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EXPRESSIONS FOR THE SOLUTIONS OF THE RIESZ-MERCIER EQUATION

by

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EXPRESSIONS FOR THE SOLUTIONS OF THE RIESZ-MERCIER EQUATION

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Abstract

Expressions for the aggregates and currents of the Riesz-Mercier equation are considered.

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Maxwell's equations can be expressed in several ways. When Clifford numbers are used, the wave function can be shown to be a bivector, whereas the current is expressed as a vector[1-3]. Quaternionic form of extended Maxwell's equations with sources and abelian monopoles was found by Majernik and Nagy[4], the method being extended by Singh[5] to the electromagnetic-current equations.

It is well known that Maxwell's equations can be written in the Dirac-like form[6-11]. In some formulations a supplementary condution is imposed on the wave function[6] in view of the transversal character of the electromagnetic wave. It is possible also to find a solution whose wave function is a two-column matrix though there appears in this case problems connected with the covariance of the theory[12]. As it should be expected each column of this solution satisfies the same equation[13]. This procedure used in the electromagnetic case can be extended to the linear equations for the gravitational field propose by Carstoiu[14] and Cattani[15], in view of the analogies between the theories. It is easy to obtain the Dirac-like form of the Carstoiu-Cattani equations[16], and it can be shown how to write the wave aggregate and the current in terms of the Dirac matrices, when use is made of a suggestion indicated by Teitler[17], due to Riesz, Sommerfeld and Sauter[18]. According to this suggestion the wave aggregate, as well as the current, are to be considered as parts of left ideals of the Clifford algebra C_4.

It is the aim of the present note to start from the equation deduced in reference [12], and use the procedure prescribed in reference [16]. It will be seen that, in general, neither the aggregate is a bivector nor the current is a vector. Nevertheless, it is possible to express the wave aggregate as a bivector in free space, if the invariance of the Maxwell equations under the transformations $\vec{E} \rightarrow -\vec{H}$, $\vec{H} \rightarrow \vec{E}$ is taken into account. Even if abelian monopoles are introduced and the above transformation is extended to the more general form $\vec{E} \rightarrow +\vec{H}$, $\vec{H} \rightarrow -\vec{E}$, $\rho \rightarrow +\rho'$, $\rho' \rightarrow -\rho$, $\vec{j} \rightarrow -\vec{j}'$, $\vec{j}' \rightarrow \vec{j}$, the current cannot be expressed as a vector and the wave is not a bivector. Here, ρ and j correspond to the usual charge density and current whereas primed quantities correspond to the monopole.

According to Eq. (4) of Ref. [12], Maxwell's equations can be written as

$$\gamma_\mu \partial_\mu \psi = (4\pi/c)\ J \quad , \tag{1}$$

where

$$\psi = \begin{bmatrix} \vec{\sigma}.\vec{E} \\ \vec{\sigma}.\vec{H} \end{bmatrix} , \tag{2}$$

and

$$J = \begin{bmatrix} i\vec{\sigma}.\vec{j} \\ c\rho \end{bmatrix} . \tag{3}$$

The following Dirac matrices are being used:

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\ (k = 1,2,3) \quad , \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 . \tag{4}$$

Following Riesz, Sommerfeld and Sauter[18] the aggregate (2) can be extended to

$$\psi_I = \begin{bmatrix} \vec{\sigma}\cdot\vec{E} & 0 \\ \vec{\sigma}\cdot\vec{H} & 0 \end{bmatrix} , \tag{5a}$$

and the current (3) can be generalized to the matrix

$$J_I = \begin{bmatrix} i\vec{\sigma}.\vec{j} & 0 \\ c\rho & 0 \end{bmatrix} \quad . \tag{5b}$$

After easy calculations ψ_I and J_I can be written as

$$\psi_I = (1/8)\varepsilon_{\mu\nu\sigma\lambda}F_{\mu\nu}\gamma_\sigma\gamma_\lambda(1 + \gamma_4) \quad , \tag{6a}$$

and

$$J_I = (1/2)\gamma_5(\gamma_\mu j_\mu)\ (1 + \gamma_4) \quad , \tag{6b}$$

where

$$E_k = iF_{k4}\ ,\ H_k = \varepsilon_{kje}F_{je}\quad ,\quad j_4 = ic\rho$$

Of course, instead of (5a) and (5b) one can have also

$$\psi_{II} = \begin{bmatrix} 0 & \vec{\sigma}.\vec{E} \\ 0 & \vec{\sigma}.\vec{H} \end{bmatrix} , \tag{7a}$$

and

$$J_{II} = \begin{bmatrix} 0 & i\vec{\sigma}.\vec{j} \\ 0 & c\rho \end{bmatrix} , \tag{7b}$$

so that ψ_{II} and J_{II} can be written as

$$\psi_{II} = (-i/4)\gamma_\mu\gamma_\nu F_{\mu\nu}\ (1 - \gamma_4) \quad , \tag{8a}$$

and

$$J_{II} = (i/2)\gamma_\mu j_\mu\ (1 - \gamma_4) \quad . \tag{8b}$$

A simple inspection of (6a,b) and (8a,b) is enough to shown that in both cases ψ is not a bivector and J is not a vector. In the case of the electromagnetic equation in free space the invariance under the transformation $E \to H$, $H \to -E$ holds so that, instead of (5a) and (8a), it is possible to envisage another form of the matrix, namely

$$\psi_{III} = \begin{bmatrix} \vec{\sigma}.\vec{E} & \vec{\sigma}.\vec{H} \\ \vec{\sigma}.\vec{H} & -\vec{\sigma}.\vec{E} \end{bmatrix} , \qquad (9a)$$

where the above mentioned invariance is taken into account. The new aggregate is

$$\psi_{III} = (1/4)\varepsilon_{\mu\nu\sigma\lambda}F_{\mu\nu}\gamma_\sigma\gamma_\lambda\gamma_4 \quad . \qquad (10)$$

If (10) is inserted into the equation (1), where J is previously made equal to zero, γ_4 can be simplified and the solution can be considered to be

$$\psi_{IV} = (1/4)\varepsilon_{\mu\nu\sigma\lambda}F_{\mu\nu}\gamma_\sigma\gamma_\lambda \quad , \qquad (11)$$

which is, of course, a bivector. The simplification, which alows the transition from (10) to (11), cannot be applied to (6a,b) or (8a,b), in view of the presence of the idempotentes $1+\gamma_4$ and $1-\gamma_4$ even in free space.

If, besides ordinary charges, abelian monopoles are introduced, it is possible to consider a more extended class of transformations, because, in this case, the invariance under $\vec{E} \to \vec{H}$, $\vec{H} \to -\vec{E}$, $\rho \to +\rho'$, $\rho' \to -\rho$, $\vec{j} \to -\vec{j}'$, $\vec{j}' \to \vec{j}$ holds. Primed quantities refered to monopoles. The new aggregate is given by (10) and the matrix corresponding to the right hand side of (1) is

$$J_V = \begin{bmatrix} c\rho + i\vec{\sigma}.\vec{j} & c\rho' + i\vec{\sigma}.\vec{j}' \\ c\rho' + i\vec{\sigma}.\vec{j}' & -c\rho - i\vec{\sigma}.\vec{j} \end{bmatrix} \qquad (12)$$

that can be written as

$$J_V = \gamma_4(-i\gamma_\mu j_\mu \gamma_4 + \gamma_5 \gamma_4 \gamma_\mu j'_\mu) \quad . \tag{13}$$

When the aggregate ψ_V, given by (10), and the current, given by (13), are inserted into (1), γ_4 cannot be simplified. Of course, if the products $\gamma_\mu j_\mu$ and $\gamma_\mu j'_\mu$ are separated into three and four-components, then γ_4 can be simplified but the compactness of the expression is lost.

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