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Polyakov's Fermi-Bose Transmutation in 3D-Abelian Thirring Model, A Path Integral Approach

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ABSTRACT

We analyze in the Path Integral Formalism the Polyakov's Fermi-Bose Transmutation in the context of the 3D Abelian Thirring model.

Key-words: Polyakov's transmutation; Three-dimensional quantum field theory; Path integral.

The Polyakov's Fermi-Bose Transmutation in the infrared regime of the (CP^1 model ([1]) has became a basic phenomenum to understand high T_c superconductivity in a Quantum Field Framework ([2],[3]). In this paper we present the above cited phenomenum in an Abelian four-fermion theory (the Thirring Model) by using the quantum field path integral formalism.

Let us start our study by considering the 3D-massive Abelian Thirring lagrangian in the Euclidean world

$$\mathcal{L}_{(a)}(\psi_1,\psi_2) = \psi_1(i\gamma\partial)\psi_2 + g^2(\psi_1\gamma^\mu\psi_2)^2 + m\psi_1\psi_2 \tag{1}$$

Here the (Euclidean) complex Fermi fields are denoted by (ψ_1,ψ_2) and g^2 is the model positive coupling constant. The 3D-Euclidean γ^μ matrices obey the usual relationship

$$\{\gamma_\mu,\gamma_\nu\} = \delta_{\mu\nu};[\gamma^\mu,\gamma^\nu] = i\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{2}$$

The Euclidean fields $\psi_1^\alpha(x)$ and $\psi_2^\beta(x)$ satisfy the Euclidean anti-commuting relations $(\alpha,\beta = 1,2,3)$

$$\{\psi_1^\alpha(x),\psi_2^\beta(x)\} = \delta_{\alpha\beta}\delta^3(x-y) \tag{3}$$

The Lagrangian (1) is invariant under the global Abelian group $\psi_1 \to e_1^{i\alpha}$; $\psi_2 \to e^{-i\alpha}\psi_2$ with the Noetherian conserved current

$$\partial_\mu(\psi_1\gamma^\mu\psi_2)(x) = 0 \tag{4}$$

In order to analyze the Polyakov's Boson-Fermion transmutation, we consider the generating functional

$$Z_{(a)}[n,\bar{n}] = \frac{1}{Z_{(a)}[0,0]}\int D^F[\psi_1(x)]D^F(\psi_2(x)]exp\left\{-\int d^3x\left(\mathcal{L}_{(a)}(\psi_1,\psi_2) + \bar{n}\psi_1 + n\psi_2\right)\right\} \tag{5}$$

By making use of the Hubbard-Stratonovich field reparametrization, we rewrite eq. (4) in a form useful for our purposes

$$\begin{aligned} Z_{(a)}[n,\bar{n}] = {} & \frac{1}{Z_{(a)}[0,0]}\int D^F[\psi_1(x)]D^F(\psi_2(x)].D^F[A_\mu(x)] \\ & exp\left(-\frac{1}{2}\int d^3x A_\mu^2(x)\right)\delta^{(F)}[(\partial_\mu A_\mu)(x)] \\ & exp\left(-\int d^3x[\psi_1(i\gamma\partial + g\gamma A + m)\psi_2 + \bar{n}\psi_1 + n\psi_2]\right) \end{aligned} \tag{6}$$

where $A_\mu(x)$ is an auxiliary Euclidean Abelian real vector field satisfying the Landau gauge as a consequence of Eq. (4).

At this point, it becomes important to remark that the fermionic measure $D^F[\psi_1(x)D^F[\psi_2(x)]$ in eq. (6) is defined in terms of the normalized eigenvectors of the self-adjoint Euclidian Dirac operator $i\gamma_\mu(\partial_\mu - igA_\mu)$ since we want to keep the model's physical local gauge invariance in the pure fermion sector of the theory

$$\begin{aligned} \psi_1(x) &\to \psi_1(x)\,e^{i\alpha(x)} \\ \psi_2(x) &\to \psi_2(x)\,e^{-i\alpha(x)} \\ A_\mu(x) &\to A_\mu(x) \end{aligned} \tag{7}$$

Note that this local Abelian gauge invariance in the fermionic parametrization eq. (1) is a consequence of the current conservation eq. (4) at the quantum level of the generating functional eq. (5) and differs from the usual local gauge invariance of the gauge models. The local invariance eq. (7) is a consequence of the following path integral identity

$$\int D^F[\psi_1(x)e^{i\alpha(x)}]D^F[\psi_2(x)e^{-i\alpha(x)}]exp\left\{-\int d^3x(\mathcal{L}_{(a)}(\psi_1\, e^{i\alpha(x)},\psi_2\, e^{-i\alpha(x)}\right\} =$$

$$\int D^F[\psi_1(x)]D^F[\psi_2(x)\; exp\left\{-\int d^3x[\mathcal{L}_{(a)}(\psi_1(x),\psi_2(x)) - \alpha(x)\partial_\mu(\psi_1\gamma^\mu\psi_2(x)]\right\} \quad (8)$$

In this quantum field path integral framework, the infrared Polyakov's Fermi-Bose transmutation ([1]) may be understood as the large fermion mass limit of the otherwise trivial 3D-Abelian Quantum Field Thirring model ([4]).

Explicitly we should introduce a cut-off in eq. (6), write the effective $m \to \infty$ field theory and at the end of this procedure remove the cut-off in the final result.

$$Z_{(b)}[n,\bar{n}] = \frac{1}{Z_{(a)}(0,0)}\int D^F[A_\mu(x)]\; exp\left(-\frac{1}{2}\int d^3x(A_\mu^2(x)\right)$$

$$det[i\gamma\partial + g\gamma A + m)]\delta^{(F)}[(\partial_\mu A_\mu)]exp\left\{+\frac{1}{2}\int d^3x d^3y\; \bar{n}(x)(i\gamma\partial + g\gamma A + m)^{-1}n(y)\right\} \quad (9)$$

The fermion vacuum loops associated to the fermion functional determinant may be easily evaluated at the limit of large mass by using the proper-time definition for this functional determinant. We have, therefore, the definition

$$log\; det[i\gamma\partial + gA\gamma + m] = -\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{dt}{2t}\, Tr_{(F)}\left(e^{-t[i\gamma\partial+g\gamma A+m]^2}\right) \quad (10)$$

where $Tr_{(F)}$ denote the functional trace.

We have thus, the following result for the family of interpolating Dirac operator $i\gamma\partial + sgA + m, 0 \le s \le 1$:

$$\frac{d}{ds}[log\; det[i\gamma\partial + sg\gamma A + m]] =$$

$$\lim_{\epsilon\to 0^+}\int_\epsilon^\infty dt\; e^{-\lambda m^2}Tr_{(F)}g\gamma A(i\gamma\partial + sg\gamma A.S + m)exp(-t[i\gamma\partial + gsA + m]^2) \quad (11)$$

By taking the limit of large fermion mass as in ref. [5], we get the result below, after integrating the interpolating parameter s in the range $0 \le s \le 1$

$$log\; det(i\gamma\partial + g\gamma A + m)/det(i\gamma\partial + m) = \frac{g^2 m}{(4\pi)^{\frac{3}{2}}}.\frac{1}{\varepsilon}\int d^3x(A_\mu^2(x))$$

$$-ig^2\frac{\sqrt{\pi}}{|2|}\frac{m}{m}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x) + 0\left(\frac{1}{m^2}\right) \quad (12)$$

It is worth point out the existence of an induced (cut-off dependent) mass term for the auxiliary vector field (this auxiliary vector field at the quantum level coincides with the Noetherian $U(1)$ global current $A_\mu(x) = (\psi_1\gamma^\mu\psi_2)(x))$).

Note that this mass term signals the dynamical breaking of the usual gauge invariance in the pure fermionic sector of eq. (9) which involves the gauge field change $A_\mu \to A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x)$ as in 2D-models (see eq. [7]) and differing from the model gauge invariance physical one eq. (7). It is instructive point out that we have evaluated the functional determinant at the limit of $m \to \infty$ by not imposing the usual gauge invariace above mentioned, in the domain of the Dirac operator in the presence of the external auxiliary field $A_\mu(x)$. If this is not the case one could rule out the first term in the right hand side of eq. (12) by using the gauge invariance of this domain.

The physical consequence of this term is a renormalization of the bare fermion mass at one loop [6]

$$\frac{m_B}{\varepsilon} = m_R. \tag{13}$$

The second term in the right-hand side of eq. (12) is the Chern-Simons Lagrangian. Substituting eq. (12)-eq. (13) int eq. (9) we get the partial result at large m_R

$$\begin{aligned} Z_{(b)}[n,\bar{n},m_R \to \infty] &= \frac{1}{Z_{(b)}(0,0)}\int D^F[A_\mu(x)] \\ &exp\left\{-\frac{1}{2}\left(1-\frac{g^2}{(4\pi)^{\frac{3}{2}}}m_R\right)\int d^3x A_\mu^2\right\} \\ &exp\left\{-ig^2\frac{\sqrt{\pi}}{2}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x)\right\}\delta^{(F)}[(\partial_\mu A_\mu)(x)] \\ &exp\left\{+\frac{1}{2}\int d^3x d^3y\ \bar{n}(x)(i\gamma\partial + g\gamma A + m)^{-1}n(y)\right\} \end{aligned} \tag{14}$$

Following closely ref. [1], now we analyse the large m_R limit of the external fermion sources by considering the Feynman path integral representation for the Feynman green function of the Dirac operator in the presence of $A_\mu(x)$

$$(i\gamma\partial + g\gamma A + m_R)^{-1}_{\alpha,\beta}(x,y) = \int_0^\infty dt\ e^{-m_R t}\int_{\substack{x(0)=x\\ x(t)=y}} D^F[x(\sigma)]e^{ig\int_0^t d\sigma A_\mu(x(\sigma))dx^\mu(\sigma)}\Phi_{\alpha,\beta}(x,y) \tag{15}$$

where the spin-factor is explicitly given by

$$\Phi_{\alpha,\beta}(x,y) = \int D^F[\pi^\mu(\sigma)].e^{i\int_0^t ds(\pi^\mu.\dot{X}^\mu)(\sigma)}\ \mathbf{P}\left\{e^{i\int_0^t ds(\pi^\mu(\sigma)\gamma_\mu)}\right\} \tag{16}$$

Here $\mathbf{P}$ means the path ordenation of the 3D-γ_μ matrices along the Feynman trajectory $\{X_\mu(\sigma)\}$.

At the limit of large m_R, only the straight-line trajectory is leading to eq. (15)-eq. (16) ($t \to 0$) and producing the result

$$\lim_{m_R\to\infty}(i\gamma\partial + g\gamma A + m_R)^{-1}_{\alpha,\beta}(x,y) = (U_\alpha^{(1)}\bar{U}_\beta^{(2)}.exp(+ig\int_x^y A_\mu(x)dx^\mu) \tag{17}$$

where $U_\alpha^{(1),(2)}$ are the usual spinerial base associated to the free massive fermion fields $\{\psi^1,\psi^2\}$.

By grouping togheter eq. (17) and eq. (14) we finally obtain our Polyakov's infrared Bosonization for the 3D-Thirring model

$$\tilde{Z}[n,\bar{n}] = \frac{1}{\tilde{Z}[0,0]} \cdot \int D^F[A_\mu(x)] exp\left\{-\frac{1}{2}\left(1-\frac{g^2 m_R}{(4\pi)^{\frac{3}{2}}}\right)\int d^3x\, A_\mu^2(x)\right\}$$
$$exp\left\{-ig\frac{g^2\sqrt{\pi}}{2}\int d^3x(A_\mu \varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x)\right\}\,\delta^{(F)}[(\partial_\mu A_\mu)(x)]$$
$$exp\left\{+\frac{1}{2}\int d^3x d^3y(\bar{n}_a(x)n_b(x))(U^a\bar{U}^b) exp\left(+ig\int_x^y A_\mu(x)dx^\mu\right)\right\} \tag{18}$$

Now it is a straightforward consequence of eq. (18) the infrared (large mass) Bosonization formulae for the 3D-Abelian Thirring model

$$\psi_a^1(x) = U_a^{(1)}.exp\left(ig\int_{-\infty}^x A_\mu dx^\mu\right)$$
$$\psi_b^2(x) = \bar{U}_b^{(2)}\, exp\left(-ig\int_{-\infty}^x A_\mu dx^\mu\right) \tag{19}$$

where $A_\mu(x)$ is the quantum field associated to the "massive" Chern-Simon theory

$$\mathcal{L}(A_\mu(x)) = -\frac{1}{2}\left(1-\frac{g^2 m_R}{(4\pi)^{\frac{3}{2}}}\right)\int d^3x A_\mu^2(x) - \frac{ig^2\sqrt{\pi}}{2}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A)(x) \tag{20}$$

Equations (19) and (20) are our main result.

In the important case of the Thirring model coupled to an external divergence free current source $J_\mu(x)$.

$$\tilde{\mathcal{L}}_{(a)}(\psi_1,\psi_2,J_\mu) = \mathcal{L}_{(a)}(\psi_1,\psi_2) + J_\mu(\psi_1\gamma^\mu\psi_2)(x) \tag{21}$$

we can proceed as above and obtain the associated Polyakov's full bosonized generating functional.

$$\tilde{Z}[J_\mu] = \frac{1}{\tilde{Z}(0)}\int D^F[A_\mu(x).\delta^{(F)}[(\partial_\mu A_\mu)(x)] exp\left\{-\frac{1}{2}\int d^3x(A_\mu+J_\mu)^2(x)\right\}$$
$$\left\{+\frac{g^2 m_R}{2(4\pi)^{\frac{3}{2}}}\int d^3x\, A_\mu^2(x) - \frac{ig^2\sqrt{\pi}}{2}\int d^3x\,(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))\,(x)\right\} \tag{22}$$

We point out that we have neglected in eq. (9) the zeros modes of the Dirac operator $(i\gamma\partial + g\gamma A + m)$ under the hypothesis that this set is a set of zero functional measure in the manifold of the (random) Euclidean fermion fields $\{\psi_1(x),\psi_2(x)\}$ contributing to the generating functional eq. (6). They may be relevant only in a situation of implementing a semi-classical approximation around these Saddole-points zero modes and being not quantized within this hypothesis. In other words we have the field decomposition

$$\psi_1(x) = \psi_1^{(0)}(x) + \hbar\psi_1^q(x)$$
$$\psi_2(x) = \psi_2^{(0)}(x) + \hbar\psi_2^q(x) \tag{23}$$
$$A_\mu(x) = A_\mu^{(0)}(x) + \hbar A_\mu^q(x)$$

with

$$(i\gamma\partial + gA^{(0)}\gamma + m)\psi^{(0)}_{(1,2)}(x) = 0 \tag{24}$$

and

$$D^F[\psi_1(x)]D^F[\psi_2(x)]D^F[A_\mu(x) \sim D^F[\psi_1^q(x)]D^F[\psi_2^q(x)]D^F[A_\mu^q(x)] \tag{25}$$

In the case one quantize these zero modes it is easy to see that the fermionic normalized generating functional eq. (6) (without the $A_\mu(x)$ - funcitonal integral) as well defined if and only if the sources $\{n(x), \bar{n}(x)\}$ are orthogonal to these zero modes due to the existence of the normalization factor in the theory generating functional.

Note that the fermion vacuum energy (the theory's partition functional) is always zero when one considers quantized zero modes, opposite to the case of Saddole-point framework (eq. (23)-eq.(24)) where it is non zero and depending on the explicit geometrical topological caracterization of the moduli space associated to these zero modes.

As a last remark we write a family of zero modes of the Euclidean 3D-Dirac operator in the presence of special external $A_\mu^{vortex}(x)$ vortex fields.

Explicitly we have the following result

$$(i\gamma\partial + g\gamma A_\mu^{vortex}(x) + m)\tilde{\psi}(x) = 0 \tag{26}$$

where

$$\begin{aligned} A_0(x,y,z) &= \bar{A}_0{}^m(x,y) \\ A_1(x,y,z) &= \bar{A}_1^{(m)}(x,y) \\ A_3(x,y,z) &= 0 \end{aligned} \tag{27}$$

and

$$\tilde{\psi}(x) = \tilde{\psi}^{(m)}(x,y).e^{im\gamma_3.z}$$

Here $\{\bar{A}_0^{(m)}(x,y), \bar{A}_1^{(m)}(x,y)\}$ are 2D-vortex field configurations and $\tilde{\psi}^{(m)}(x,y)$ the associated 2D-fermion (vortex) zero modes associated to the 2D-Chern number $m(-\infty \leq m \leq \infty)$ ([7]).

Finally we remark that in the case of evaluating correlation functions of bilinear fermion fields on the non-Abelian case at large number of colors one arrive at evaluations of punctured Wilson loops in the gauge invariant non-Abelian Chern-Simons Theory which by its turn are exactly given by the fermion vertexs associated to the proposed topological fermionic string representation of refs. [3], [8] for the Thirring bosonized eq. (8) non-Abelian Chern-Simon Theory with the gauge invariant fining tuning $g^2 m_R = (4\pi)^{\frac{3}{2}}$ parametrization (see eq. (18)).

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