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A NEW APPROACH FOR MULTICRITICALITY IN DIRECTED AND DIODE PERCOLATION

by

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A new approach for multicriticality in directed and diode percolation

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We present a new and very simple model for treating directed and more general diode percolation problems, by allowing neighboring sites to be joined by up to two independent bonds of opposite orientations. A generalized "break-collapse" method is developed to calculate renormalization group recursion relations. On the square lattice, a very symmetric phase diagram is obtained which displays multicritical percolation phenomena, and a variety of interesting conductivity transitions are predicted.

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In their original work on percolation, Broadbent and Hammersley[1] proposed models in which neighboring lattice sites were connected by directed bonds, or diodes, which allow information "flow" in one direction only. This is in contrast to the percolation models generally considered[2] where sites may be connected by two-way bonds, or resistors. Recent interest in directed percolation stems from their rich phenomenology and diverse applications.[3] The directionality constant gives rise to anisotropic critical phenomena[3-5]

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and shifts in the upper critical dimension from 6 in pure percolation to 5. More general models can be defined where the bonds may be either resistors, diodes of an arbitrary orientation, or even bonds that conduct in both directions, but with different conductivities.[5] Such networks exhibit novel critical phenomena driven by either concentration or orientational fields.

Unfortunately, the description of these more general models requires a large parameter space and tedious calculations in the renormalization group framework. In this Letter, we show that the rich geometrical and conductivity properties of such networks can be satisfactorily described by a very simple model in which neighboring lattices sites may be joined by up to two independent directed bonds of opposite orientations (Fig. 1). The interplay between percolation in one direction and the opposite direction gives rise to an interesting multicritical behavior in our model. Moreover, our study indicates that results of pure percolation are recovered if a network is globally isotropic rather than each bond being isotropic.

We consider the square lattice where "positively" oriented bonds which point either upward or to the right occur with probability p, while "negatively" oriented diodes occur with an independent probability q. When one of p or q is zero, directed percolation is recovered,[3,4] while for $p = q$, each bond is isotropic, on average, and the results of pure percolation are obtained. From a position-space renormalization group (PSRG) approach, we find a multicritical point at $p = q = 1/2$, where a two-way percolating phase (denoted + - in Fig. 2) is simultaneously critical with one-way percolating phases (either + or -) and a non-percolating phase.

To perform the PSRG calculation, we partition the square lattice into 2×2 cells[6] as indicated in Fig. 1. Each such cell is rescaled into a single bond pair with renormalized values of p' and q'. The recursion

relations for these two quantities are determined by calculating the respective probabilities of traversing the cell upward or downward. A priori, this calculation entails a tedious evaluation all of the 2^{10} configurations on the 2 x 2 cell. To avoid this, we have developed an extension of the break-collapse[7] method to calculate traversing probabilities by performing simple topological operations without evaluating all cell configurations.

To illustrate the method, notice as a preliminary that the probabilities of traversing two bond pairs, (p_1,q_1) and (p_2,q_2), which are either in series or parallel respectively, are

$$p_s = p_1 p_2 \tag{1a}$$

$$p_p = p_1 + p_2 - p_1 p_2 \ , \tag{1b}$$

and analagously for the q's. This can be applied to an arbitrary graph that is reducible by series and parallel operations. However, if the graph is irreducible, such as the Wheatstone bridge of Fig. 3, we need the following

$$G = (1-p_i)(1-q_i)G_i^{bb} + (1-p_i)q_i G_i^{bc} + p_i(1-q_i)G_i^{cb} + p_i q_i G_i^{cc} \tag{2}$$

where G is the probability of traversing the graph comprised of the bond set $\{(p_i,q_i)\}$, and the superscripts on G_i denote that the i^{th} bond pair (p_i,q_i) has its constituents either "broken" (b) or "collapsed" (c). We illustrate the use of Eq. (2) in Fig. 3. The dashed line represents a "pre-collapsed" bond which is present with probability unity. It is necessary to keep such a bond in this state, until it is ascertained that a path through the cell actually traverses via this particular bond.

By using the property

$$G_i^{bc} + G_i^{cb} = G_i^{bb} + G_i^{cc} \tag{3}$$

Eq. (2) can be rewritten more simply as

$$G = (1-p_i)G_i^{bb} + q_i G_i^{cc} + (p_i-q_i)G_i^{cb} \tag{4}$$

The combined use of Eqs. (1) and (4) constitute a generalized break-collapse procedure for calculating traversing probabilities, and the break-collapse method for pure percolation[7] is recovered when $p_i = q_i$ for all i.

For our model, we find the following recursion relations on the square lattice after a simple calculation on the 2 x 2 cell

$$\begin{aligned} p' &= p^5 - 3p^4 + p^3 + 2p^2 + (p^4 - 2p^3 + p^2)q \equiv f(p,q) \\ q' &= f(q,p) \end{aligned} \tag{5}$$

which reduce to the recursion relation for pure bond percolation[6] when p = q, and to that of directed percolation[8] for one of p or q equal to zero.

The resulting phase diagram is shown in Fig. 2. The p-q plane is divided by second-order transition lines into four phases characterized by two-way percolation (+ −), one-way percolation (+ or −), or no percolation. These four phases meet at a multicritical point, M, defined by $p_c = q_c = 1/2$. The location of M at (1/2,1/2) is a consequence of self-duality, a property of the square lattice. The line p = q renormalizes into itself under rescaling and along it the results of pure percolation are reproduced. Therefore the critical properties of our globally isotropic network (though

not necessaarily locally isotropic) is the same as pure percolation. The point (p_c,q_c) is completely unstable with an eigenvalue in the (1,1) direction equal to 13/8, just as in pure percolation. This leads to a correlation length exponent of $\nu = \ln 2/\ln 1.625 \cong 1.43$, compared to the conjectured[9] exact value of $\nu = 4/3$. The eigenvalue in the (1,-1) direction is equal to 3/2, yielding an exponent of 1.71. Physically, this exponent describes how the average length of connected paths, running opposite to the average orientation of the diodes, diverges as the orientation becomes random. This is somewhat analagous to the divergence of the spin correlation length, $\xi \sim H^{-\nu/\beta\delta}$, as a function of the magnetic field H at the critical temperature.

Let us now turn to treating the conductivity of this network. We consider a model in which each directed bond has a finite conductance in the direction of the arrow, and zero conductance in the opposite direction. The conductance distribution for each bond is

$$P_+(g) = (1-p)\delta(g) + p\delta(g-g_+)$$
$$P_-(g) = (1-q)\delta(g) + q\delta(g-g_-) \qquad (6)$$

where + and - refer to the positive and negative directions respectively, and g is the value of the bond conductance. We wish to investigate the conductivity of a network containing these circuit elements in the vicinity of the multicritical point M. At M, the network is isotropic and the usual scaling laws relating eigenvalues to exponents should be valid. For the four anisotropic fixed points on the edges of the phase diagram, anisotropic scaling laws or a different rescaling procedure than the one used here would be needed.

Upon rescaling, a new conductance distribution is obtained which is a sum over many delta functions, and we approximate it by the original binary form.[10,11] Recursion relations for g_+ and g_- are calculated by imposing that their average values remain invariant. This leads to

$$g_+' = g_+[p^5 + \frac{16}{5}p^4(1-p) + \frac{10}{3}p^3(1-p)^2 + p^2(1-p)^3 + p^3(1-p)q/(3+2\Psi) + p^2(1-p)^2q/(2+\Psi)]/p'$$

$$\equiv h(p,q,g_+,g_-)$$

$$g_-' = h(q,p,g_-,g_+) \tag{7}$$

where $\Psi \equiv g_+/g_-$. From Eqs. (5) and (7), we find the mean conductivities of the lattice, $\sigma_+(p,q)$ and $\sigma_-(p,q)$, by using the fact[10] that these quantities scale respectively as $b^{d-2}g_+^{-1}$, where b is the rescaling factor in the PSRG. From this, we may numerically calculate σ_+ and σ_-, and the values obtained reflect the geometry of the lattice. In the non-percolating phase, both σ_+ and σ_- are zero, while in the one-way percolating phases one of σ_+ or σ_- is non-zero, and in the two-way percolating phase both σ_+ and σ_- are non-zero.

In order to study the critical behavior of the conductivity, we need to specify the initial ratio of Ψ. We shall treat the special case $\Psi = 1$ as this leads to a network which is both geometrically and electrically isotropic at M. Two independent exponents govern the behavior of the conductivity as M is approached. If the point is approached from the two-way percolating phase along a path with $p = q$ (see Fig. 2), then the conductivity is isotropic and it vanishes as $(p - 1/2)^t$. The 2 x 2 rescaling gives a conductivity exponent of $t = 1.17$,[11] to be compared with the recent numerical estimate of $t = 1.28 \pm 0.02$.[12] On the other hand, if the multicritical point

is approached along the path $p = 1 - q$, then the directed conductivity vanishes with a new exponent which is equal to 1.40 in our 2 x 2 cell approximation. Just as for percolation properties, the exponent t measures how the conductivity vanishes when a temperature-like field is varied, while the new exponent measures how the directed conductivity vanishes when a magnetic-field-like variable vanishes. It would be interesting to obtain a more accurate estimate of this new exponent perhaps by treating larger cells[6], as well as providing an interpretation for the exponent in terms of cluster structure.

A second interesting feature is the behavior of the conductivity as a function of p for various values of q (Fig. 4). For $q < q_d$ (which equals 0.444 in the 2 x 2 approximation), σ_+ becomes non-zero at some value of p, and this conductivity increases smoothly until $p = 1$. However for $0.444 < q < 0.5$, σ_+ displays two singularities. At the first one, σ_+ becomes non-zero, while at the second one, there is percolation of connected paths running in the negative sense which cause σ_- to become non-zero. These new paths provide additional current-carrying contributions in the positive direction, thereby causing a non-analyticity in σ_+. This property, and related features, such as negative resistance, appear to be a characteristic of biased networks which are between the isotropic and directed percolation thresholds.[13] When $q = 1/2$, the two thresholds coalesce, and for $q > 1/2$, only a single threshold occurs.

In conclusion, we have introduced a new model to describe the geometry and conductivity of random networks with directed bonds. This model has the advantage of providing a large amount of qualitative information with very simple calculations. A symmetric phase digram was found which displays multicriticality between two-way-percolating, one-way-percolating, and

non-percolating phases. In treating the conductivity , a new exponent was introduced to describe how the directed conductivity becomes isotropic near the multicritical point. It would be worthwhile to extend the present study to the case $\psi = g_+/g_- \neq 1$, where even more interesting network responses are possible.

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Figure Captions

Figure 1 Definition of our percolation model, on a 2 x 2 cell of the square lattice. Each pair of neighboring sites may be connected by a "positive" bond which occurs with probability p, or a "negative" bond which occurs with an independent probability q. Only the vertical bonds and the upper-left horizontal bond pair are required for the PSRG calculation, leading to the Wheatstone bridge of Fig. 3.

Figure 2 Phase diagram of the the model on the square lattice. Fixed points of the renormalization transformation are shown by heavy dots, and the arrows indicate the direction of flow under renormalization. The four phases of the network are indicated. At M, two independent exponents may be defined by approaching the point along either the (1,1) diagonal or along the (1,-1) diagonal.

Figure 3 Break-collapse reduction of the Wheatstone bridge. The dashed lines indicate a bond in a "pre-collapsed" state, and the probabilities of the various graphs are shown on the figure.

Figure 4 Schematic behavior of the normalized positive conductivity, σ_+, as a function of p for various values of q. Two distinct singularities occur if $0.444 < q < 0.5$ (dashed line of Fig. 2).

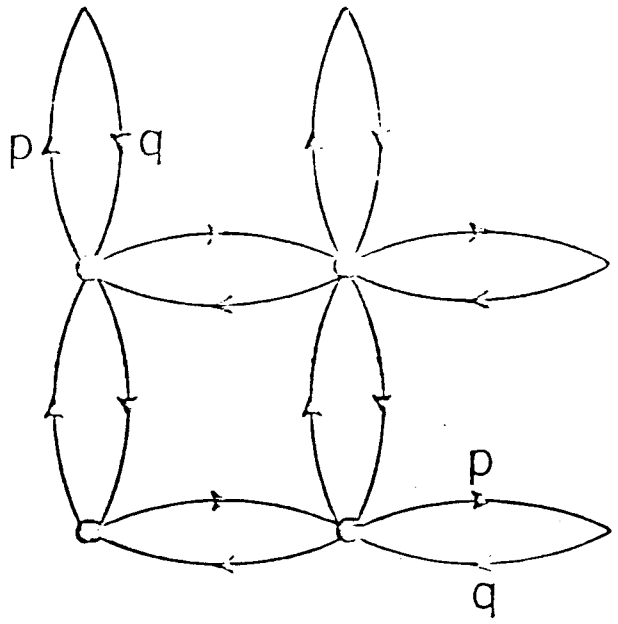

FIG. 1

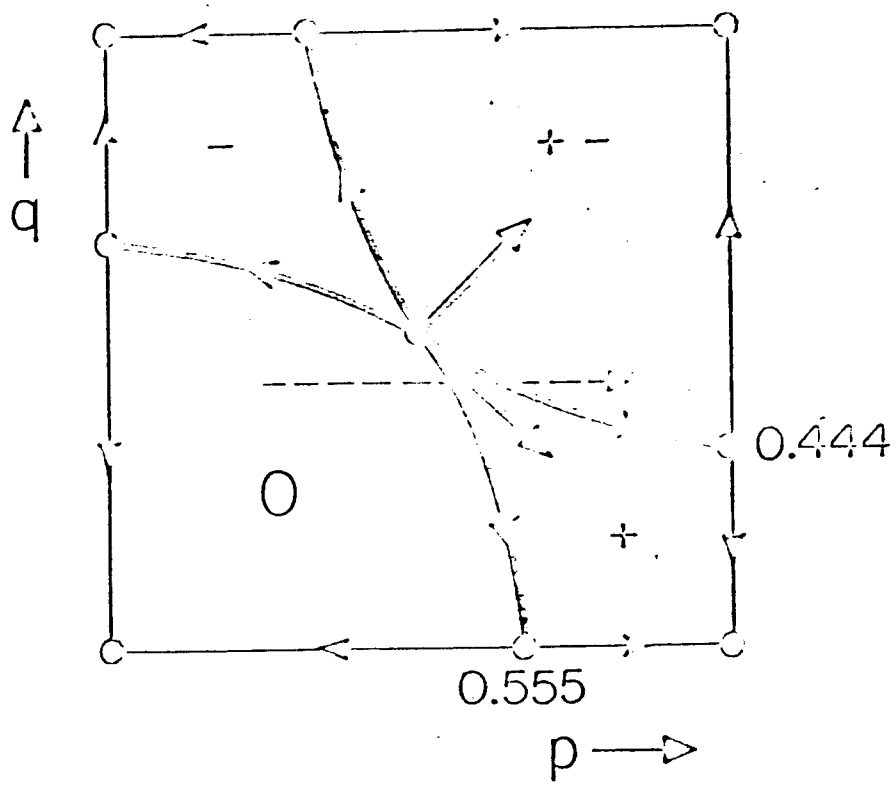

FIG. 2

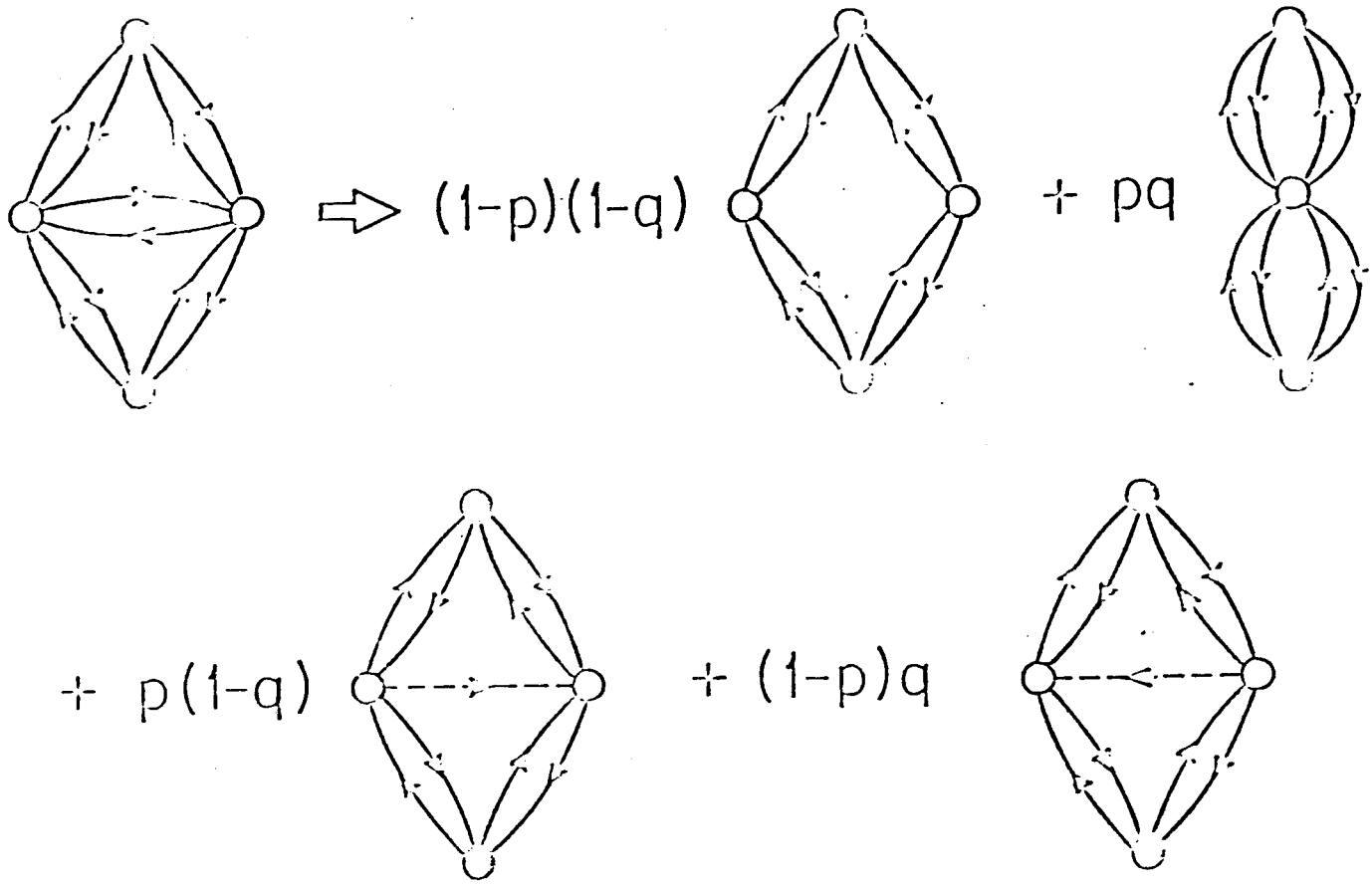

FIG. 3

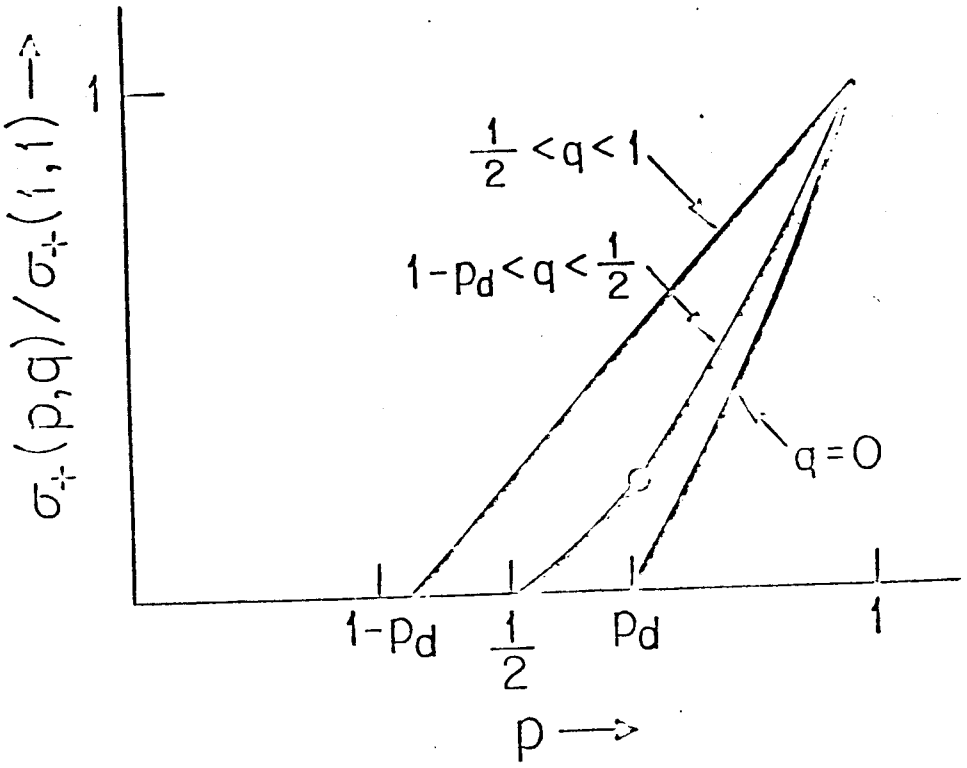
$\sigma_+(p,q)/\sigma_+(1,1) \rightarrow$
1
$\frac{1}{2} < q < 1$
$1-p_d < q < \frac{1}{2}$
$q = 0$
$1-p_d$
$\frac{1}{2}$
p_d
1
$p \longrightarrow$

FIG. 4