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Functional Invariance of the Ernst Equation

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Abstract

For the stationary axially symmetric vacuum spacetime we study the functional invariance of the Ernst equation with the object of finding new solutions of this complex equation. We present two classes of solutions. The first is obtained from a three parameters $SU(1,1)$ transformation. While the second arises from the assumption that there exists a functional dependence between the real and imaginary parts of the complex Ernst potential.

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I Introduction

Einstein vacuum field equations for a stationary axially symmetric spacetime reduce to the Ernst equation [1]

$$(\xi\bar{\xi}-1)\nabla^2\xi = 2\bar{\xi}\nabla\xi\cdot\nabla\xi, \tag{1}$$

where ∇ and ∇^2 are the gradient and the three-dimensional Laplacian operators respectively, $\bar{\xi}$ is the conjugated of the complex potential ξ. The solution of (1) is usually written in terms of prolate spheroidal coordinates, λ a radial coordinate and $-1 \le \mu \le 1$ an angular coordinate , which are linked to the Weyl coordinates, ρ and z, by

$$\rho = k(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}, \;\; z = k\lambda\mu, \tag{2}$$

where $k > 0$ is an arbitrary constant. The general solution of (1) can be expressed as

$$\xi(\lambda,\mu) = P(\lambda,\mu) + iQ(\lambda,\mu), \tag{3}$$

where P and Q are real functions of λ and μ. Substituting (3) into (1) we obtain

$$\begin{aligned}(P^2+Q^2-1)&\left[(\lambda^2-1)P_{,\lambda\lambda}+2\lambda P_{,\lambda}+(1-\mu^2)P_{,\mu\mu}-2\mu P_{,\mu}\right]\\ &= 2P\left[(\lambda^2-1)(P_{,\lambda}^2-Q_{,\lambda}^2)+(1-\mu^2)(P_{,\mu}^2-Q_{,\mu}^2)\right]\\ &\quad +4Q\left[(\lambda^2-1)P_{,\lambda}Q_{,\lambda}+(1-\mu^2)P_{,\mu}Q_{,\mu}\right],\end{aligned} \tag{4}$$

$$\begin{aligned}(P^2+Q^2-1)&\left[(\lambda^2-1)Q_{,\lambda\lambda}+2\lambda Q_{,\lambda}+(1-\mu^2)Q_{,\mu\mu}-2\mu Q_{,\mu}\right]\\ &= -2Q\left[(\lambda^2-1)(P_{,\lambda}^2-Q_{,\lambda}^2)+(1-\mu^2)(P_{,\mu}^2-Q_{,\mu}^2)\right]\\ &\quad +4P\left[(\lambda^2-1)P_{,\lambda}Q_{,\lambda}+(1-\mu^2)P_{,\mu}Q_{,\mu}\right],\end{aligned} \tag{5}$$

where the commas stand for differentiation with respect to the indexes.

In this paper we propose to study the invariance of the system (4) and (5) following the action of a functional transformation. More precisely, the question that we want to answer is the following. If $[P(\lambda,\mu), Q(\lambda,\mu)]$ and $[R(\lambda,\mu), S(\lambda,\mu)]$ are two solutions of this system, under which conditions can we find transformations of the form $P(R,S)$ and $Q(R,S)$ that can allow to pass from one system to the other? Then, if

$$\xi_{PQ} = P(\lambda,\mu)+iQ(\lambda,\mu), \;\; \xi_{RS} = R(\lambda,\mu)+iS(\lambda,\mu), \tag{6}$$

are two solutions of the Ernst equation (1) we look for

$$\xi_{PQ} = P(R,S)+iQ(R,S) = \xi_{PQ}(\xi_{RS}). \tag{7}$$

In two recent preprints [2, 3] the invariance of Ernst equation is considered. The effect of coordinate transformations on the Ernst equation and the conditions of its invariance are studied in [2]. In the second preprint [3], transformations on the unknown functions written explicitly with the Ernst twist potential leaving the complex field equation invariant are obtained. Here we proceed in a slightly different way, we consider the conditions for the invariance of the form of Ernst equation itself written with P and Q, which means that we look for the invariance of the system (4) and (5). This method produces more symmetrical results.

II The method

In (4) and (5) we assume a functional transformation

$$P(\lambda,\mu)=P_1\left[R(\lambda,\mu),S(\lambda,\mu)\right],\quad Q(\lambda,\mu)=Q_1\left[R(\lambda,\mu),S(\lambda,\mu)\right], \tag{8}$$

where $[R(\lambda,\mu),S(\lambda,\mu)]$ is a solution of this system. We find that this functional dependence has to verify six second order partial differential equations with unknowns P_1 and Q_1 and variables R and S. In principle two equations are sufficient since there are two unknowns.

To present the differential equations we first define the following quantities,

$$A_1 \equiv (P_1^2+Q_1^2-1)P_{1,RR}-4Q_1P_{1,R}Q_{1,R}-2P_1(P_{1,R}^2-Q_{1,R}^2), \tag{9}$$

$$B_1 \equiv (P_1^2+Q_1^2-1)P_{1,SS}-4Q_1P_{1,S}Q_{1,S}-2P_1(P_{1,S}^2-Q_{1,S}^2), \tag{10}$$

$$\begin{aligned}C_1 \equiv{}& (P_1^2-Q_1^2-1)P_{1,RS}\\ &-2Q_1(P_{1,R}Q_{1,S}+P_{1,S}Q_{1,R})-2P_1(P_{1,R}P_{1,S}-Q_{1,R}Q_{1,S},\end{aligned} \tag{11}$$

$$A_2 \equiv (P_1^2+Q_1^2-1)Q_{1,RR}-4P_1P_{1,R}Q_{1,R}+2Q_1(P_{1,R}^2-Q_{1,R}^2), \tag{12}$$

$$B_2 \equiv (P_1^2+Q_2^2-1)Q_{1,SS}-4P_1P_{1,S}Q_{1,S}+2Q_1(P_{1,S}^2-Q_{1,S}^2), \tag{13}$$

$$\begin{aligned}C_2 \equiv{}& (P_1^2+Q_1^2-1)Q_{1,RS}\\ &-2P_1(P_{1,R}Q_{1,S}+P_{1,S}Q_{1,R})+2Q_1(P_{1,R}P_{1,S}-Q_{1,R}Q_{1,S}),\end{aligned} \tag{14}$$

which have to satisfy

$$A_1+B_1=0,\quad A_2+B_2=0, \tag{15}$$

in order to preserve the invariance of (4) and (5). Now we can write the system of six partial differential equations which we present in two groups, the first arising from (4),

$$\begin{aligned}(P_1^2+Q_1^2-1)(P_{1,RR}+P_{1,SS})={}&2P_1(P_{1,R}^2-Q_{1,R}^2+P_{1,S}^2-Q_{1,S}^2)\\ &+4Q_1(P_{1,R}Q_{1,R}+P_{1,S}Q_{1,S}),\end{aligned} \tag{16}$$

$$2(P_1^2+Q_1^2-1)\frac{RP_{1,R}-SP_{1,S}}{R^2+S^2-1}+A_1=0, \tag{17}$$

$$2(P_1^2+Q_1^2-1)\frac{SP_{1,R}+RP_{1,S}}{R^2+S^2-1}+C_1=0; \tag{18}$$

and the second arising from (5),

$$(P_1^2+Q_1^2-1)(Q_{1,RR}+Q_{1,SS}) = -2Q_1(P_{1,R}^2-Q_{1,R}^2+P_{1,S}^2-Q_{1,S}^2) +4P_1(P_{1,R}Q_{1,R}+P_{1,S}Q_{1,S}), \tag{19}$$

$$2(P_1^2+Q_1^2-1)\frac{RQ_{1,R}-SQ_{1,S}}{R^2+S^2-1}+A_2=0, \tag{20}$$

$$2(P_1^2+Q_1^2-1)\frac{SQ_{1,R}+RQ_{1,S}}{R^2+S^2-1}+C_2=0. \tag{21}$$

We can verify that $P_1 = R$ and $Q_1 = S$, as well as $P_1 = S$ and $Q_1 = R$, are solutions of the system (16-21), which must be true since (R, S) is a solution by construction of the method itself.

In the following sections we build two kinds of different solutions of the systems (16-18) and (19-21) which satisfy the invariance of Ernst equation. In the next section we obtain the solution of the first kind satisfying a $SU(1,1)$ transformation with three parameters. In section IV the second kind of solutions is presented which satisfy a functional dependence between P_1 and Q_1.

III Solutions arising after $SU(1,1)$ transformation

Since we know that a particular solution of the system (16-18) and (19-21) is $P_1 = R$ and $Q_1 = S$ we can consider the complex potential ξ_{RS},

$$\xi_{RS} = R + iS, \tag{22}$$

and build a new solution, which we call P and Q, after using a three parameters transformation [5, 6],

$$\xi = P + iQ = \frac{c_1\xi_{RS}+d_1}{\bar{d}_1\xi_{RS}+\bar{c}_1}, \tag{23}$$

with c_1 and d_1 complex and

$$\begin{pmatrix} c_1 & d_1 \\ \bar{d}_1 & \bar{c}_1 \end{pmatrix} \in SU(1,1), \;\; |c_1|^2-|d_1|^2=1. \tag{24}$$

We call the attention that (23) is not, strictly, an Ehlers transformation (which is wrongly stated as such in [5, 6]), which belongs to the group $SU(2)$ and has only one parameter. The transformation (23) belongs to the group $SU(1,1)$ or $QU(2)$ (see [4] for the group classification).

Now we can calculate $P(R, S)$ and $Q(R, S)$ and verify easily that they are solutions of the system (16-21). The inverse solution of (23) is obviously also a solution of the same system. The solution (23) is more general than the one obtained in [3] but still not, of course, the most general. We can find other types of solutions as we show in the next section.

IV Solution with functional dependence between P and Q

Here we look for a solution of (16-21) with the form of a functional dependence between P and Q through a new unknown function σ,

$$P = P\left[\sigma(R,S)\right], \quad Q = Q\left[\sigma(R,S)\right]. \tag{25}$$

Then (16-18) with (25) becomes,

$$\begin{aligned}
&(P^2+Q^2-1)P_{,\sigma}(\sigma_{,RR}+\sigma_{,SS}) \\
&+(\sigma_{,R}^2+\sigma_{,S}^2)\left[(P^2+Q^2-1)P_{,\sigma\sigma}-2P(P_{,\sigma}^2-Q_{,\sigma}^2)-4QP_{,\sigma}Q_{,\sigma}\right]=0, && (26)\\
&(P^2+Q^2-1)P_{,\sigma}\left[\sigma_{,RR}+2\frac{R\sigma_{,R}-S\sigma_{,S}}{R^2+S^2-1}\right] \\
&+\sigma_{,R}^2\left[(P^2+Q^2-1)P_{,\sigma\sigma}-2P(P_{,\sigma}^2-Q_{,\sigma}^2)-4QP_{,\sigma}Q_{,\sigma}\right]=0, && (27)\\
&(P^2+Q^2-1)P_{,\sigma}\left[\sigma_{,RS}+2\frac{S\sigma_{,R}+R\sigma_{,S}}{R^2+S^2-1}\right] \\
&+\sigma_{,R}\sigma_{,S}\left[(P^2+Q^2-1)P_{,\sigma\sigma}-2P(P_{,\sigma}^2-Q_{,\sigma}^2)-4QP_{,\sigma}Q_{,\sigma}\right]=0. && (28)
\end{aligned}$$

This system (26-28) is satisfied if

$$(P^2+Q^2-1)P_{,\sigma\sigma}-2P(P_{,\sigma}^2-Q_{,\sigma}^2)-4QP_{,\sigma}Q_{,\sigma}=0, \tag{29}$$

with

$$\begin{aligned}
\sigma_{,RR}+\sigma_{,SS}&=0, && (30)\\
\sigma_{,RR}+2\frac{R\sigma_{,R}-S\sigma_{,S}}{R^2+S^2-1}&=0, && (31)\\
\sigma_{,RS}+2\frac{S\sigma_{,R}+R\sigma_{,S}}{R^2+S^2-1}&=0. && (32)
\end{aligned}$$

While for (19-21) with (25) the system is satisfied if

$$(P^2+Q^2-1)Q_{,\sigma\sigma}+2Q(P_{,\sigma}^2-Q_{,\sigma}^2)-4PP_{,\sigma}Q_{,\sigma}=0, \tag{33}$$

plus again the same set (30-32). We see that (30-32), which determines $\sigma(R,S)$, is a linear system of partial differential equations, while the system (29) and (33), that determines

$P(\sigma)$ and $Q(\sigma)$, is non linear. We observe too that this last system is invariant under the transformation $P[\chi(\sigma)]$ and $Q[\chi(\sigma)]$ with

$$\chi = \alpha\sigma + \beta, \tag{34}$$

where α and β are constants. Furthermore, if we introduce a complex function X, like

$$X = P + iQ, \tag{35}$$

in the system (29) and (33) then it can be written

$$(X\bar{X} - 1)X_{,\sigma\sigma} = 2\bar{X}X_{,\sigma}^2, \tag{36}$$

which has a first integral

$$X_{,\sigma}\bar{X}_{,\sigma} = (1 - X\bar{X})^2, \tag{37}$$

or

$$P_{,\sigma}^2 + Q_{,\sigma}^2 = (P^2 + Q^2 - 1)^2. \tag{38}$$

IV.1 Solution for $\sigma(R, S)$

The expression (31) can be written in two different ways,

$$\Big[(R^2 + S^2 - 1)\sigma_{,R}\Big]_{,R} - 2S\sigma_{,S} = 0, \tag{39}$$

$$\Big[(R^2 + S^2 - 1)\sigma_{,S}\Big]_{,S} - 2R\sigma_{,R} = 0; \tag{40}$$

as well as (32) can be written similarly,

$$\Big[(R^2 + S^2 - 1)\sigma_{,R}\Big]_{,S} + 2R\sigma_{,S} = 0, \tag{41}$$

$$\Big[(R^2 + S^2 - 1)\sigma_{,S}\Big]_{,R} + 2S\sigma_{,R} = 0. \tag{42}$$

The integration of (39-42) produces a homogeneous differential equation of first order,

$$R\sigma_{,R} + S\sigma_{,S} + \sigma = 0, \tag{43}$$

where we have suppressed the integration constant. Integrating once more we obtain from (43)

$$\sigma(R, S) = \frac{f(\tau)}{R}, \tag{44}$$

where f is an arbitrary function of $\tau = S/R$. In order that σ, as given by (44), behaves as an harmonic function, since (30) must hold, then f has to satisfy

$$(1 + \tau^2)f_{,\tau\tau} + 4\tau f_{,\tau} + 2f = 0, \tag{45}$$

which has the solution

$$f = \frac{a + b\tau}{1 + \tau^2}, \tag{46}$$

where a and b are two arbitrary constants. Hence from (44) and (46) we have the solution for $\sigma(R, S)$,

$$\sigma = \frac{aR + bS}{R^2 + S^2}. \tag{47}$$

IV.2 Solutions for $P(\sigma)$ and $Q(\sigma)$

To find solutions of the system (29) and (33) is more difficult. One way of tackling this problem is to assume $P(Q)$, then the compatibility condition of this system becomes

$$(P^2 + Q^2 - 1)P_{,QQ} + 2(1 + P_{,Q}^2)(P - QP_{,Q}) = 0. \tag{48}$$

We found two different solutions for (48).

IV.2.1

A simple particular solution is

$$P = kQ, \tag{49}$$

where k is a constant. Inversely, if we assume $Q(P)$ then the same system reduces to

$$(P^2 + Q^2 - 1)Q_{,PP} + 2(1 + Q_{,P}^2)(Q - PQ_{,P}) = 0, \tag{50}$$

which still satisfies the same particular solution (49).

Substituting (49) into (29) or (33) we obtain

$$\frac{Q_{,\sigma\sigma}}{Q_{,\sigma}} + 2\frac{(1 + k^2)QQ_{,\sigma}}{1 - (1 + k^2)Q^2} = 0, \tag{51}$$

which after integration produces the solution

$$Q = \frac{\tanh\left[(1 + k^2)^{1/2}\sigma\right]}{(1 + k^2)^{1/2}}, \tag{52}$$

with P obtained from (49). We observe that the inverse solution Q_I,

$$Q_I = -\frac{\coth\left[(1 + k^2)^{1/2}\sigma\right]}{(1 + k^2)^{1/2}}, \tag{53}$$

is also a solution of (29) and (33).

IV.2.2

Another solution of (48) for $P(Q)$ is obtained from the quadratic expression, a circle,

$$(P-c)^2 + (Q-d)^2 = c^2 + d^2 - 1, \tag{54}$$

where c and d are arbitrary constants. Now the system (29) and (33) can be integrated, but the simplest way is to differentiate (54) with respect to σ and substitute it into (38) yielding

$$P_{,\sigma} = \frac{(P^2+Q^2-1)(Q-c)}{(c^2+d^2-1)^{1/2}}, \tag{55}$$

where Q is the solution $Q(P)$ of (54). Following a well known method, by considering (55) as an equation of the type

$$P_{,\sigma\sigma} = -\frac{\partial U}{\partial P}, \tag{56}$$

where $U(P)$ is a potential, one can obtain a number of different solutions. These solutions depend upon the chosen values of the constants, they can be periodical or non periodical, like soliton waves of the form $l\cosh^{-2}(l\sigma)$ where l is a constant.

V Conclusion

We reduce the study of invariance of the complex Ernst equation to the study of a system of six second order partial differential equations (16-21) independent of the coordinate system. Two classes of solutions of this system are presented. One stemms from a three parameter $SU(1,1)$ transformation and the other from the assumption that there exists a functional dependence of the real and imaginary parts of the Ernst complex potential.

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