac{\beta}{m}Z\right]\psi'+\imath\hbar\frac{\partial\psi'}{\partial t}-\frac{\imath\hbar}{2m}\frac{\partial^2\xi}{\partial q^2}\psi'-\frac{\hbar}{m}\frac{\partial\xi}{\partial q}\frac{\partial\psi'}{\partial q}+\frac{\hbar^2}{2m}\frac{\partial^2\psi'}{\partial q^2}=0. \tag{79}$$

Since

$$\lim_{\hbar\to 0}\psi'\sim\psi''\neq 0 \tag{80}$$
$$\lim_{\hbar\to 0}\hbar\frac{\partial\psi'}{\partial x}\sim 0,(x=q,t) \tag{81}$$
$$\lim_{\hbar\to 0}\hbar^l\frac{\partial^l W'}{\partial p^l}\sim 0,(l=1,2) \tag{82}$$
$$\tag{83}$$

we arrive at

$$\frac{\partial S}{\partial t}-\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2-V-V_R-\frac{\beta}{m}Z=0 \tag{84}$$

which is a conservative Hamilton-Jacobi equation; while the evaluation of the classical limit of Eq.(50) in phase space yields

$$\frac{\partial F}{\partial t}+\frac{p}{m}\frac{\partial F}{\partial q}-\frac{\partial}{\partial q}\left[V+V_R+\frac{\beta}{m}Z\right]\frac{\partial F}{\partial p}=0, \tag{85}$$

Analogously, we obtain

$$\frac{\partial S}{\partial t}+\frac{e^{-2\beta t}}{2m}\left(\frac{\partial S}{\partial q}\right)^2+e^{2\gamma t}\left(V+V_R+\frac{\beta}{m}Z\right)=0 \tag{86}$$

as the classical limit of Eq.(55). Already from limit $\hbar\to 0$ about (55) in quantum phase space we have

$$\frac{\partial F}{\partial t}+e^{-2\gamma t}\frac{p}{m}\frac{\partial F}{\partial q}-e^{2\gamma t}\frac{\partial}{\partial q}\left[V+V_R+\frac{\beta}{m}Z\right]\frac{\partial F}{\partial p}=0. \tag{87}$$

It is interesting to note that from (86) and (87) with $V_R, Z=0$ we find the Bateman Hamiltonian [83]

$$\frac{e^{-2\gamma t}}{2m}p^2+e^{2\gamma t}V \tag{88}$$

as the classical limit of a quantum system whose dissipation is introduced *kinematically* only. That means that in classical mechanics this Hamiltonian does not describe actually a dissipative system [29,30], although it leads to the same equations of motion.

Finally the classical limit of Eq.(60) and its correspondent in phase space are given by

$$\frac{\partial S'}{\partial t}+\frac{e^{-4\gamma t}}{2m}\left(\frac{\partial S'}{\partial Q}\right)^2+e^{2\gamma t}\left(V'+V'_R+\frac{\beta}{m}Z'\right)=0, \tag{89}$$

and

$$\frac{\partial F'}{\partial t}+e^{-4\gamma t}\frac{p}{m}\frac{\partial F'}{\partial Q}-\frac{\partial}{\partial Q}\left[V'+V'_R+\frac{\beta}{m}Z'\right]\frac{\partial F'}{\partial P}=0, \tag{90}$$

respectively.

VII Summary and concluding remarks

In this paper we saw that the use of the classical Wigner transformation (a Fourier transform) leads the dynamical description of a physical system in phase space to one in the Wigner space (the Wigner representation of classical mechanics). In this new representation we have also shown that it is possible to build an operator structure and to find uncertainty relations to classical mechanics similar those of quantum mechanics. Hence, the main object of this paper was to defend the point of view that the Wigner representation of classical mechanics does constitute the suitable *locus* for quantizing classical systems because it starts directly from the equations of motion, thus avoiding all ambiguities, difficulties and lack of generality, inherent in the quantization methods based on Lagrangians and Hamiltonians.

For example, by considering the thermal bath having a classical nature we have quantized the motion of a Brownian particle with non-linear friction described by the Fokker-Planck equation in phase space, and obtained a quantum master equation that, as a particular case, reduces to the master equation of Caldeira-Leggett for linear friction. It is worth remarking that there exists no wave function associated with the Brownian particle; nevertheless, when the environmental influence is neglected it is possible to describe approximately the motion of this particle either by means of the conservative Schrödinger-Langevin equation, or by means of equations of motion of the Caldirola-Kanai type whose dissipative nature arises only kinematically. In addition, in order to become our approach self-consistent we have successfully calculated the classical limit $\hbar \to 0$ of these quantum equations.

On the other hand, by quantizing our particle under the influence of a quantum reservoir we arrived to the equation obtained by Caldeira *et al.* in Ref. [85]. In contrast, we did not make considerations about the weakness of the damping.

In brief, the (dynamical) quantization occurs only for the particle variables, while the thermal bath may be described classically or quantally. The quantum equations are obtained without the construction of Hamiltonian models, without requiring regimes of low or high temperature, and weak or strong damping.

In general one believes that quantum mechanics is the universal theory of matter, classical mechanics being, therefore, a mere particular case from that (see, e.g., the classical limiting methods: Ehrenfest's theorem and the WKB approximation). On the other hand, using other quantization methods (see Refs. [52,79,84]) one attempts to reduce quantum mechanics to classical mechanics. In the present approach, quantization and classical limit generate a *circular* relationship between these two mechanics. Here, we have chosen classical mechanics as the starting point. A deeper study of these epistemological questions will appear elsewhere.

Even though the quantization and classical limiting processes of our paper are *consistent*, i.e., lead to the respective quantum and classical equations, we should note that two questions still remain open:

(a) What is the physical reason behind the condition (32) which in turn allows to define a quantization method? For isolated systems the condition (32) is related to the concept of equilibrium entropy [84]; however, for nonconservative systems (out the equilibrium) we cannot use this procedure.

(b) Why the parameter α in Eq.(92) (see Appendix) must be arbitrary?

In spite of these two open questions, in Ref.[86] we have also quantized and calculated the classical limit of the following nonconservative deterministic systems: a linearly damped particle, a van der Pol system and a Duffing system; while in Ref.[87] we have evaluated the classical limit of the Pauli and the Dirac equations in quantum phase space.

Acknowledgements

The author thanks A. O. Caldeira and the Referee for critical comments. This work was financially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq/Brazil).

Appendix

Let

$$D_\hbar \Psi_\hbar = 0 \tag{91}$$

be a differential equation describing a quantum system. We perform the transformation

$$\Psi'_\hbar = e^{-\imath\alpha\xi/\hbar}\Psi_\hbar, \tag{92}$$

so that (91) turns out to be given by

$$D'_\hbar \Psi'_\hbar = 0 \tag{93}$$

with $D'_\hbar \equiv e^{-\imath\alpha\xi/\hbar} D_\hbar e^{\imath\alpha\xi/\hbar}$, where α is an arbitrary parameter characterizing the transformation (92). The classical limit $\hbar \to 0$ about (93) yields the classical equation of motion for ξ (independent of $\hbar$): $D^\xi = 0$, since asymptotic conditions can be imposed on the behaviour of the functions $\Psi'_\hbar, \hbar\Psi'_\hbar$, etc., and their derivatives. As an example, let us consider the one-dimensional Schrödinger equation for a particle with mass m subjected to an external scalar potential $V = V(q,t)$

$$\Theta_\hbar \psi_\hbar(q,t) = 0, \tag{94}$$

where $\Theta_\hbar = (-\hbar/2m)(\partial^2/\partial q^2) + V - \imath\hbar\partial/\partial t$. We perform the following unitary transformation

$$\psi'_\hbar = e^{-\imath\xi/\hbar}\psi_\hbar \quad ; \quad \Theta'_\hbar = e^{-\imath\xi/\hbar}\Theta_\hbar e^{\imath\xi/\hbar}, \tag{95}$$

so that Eq.(94) becomes

$$\left\{ \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial q^2} - \frac{\imath\hbar}{2m}\left[\frac{\partial^2\xi}{\partial q^2} + 2\frac{\partial\xi}{\partial q}\frac{\partial}{\partial q} + 2m\frac{\partial}{\partial t}\right] + \left[\frac{1}{2m}\left(\frac{\partial\xi}{\partial q}\right)^2 + V + \frac{\partial\xi}{\partial t}\right]\right\}\psi'_\hbar = 0. \tag{96}$$

We take $\hbar \to 0$ about Eq.(96) and obtain (with $\xi = S(q,t)$)

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V + \frac{\partial S}{\partial t} = 0 \tag{97}$$

which in turn may be put in the form $-\partial S/\partial t = H(\partial S/\partial q, q, t)$, because

$$\lim_{\hbar\to 0}\left\{\hat{p}' = e^{-\imath S/\hbar}\left(-\imath\hbar\partial/\partial q\right)e^{\imath S/\hbar}\right\} = \partial S/\partial q \equiv p.$$

Equation (97) is exactly the well-known Hamilton-Jacobi partial differential equation of classical mechanics. An important point to be emphasized is that the limit $\hbar \to 0$ about (96) only there exists if the following *asymptotic* conditions are obeyed:

$$\lim_{\hbar\to 0} \psi'_\hbar \sim \psi''_\hbar \neq 0 \tag{98}$$

$$\lim_{\hbar\to 0} \hbar\psi'_\hbar \sim 0 \tag{99}$$

$$\lim_{\hbar\to 0} \hbar\psi'_\hbar \sim 0 \tag{100}$$

$$\lim_{\hbar\to 0} \hbar\frac{\partial\psi'_\hbar}{\partial x} \sim 0, (x = q, t) \tag{101}$$

$$\lim_{\hbar\to 0} \hbar^2\frac{\partial^2\psi'_\hbar}{\partial q^2} \sim 0. \tag{102}$$

A non-trivial consequence of our classical limiting method is the following. For the superposition of WKB functions $\psi_\hbar = e^{\imath A/\hbar} + e^{\imath B/\hbar}$, with $A = S_0 + (\hbar/\imath)S_1 + (\hbar/\imath)^2 S_2 + ...$ and $B = S_0 - (\hbar/\imath)S_1 - (\hbar/\imath)^2 S_2 - ...$, our conditions (98-102) are also satisfied. This means that the validity conditions of the WKB approximation are not necessary to obtain the classical limit of the Schrödinger equation.

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