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ON A NEW FORMULATION OF THE HAMILTONIAN THEORY OF GRAVITY

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On a New Formulation of the Hamiltonian Theory of Gravity

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Abstract: We present an Hamiltonian formulation of Einstein's theory of gravity in Jordan's quasi-Maxwellian formulation.

Abstract: We present an Hamiltonian formulation of Einstein's theory of gravity in Jordan's quasi-Maxwellian formulation.

Gauge theories have been very successful among physicists in the last years. This success stimulated many authors to look for a description of gravitation which could be understood as a gauge theory. Although failing to it, these works have obtained interesting subsidiary results, mainly the understanding of the real importance of the quasi-Maxwellian equations of gravitation, firstly proposed by Jordan and co-workers[1] and later developped by many authors[2,3,4,5].

Recently Camenzind[4] proposed a generalization of Jordan's formulation of the theory of gravity by imposing to the Weyl conformal tensor $W^{\alpha\beta\mu\nu}$ the equation

$$(1) \qquad W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu}$$

in which the current $J^{\alpha\beta\mu}$ is to be written in terms of the energy-momentum tensor. It has been shown by Lichnerowicz[6] that under convenient initial conditions equation (1) is equivalent to Einstein's dynamics for the gravitational field.

Besides, equation (1) is structurally similar to Maxwell's equations of eletrodynamics. This remarkable property led us to conceive the idea that the Hamiltonian formulation of gravitation could be naturaly developped in the framework of Jordan's theory (eq. (1)) if we were able to write the Weyl tensor in terms of first derivatives of a third order tensor potential. This problem was solved by Cornelius Lanczos[7] about twenty years ago. Quite surprisingly such potential theory of the Weyl conformal tensor remained almost forgotten by the scientific comunity. As far as I know very few works have been published dealing with Lanczos' potentials. This seems to have been caused by a general inability of realizing how fair was Lanczos' formulation, due to the lack of a mathematical rigorous demonstration of this result. Recently, this situation was remedied with the publication of the work of Bampi and Caviglia[8].

We write the Weyl conformal tensor as:

(2)
$$-W_{\alpha\mu\beta\nu} = A_{\alpha\mu\beta;\nu} - A_{\alpha\mu\nu;\beta} + A_{\beta\nu\alpha;\mu} - A_{\beta\nu\mu;\alpha} - \frac{1}{2}(A_{\alpha\beta}+A_{\beta\alpha})g_{\mu\nu} - \frac{1}{2}(A_{\mu\nu}+A_{\nu\mu})g_{\alpha\beta} + \frac{1}{2}(A_{\alpha\nu}+A_{\nu\alpha})g_{\mu\beta} + \frac{1}{2}(A_{\mu\beta}+A_{\beta\mu})g_{\alpha\nu}$$

in which (;) means co-variant derivative and $A_{\mu\nu} \equiv A_{\mu\ \ \nu;\lambda}^{\ \lambda}$.

Tensor $A_{\alpha\beta\mu}$ has only 10 degrees of freedom in the Lanczo's gauge by imposing the conditions

(3a)
$$A_{\alpha\beta\mu} = - A_{\beta\alpha\mu}$$

(3b)
$$A_{\alpha*\beta\mu}g^{\alpha\mu} = 0$$

(3c)
$$A_{\alpha\beta\mu}g^{\alpha\mu} = 0$$

(3d)
$$A_{\alpha\beta\ \ ;\lambda}^{\ \ \lambda} = 0 \quad .$$

The star symbols means the dual operation.

Bampi and Caviglia have shown that it is enough to have conditions (3a),(3b) in order to be able to write the decomposition (2). In this case $A_{\alpha\beta\mu}$ has 20 degrees of freedom. We have worked in Lanczo's gauge (3c),(3d) in order to simplify our exposition here. This does not effect any of our main results.

In our theory we choose the density Lagrangian L_0 to be a non-trivial invariant form of lowest order constructed with Weyl tensor, that is, we set

$$L_0 = \frac{1}{8} \sqrt{-g} \, W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} \quad . \tag{4}$$

In terms of the electric and magnetic tensors, that is, $E_{\mu\nu} \equiv -W_{\mu\alpha\nu\beta}V^{\alpha}V^{\beta}$ and $H_{\mu\nu} \equiv -\overset{*}{W}_{\mu\alpha\nu\beta}V^{\alpha}V^{\beta}$ for an arbitrary observer with four-velocity V^{μ}, it takes the form:

$$L_0 = \sqrt{-g} \, (E_{\mu\nu}E^{\mu\nu} - H_{\mu\nu}H^{\mu\nu}) \equiv \sqrt{-g} \, (E^2 - H^2) \quad . \tag{4'}$$

We will present our theory here in a Gaussian system of coordinates co-moving with the velocity field V_{μ}, that is we set $V^{\mu} = \delta^{\mu}{}_0$ and $ds^2 = dt^2 - g_{ij}(x^i,t)dx^i dx^j$. It is important to notice that in this system

$$V_{\mu;\nu} = V_{\nu;\mu} \equiv \theta_{\mu\nu}$$

with

$$\theta_{\mu\nu} \equiv \sigma_{\mu\nu} + \frac{\theta}{3} h_{\mu\nu} \quad ,$$

in which $h_{\mu\nu} = g_{\mu\nu} - V_{\mu}V_{\nu}$ is the metric of the space section orthogonal to V^{μ} and $\sigma_{\mu\nu}$ is the shear of the congruence. The choice of this system of coordinates is not essential, but has the advantage of greatly simplifying our formulas.

A general formulation of the theory and its manifest covariance will be presented elsewhere, in more details.

From Lagrangian (4) we define the momentum canonically conjugated to A_{ijk} and $B_{iok} \equiv A_{iok} + A_{koi}$, which are the only dynamical variables as one can see from the Hamiltonian below. We obtain:

$$\pi^{ijk} = -\sqrt{-g}\, W^{ijko} = \sqrt{-g}\, \eta^{ijom}\, H_m{}^k$$

$$\pi^{iok} = -\sqrt{-g}\, W^{ioko} = \sqrt{-g}\, E^{ik}$$

The other momenta π_{iko} and π_{ioo} vanish identically and have to be considered as constraints (primary constraints in Dirac's[8] notation).

After a rather long but direct calculation we obtain the density Hamiltonian

$$H = \pi^{ijk} A_{ijk,o} + \pi^{iok} B_{iok,o} - L_0$$

as given by

$$H = H_C + H_M + H_N \quad ,$$

in which the canonical Hamiltonian H_c and H_m, H_n are given by

$$H_C = \sqrt{-g}\, (E^2 - H^2)$$

$$H_M = H_M\, (E^{ik}, H^{mk}, {}^{(3)}g_{ij})$$

$$H_N = H_N\, (E^{ik}, H^{nk}, \theta_{ij})$$

The propagation of the constraints $\pi^{ioo} \cong 0$ and $\pi^{iko} \cong 0$ gives origin to the divergence equations of E_{ik} and H_{ik}. The evolution equations $E_{\ell k,o}$ and $H_{\ell k,o}$, are obtained by means of the Poisson bracket with H in the standard way.

We remark that, as the usual procedure in the quasi-

-Maxwellian equations of gravity, the system must be complemented by the evolution equations for the parameters θ, σ_{ij} of the congruence of the observers. These equations (of first order) propagate these parameters to the future of a spacelike hypersurface Σ , once we know $R_{\mu\nu}(\Sigma) = 0$ and the evolution of Weyl tensor, which is obtained from the Hamiltonian H .

We obtain thus the equation $W^{\alpha\beta\mu\nu}{}_{;\nu} = 0$ which, when added to the initial condition $R_{\mu\nu}(\Sigma) = 0$, is equivalent to the dynamics of Einstein's theory of gravity. The generalization of this procedure for the case in which matter is present is straightforward and has been done elsewhere[10].

The present formalism can be thought as an alternative to the usual canonical formulation of the theory of gravity. The quantum version of our present theory is in preparation.

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