CBPF-NF-044/03

Jost function, prime numbers and Riemann zeta function

Sergio Joffily

Centro Brasileiro de Pesquisas Físicas,
Laboratório de Cosmologia e Física Experimental de Altas Energias
Rua Dr. Xavier Sigaud, 150,
22290-180, Rio de Janeiro, Brazil
E-mail: joffily@cbpf.br

Dedicated to Professor Roberto A. Salmeron
on the occasion of his 80th birthday.

Abstract

The large complex zeros of the Jost function (poles of the S matrix) in the complex wave number-plane for s-wave scattering by truncated potentials are associated to the distribution of large prime numbers as well as to the asymptotic behavior of the imaginary parts of the zeros of the Riemann zeta function on the critical line. A variant of the Hilbert and Polya conjecture is proposed and considerations about the Dirac sea as "virtual resonances" are briefly discussed.

There is an old conjecture attributed to Hilbert and Polya about the zeros of the Riemann zeta function $\zeta(z)$ on the critical line as eigenvalues of a self-adjoint linear operator H in some Hilbert space. Ever since Montgomery's [1] discovery of these zeros behaving like the eigenvalues of a random hermitian matrix, attempts have been made in order to find a quantum system with the Hamiltonian represented by such an operator (see Berry and Keating [2], and references therein). We report in this Letter a variant for the above conjecture: instead of looking for H, whose spectrum coincides with the Riemann zeta zeros, we are looking for complex momenta poles of the scattering S matrix such that by a given transformation they are all mapped into the axis $Re\ z = 1/2$ in coincidence with the Riemann zeroes. The associated quantum system could be the "vacuum", interpreted as an infinity of "virtual resonances", described by the corresponding S matrix poles.

The first to associate the Riemann hypothesis in terms of transient states were Pavlov and Faddeev [3] by relating the nontrivial zeros of the zeta function to the complex poles of the scattering matrix of a particle on a surface of negative curvature. Here the same problem is discussed in the usual scattering by a potential. We begin by showing that complex momenta zeros of the Jost function for s-wave non-relativistic scattering by repulsive cutoff potentials, after appropriate transformations, correspond to the global behavior of heights $\{t_n\}$ of the zeros of $\zeta(z)$ on the critical line, $z_n = 1/2 + i\ t_n$, and surprisingly to the global behavior of the prime sequence $\{p_n\}$. With the aid of these transformations it is possible to obtain an approximate formula connecting the nth prime and the nth zeta zero, which we believe will be useful to strategies for primality. The local behavior, defined as deviations from the average density of the zeta zeros, are not obtained by the potential considered here. This will be the object of a forthcoming work where a statistical hypothesis with respect to some residual interaction will be introduced [4]. Actually fluctuations in $\{t_n\}$ are interesting for their universality, being observed in quantal spectra in different physical systems (see Mehta [5]), and by the connection with chaotic dynamics, (for a review see Bohigas [6]).

The distribution of primes, $\{p_n\}$, among all natural numbers, n, in spite of their local deviation of any known order when viewed at large possesses regularities that can be approximated by some formulas. The approximate number of primes $\pi(x)$ less than a given x, also called the prime counting function, is given by the prime number theorem $\pi(x) \sim x/\ln x$ (see Titchmarsh [7], Chapter- III), where $\ln x$ is the natural logarithm of x. This relation gives the asymptotic approximation for nth prime p_n,

$$p_n \sim n \ln\ n \quad , \qquad \text{as } n \to \infty \ . \tag{1}$$

The connection between the distribution of prime numbers $\pi(x)$ and complex zeros of the zeta function, a kind of duality between the continuum and discrete in number theory, started with Riemann's 1859 paper (see Edwards [8], p.299) by introducing methods of analytic function into number theory. Riemann's zeta function is defined (Ref. [7], p.1) either by the Dirichlet series or by the Euler product

$$\zeta(z) = \sum_n n^{-z} = \prod_p (1-p^{-z})^{-1} \ , \qquad Rez > 1 \ , \tag{2}$$

where n runs through all integers and p runs over all primes. $\zeta(z)$ can be analytically continued to the whole complex plane, except at $z = 1$ where it has a simple pole with residue 1. It satisfies the functional equation $\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$, called the reflection formula, where $\Gamma(z)$ is the gamma function. It is known that $\zeta(z)$ has simple zeros at points $z = -2n,\ n = 1, 2, \ldots$, which are called trivial zeros, with an infinity of complex zeros lying in the strip $0 < Re\ z < 1$. From the reflection formula they are symmetrically situated with respect to axis $Re\ z = 1/2$, and since $\zeta(z^*) = \zeta^*(z)$, they are also symmetric about the real axis, so, it suffices to consider the zeros in the upper half of strip $1/2 \leq Re\ z < 1$. It is possible to enumerate these complex zeros as $z_n = s_n + i\, t_n$, with $t_1 \leq t_2 \leq t_3 \leq \ldots$, and the following result can be proven (Titchmarsh [7] , p.214)

$$|z_n| \sim t_n \sim \frac{2\pi n}{\ln n} \ , \qquad \text{as } n \to \infty. \tag{3}$$

The Riemann hypothesis is the conjecture, not yet proven, that all complex zeros of $\zeta(z)$ lie on axis $Re\ z = 1/2$, called the "critical line". Based on this conjecture Riemann improved on Gauss's suggestion that $\pi(x)$ approximate the logarithmic integral as $x \to \infty$ with a new prime number formula, taking into account local prime fluctuations in terms of nontrivial zeta zeros (Ref. [8], p.299).

The Jost function [9] has played a central role in the development of the analytic properties of the scattering amplitudes. In order to recall its properties, let us consider the scattering of a non-relativistic particle, without spin, of mass m by a spherically symmetric local potential, $V(r)$, everywhere finite, behaving at infinity as

$$V(r) = O(r^{-1-\epsilon}), \quad \epsilon > 0, \quad r \to \infty \ . \tag{4}$$

The Jost functions $f_\pm(k)$ are defined (see Newton [10], p.341) as the Wronskian W, $f_\pm(k) = W[f_\pm(k,r),\ \varphi(k,r)]$, where $\varphi(k,r)$ is the regular solution of the radial Schrödinger equation

$$[\frac{d^2}{dr^2} + k^2 - V(r)]\ \varphi(k,r) = 0 \ , \tag{5}$$

(in units for which $\hbar = 2m = 1$) k being the wave number and the Jost solutions, $f_\pm(k,r)$, are two linearly independent solutions of equation (5). They satisfy the boundary conditions $\lim_{r\to\infty}[e^{\mp ikr} f_\pm(k,r)] = 1$, corresponding to incoming and outgoing waves of unit amplitude.

The properties of the solution of differential equation (5) define the domain of analyticity of the Jost functions $f_\pm(k)$ on the complex k-plane as well as its symmetry properties, such as for real potentials, $f_+^*(k^*) = f_-(k)$. The phase of the Jost function is just minus the scattering phase shift $\delta(k)$, that is $f_\pm(k) = |f_\pm(k)|\ \ e^{\mp i\delta(k)}$, so that the usual S matrix is given by

$$S(k) \equiv e^{2i\delta(k)} = \frac{f_-(k)}{f_+(k)}\ . \tag{6}$$

The complex poles ($Re\ k \neq 0$) of $S(k)$, or zeros of $f_+(k)$, correspond to the solutions of the Schrödinger equation with purely outgoing, or incoming, wave boundary conditions. Resonances show up as complex poles with negative imaginary parts, their complex energies being

$$k_n^2 = \Xi_n - i\ \frac{\Gamma_n}{2}, \tag{7}$$

where Ξ_n and Γ_n represent the energy and the width, respectively, associated with nth resonance state. For small Γ_n, resonances appear as long-lived quasistationary states populated in the scattering process. If the width is sufficiently broad no resonance effect will be observed, as if the lifetime of state $1/\Gamma_n$ is smaller than the time spent by the particle to traverse the potential: we will call this kind of S matrix pole as "virtual resonances" throughout. "Virtual resonances" like the long lived observed ones, in fact, are represented by pairs of symmetrical S matrix poles in the complex k-plane, a capture state pole in the third quadrant and the decaying state in the fourth quadrant, since they give to the asymptotic solution an incoming growing wave and an outgoing decaying wave, respectively, exponential in time [11]. That could be described by Gamow vectors treated by Bohm and Gadella [12] as pairs of S matrix poles corresponding to decay and growth states. Examples of broad resonances are the well known large poles, related basically to the cutoff in the potential considered to be without any physical significance (see Nussenzveig [13], p.178).

Condition (4), which together with differential equation (5) establishes domains of analyticity for $f_+(k)$, is not sufficient to determine the asymptotic behavior for its zeros; for that we would need more information about the interaction. Then the potential is set equal to zero for $r \geq R > 0$, which is the cutoff of the potential at arbitrarily large distances R. With this restriction it can be shown that $f_+(k) = 0$ is an entire equation of order 1/2 , and according to Piccard's theorem, has infinitely many roots for arbitrary values of the potential. Position k_n of these zeros determines $f_+(k)$ uniquely in the whole complex plane, as a consequence of Hadmard's theorem, which provides [14] $f_\pm(k) = e^{\pm ikR} f(0) \prod_{n=1}^{\infty}(1 - k/k_n)$. The asymptotic expansion of k_n, for large n, after introducing dimensionless parameter $\beta = kR$, is given by (Ref. [10], p.362)

$$\beta_n = n\pi - i\frac{(\sigma+2)\ln|n|}{2} \tag{8}$$

where $n = \pm 1, \ \pm 2, \ \pm 3 \cdots$ and σ is to be defined by the first term of the potential asymptotic expansion, near $r = R$, through $V(r) = C(R-r)^\sigma + \cdots$, $\sigma \geq 0$ and $r \leq R$.

The connection between the complex zeros of the Jost function and those of the Riemann zeta function is obtained by means of the transformation:

$$z = -i\frac{\beta}{2\ Im\ \beta}\ , \tag{9}$$

by which the lower half of complex β-plane ($Re\ \beta \neq 0$) is mapped into the critical axis, $Re\ z = 1/2$, of complex z-plane. This suggests a variant for the Hilbert and Polya conjecture, namely, looking for a potential that gives a Jost function with all zeros on the lower half of complex β-plane ($Re\ \beta \neq 0$) that coincide with complex zeros of the Riemannn zeta function after the transformation (9). In this way the Riemann hypothesis follows. For real potentials, these complex β zeros are located symmetrically about the imaginary axis, then by (9) they will be mapped symmetrically about the real axis into the critical line.

Now we show that for cutoff potentials transformation (9) gives rise to complex Jost zeros with the same asymptotic behavior as the complex Riemann zeta zeros, being all in the critical line. If $\{\beta_n\}$ are the zeros of $f_+(\beta)$ then by (9) we get

$$z_n = \frac{1}{2} - i\frac{Re\ \beta_n}{2\ Im\ \beta_n}\ , \tag{10}$$

from (8), with $\sigma = 0$, we see that $\{Im\ z_n\}$ has the same asymptotic expansion as $\{t_n\}$, given by (3), i.e.,

$$Im\ z_n = \frac{t_n}{4} \quad \text{as} \quad n \to \infty\ , \tag{11}$$

which means that for each resonance, ratio $\{2Re\ \beta_n/Im\ \beta_n\}$ corresponds to the height of the zeta zero on the critical line.

On the other hand, after introducing dimensionless quantities, energy $E_n = R^2\Xi_n$ and widths $G_n = R^2\Gamma_n$, equation (7) is written as $\beta_n^2 = E_n - iG_n/2$, the dimentionless widths $\{G_n\}$, defined as

$$G_n = 4\ Re\beta_n\ \ Im\beta_n\ , \tag{12}$$

after taking into account (8), when $\sigma = 0$, shows the same asymptotic expansion for large primes (1), given by the prime number theorem,

$$G_n = 4\pi p_n \quad \text{as} \quad n \to \infty. \tag{13}$$

Then nth large complex Jost zeros are also related to nth large primes, showing an asymptotic connection between primes and complex Riemann zeta zeros, in a one-to-one correspondence.

In the scattering on a surface of constant negative curvature [3] the potential is replaced by imposing that the particle move on the given surface in order to obtain the scattering function $S(k)$ in terms of the Riemann zeta function. Provided the Riemann hypothesis is true the poles of S-function in the complex k-plane are given by $k_n = t_n/2 - i/4$ [3,16]. By using transformation (9) we obtain $z_n = 1/2 + i\, t_n$ in coincidence with the complex zeros of the zeta function as expected. It follows also from the usual resonance width definition (7) that $\Gamma_n = t_n/2$, is therefore not constant as considered by Wardlaw and Jaworski [16], exhibiting the same fluctuation of the height of the zeta zero in the critical line in accordance with the results on the time delay fluctuation [16] for this kind of unusual scattering.

In order to look for physical systems associated to the proposed S-function, we will return to the conventional scattering. In this connection it is interesting to examine more explicitly the potential parameter dependence in the above asymptotic expression (8). Khuri [17] has recently proposed a modification to the inverse scattering problem in order to obtain the potential whose coupling constant spectrum coincides with the Riemann zeta zeros. The model we have chosen is the non-relativistic s-wave scattering by a spherically symmetric barrier potential, $V(r) = V_0$ for $r < R$ and 0 for otherwise. From the stationary scattering solution with this potential one obtains the Jost function $f_+(k) = e^{ikR}[k' cos(k'R) - ik\sin(k'R)]$, where $k'^2 = V_0 - k^2$. Introducing dimensionless parameters: $\alpha = k'R$, $\beta = kR$ and $v = V_0 R^2$, the zeros of the Jost function are given by the solution of the complex transcendental equation

$$\sqrt{\beta^2 - v}\ \cot\sqrt{\beta^2 - v} = i\beta \tag{14}$$

for each value of potential strength v. The displacements of the roots, β_n, in the β-plane with the variation of the potential strength are shown by Nussenzveig [15]. These zeros are all in the lower half of complex β-plane and located symmetrically about the imaginary axis. For finite $v \neq 0$, in equation (14), the large values of the β_n zeros are asymptotically determined, as [11,15]

$$\beta_n = \pm n\pi \mp \frac{\ln(2\ n\pi/|\sqrt{v}|)}{n\pi} - i(\ln(2\ n\pi/|\sqrt{v}|)\ \mp \frac{1}{n\pi})\ \ , \tag{15}$$

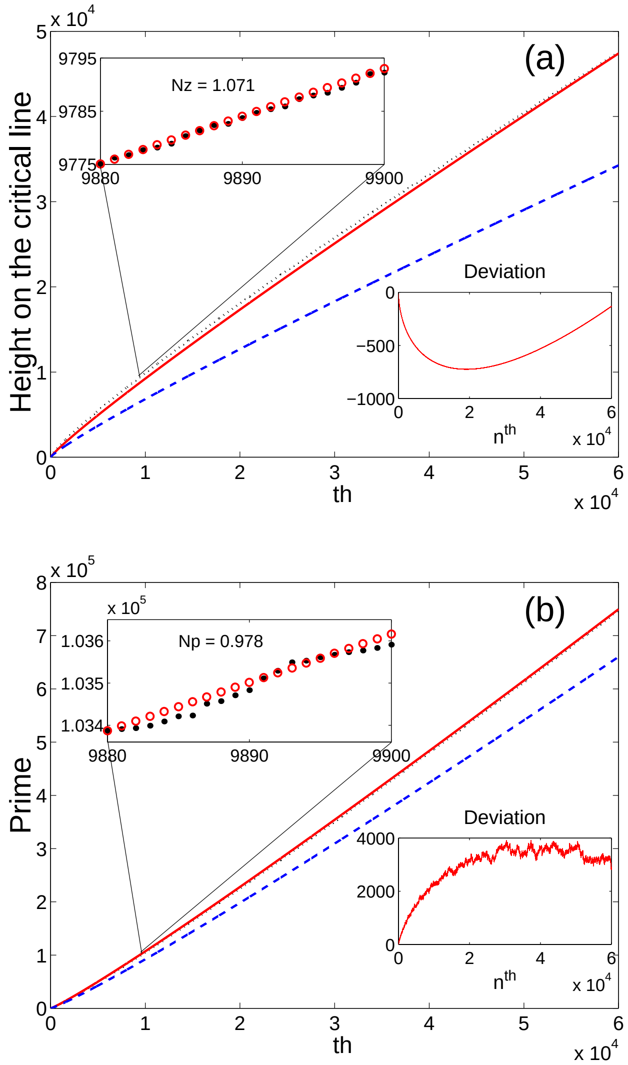

FIG. 1. (a) The height of nontrivial zeta zeros on the critical line. The full line according to energy/width ratios $\{4\ \pi E_n/G_n\}$ from the Jost zeros; the dashed line from the asymptotic formula (3); the points are computed by Odlyzko [18]. (b) Primes in order of size. The full line according to dimensionless width $\{G_n/4\pi\}$ from the Jost zeros; the dashed line according to the prime number theorem (1); the dotted line taken from the table of Caldwell [19]. Insets are defined in the text.

n being a large positive integer. This formula gives the dependence of the potential strength v, considered as a free parameter, in the determination of the asymptotic Jost zeros β_n to be compared, after the transformation (10), with the zeta heights t_n on the critical line, and after (12) with asymptotic prime numbers p_n. By using asymptotic expansion (15) we calculate the energy/width ratios $\{4\pi E_n/G_n\}$, where $4\pi E_n/G_n \sim \pi Re\ \beta_n/Im\ \beta_n$, which are compared to $\{t_n\}$ computed by Odlyzko [18] and shown in figure 1(a), for $v = 2$. Then we see the approximate agreement from the beginning, for n up to 6×10^4. The same

Jost zeros β_n, for $v = 2$, after transformation (12), give dimensionless widths $\{G_n/4\pi\}$ that are compared with prime sequence $\{p_n\}$ taken from the table of Caldwell [19], as shown in figure 1(b). The numerical factor $1/4\pi$ was imposed to G_n by the asymptotic limit (13) from the prime number theorem. Here we see the agreement with the global behavior of the sequence of primes, from the beginning in the same range. In the upper left hand corner in both figures the local zeta zero and prime fluctuactions shown by dots are not present when obtained by the corresponding Jost zeros (open circles), here normalized on the 9880-th zero; the normalization factor corresponding to the height on the critical line is $Nz = 1.071$ and to the prime being $Np = 0.978$. The inset of each figure in the lower right hand corner shows deviation in the height of zeta zeros figure 1(a) and prime figure 1(b), obtained by the corresponding Jost zeros, from true values taken from tables [18,19] in the range considered.

Finally, a conjecture is made in order to associate the Dirac sea as "virtual resonances". The hole theory is the interpretation of the negative energy solution of the relativistic single-particle Dirac equation in which the vacuum consists of all these negative energy states being filled with electrons, such that it could be considered from the beginning as a many-particle system being described by the formalism of the single-particle theory. According to Dirac (Ref. [20], p.34), "The vacuum must be a state with a lot of particles present corresponding to some stationary solutions of the Schrödinger equation. But there are no known solutions of this Schrödinger equation - not even a solution which could represent the vacuum". The question about the vacuum structure was bypassed by the second quantization, where a vacuum state is assumed and an operator defined, in order to create an electron when applied to this state, without knowing what the vacuum state really is. We would like to suggest a description of the vacuum structure as being a dynamical system described by "virtual resonances", completely independent of the second quantization. Instead of looking for a stationary solution, we look for a transient state of very short time duration. Specifically we propose a dynamical model for the vacuum described by infinite denumbered virtual resonances, with discrete widths and energies, which could be useful in the description of quantum chaos, which is under investigation. It could perhaps be verified experimentally by some devise that amplifies the vacuum fluctuation; the fluctuating nature of the Casimir force was recently discussed by Bartolo et al. [21]. In this case, the universality of level fluctuation laws of the spectra of different quantum systems (nuclei, atoms and molecules) [5,6] could be understood by the "vacuum" role as a dissipative system [22]. It had already been shown by Maier and Dreizler [23], in a Dirac particle scattering in (1+1) dimensions by an electrostatic square well potential, that complex momenta poles of the S matrix exhibit the particleantiparticle content of the Dirac theory and in the limit of weak potential strength the poles distribution are similar to the nonrelativistic case.

In summary, we have shown that the zeros of Riemann's zeta function are related to the zeros of the Jost function for cutoff potentials in the complex momenta plane in a one-to-one correspondence. The energy/width ratios of the large "virtual resonances" are associated to the nontrivial zeta zeros and the corresponding widths related to the prime sequence. In analogy to the mean field it is expected, by means of a statistical hypothesis, that the above relationship would be improved and the distribution of the virtual resonances would reflect the chaotic nature of the vacuum.

ACKNOWLEDGMENTS

We thank C. A. Garcia Canal, M. Novello, J. M. Salim, B. Schroer and I. D. Soares, for their interest in the work and useful conversations.

---

[1] H.L. Montgomery, Proc. Symp. Pure Math., **24**, 181 (1973).

[2] M.V. Berry and J.P. Keating, SIAM Review, **41**, No.2, 236 (1999).

[3] B.S. Pavlov and L.D. Faddeev, Sov. Math., **3**, 522 (1975).

[4] R. Bonetti et al. Physicas Reports, **247**, 1 (1994); C. Mahaux, K.T.R. Davies and G.R. Satchler, ibid. **224**, 237 (1993).

[5] M.L. Mehta, "Random Matrices and the Statistical Theory of Energy Levels" (2^{nd} ed.)(New York: Academic Press, 1991).

[6] O. Bohigas, Proceedings of the 1989 Les Houches Summer School, eds. M.J. Giannoni et. al., (Amsterdam: Elsevier) p 87 (1991).

[7] E.C. Titchmarsh,"The Theory of Riemann Zeta-Function" (2^{nd}. ed. Revised by D.R. Heath-brown, Oxford University Press Inc., N.Y. 1999).

[8] H.M. Edwards, " Riemann's Zeta Function" (Dover Publications, N.Y. 2001).

[9] R. Jost, Helv. Phys. Acta, **20**, 256 (1947).

[10] R. G. Newton, "Scattering Theory of Waves and Particles" (Springer-Verlag, New York, 1982).

[11] S. Joffily, Nuclear Physics, **A215**, 301 (1973).

[12] A. Bohm and M. Gadella, 'Dirac Kets, Gamow Vectors and Gel'fant Triplets" (Lecture Notes in Physics, Vol.**348**, Springer, 1989).

[13] H.M. Nussenzveig, "Causality and Dispersion Relations" (Academic Press, N.Y. 1972).

[14] T. Regge, Il Nuovo Cimento **8**, 671 (1958).

[15] H.M. Nussenzveig, Nuclear Physics, **11**, 499 (1959).

[16] D.M. Wardlaw and W. Jaworski, J. Phys. A: Math. Gen., **22**, 3561 (1989).

[17] N.N. Khuri, Mathematical Physics Analysis and Geometry, **5**, 1 (2002).

[18] A. Odlyzko, http://www.research.att.com/∼amo/zeta_tables/

[19] C.K. Caldwell, http://www.utm.edu/research/primes/

[20] P.M. Dirac, in "Directions in Physics" (eds. H.Hora and J.R.Shepanski, John Wiley 1978).

[21] D. Bartolo et al., Phys. Rev. Lett., **89**, 230601 (2002).

[22] H.B. Callen and T.A. Welton, Phys. Rev., **83**, 34 (1951).

[23] T.J. Maier and R.M. Dreizler, Phys. Rev., **A45**, 2974 (1992).