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Octonionic Realizations of 1-dimensional Extended Supersymmetries. A Classification.

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Abstract

The classification of the octonionic realizations of the one-dimensional extended supersymmetries is here furnished. These are non-associative realizations which, albeit inequivalent, are put in correspondence with a subclass of the already classified associative representations for $1D$ extended supersymmetries. Examples of dynamical systems invariant under octonionic realizations of the extended supersymmetries are given. We cite among the others the octonionic spinning particles, the $N = 8$ KdV, etc. Possible applications to supersymmetric systems arising from dimensional reduction of the octonionic superstring and M-theory are mentioned.

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1 Introduction.

In this work we extend the results of [1] and present a classification of the non-associative realizations of the 1-dimensional extended supersymmetries which are based on the division algebra of the octonions.

In [1] it was proven that the linear multiplets of representations of $1D$-Extended Supersymmetry (i.e. the basic ingredients in Supersymmetric Quantum Mechanics) fall into classes of equivalence each one characterized by a given "short multiplet", denoted as $\{n, n\}$, such that all the n bosonic (and all the n fermionic) states are grouped together with the same spin, the spin of the fermions differing by $\frac{1}{2}$ w.r.t. the spin of the bosons. This first result of [1] was obtained by noticing that the supersymmetry transformations of the higher-spin (let's say at the level s) components in a given multiplet (these transformations in any D dimension are total derivatives) in $D = 1$ coincide with time derivatives. In its turn this implies that the original supersymmetry multiplet can be algebraically replaced by a shorter one, with the original higher spin states now accommodated together with the $s - 1$ spin states in this level. The iteration of such a procedure produces at the end a short multiplet as defined above.

The second part of the [1] classification consisted in proving that all short multiplets are in one-to-one correspondence with a special class (which can be named either of "Weyl type" or "of supersymmetric type") of irreducible representations of Clifford algebras. These are the Clifford algebra representations, splitted into 2×2 block matrices, which are non-vanishing only in the antidiagonal blocks. The restriction to this class of Clifford algebras can be intuitively understood when thinking to them as the Clifford algebras which can be "promoted" to be fermionic matrices (as it would be expected for supersymmetry), realizing a superalgebra.

The one-to-one correspondence between Weyl-type Clifford algebras on one side and short multiplets of $1D$ extended supersymmetries on the other side implies the following identifications

$$D = N \tag{1}$$

and

$$d = n \tag{2}$$

where on the l.h.s. D denotes the spacetime dimensionality (here, for simplicity, assumed Euclidean) of the given Clifford algebra and $2d$ the matrix size of the representation. For what concerns the r.h.s. N denotes the number of extended supersymmetries, while n is the number of bosonic (or fermionic) states in the given short multiplet.

On the other hand, not all extended supersymmetries admit a matrix representation. There are several examples, some of them discussed in the following, of dynamical systems admitting a non-associative realization of the extended supersymmetries. Of course the non-associativity prevents representing the supersymmetry transformations through standard supermatrices and therefore these supersymmetric systems are outside the [1] classification. It is worth mentioning that in all the known examples the non-associativity enters through the octonionic structure constants.

In this paper, mimicking the approach of [1], we extend its results relying this time upon the classification of the octonionic realizations for Clifford algebras. We are able to classify the octonionic-valued extended supersymmetries carried by octonionic-valued short multiplets.

The first non-trivial example of a non-associative realization of supersymmetry involves the octonionic realization of the $N = 8$ supersymmetry. A dynamical system admitting invariance under a global octonionic $N = 8$ is e.g. given by the $N = 8$ super-KdV [2], whose Poisson brackets coincide with the Non-associative $N = 8$ Superconformal Algebra introduced in [3]. The superKdVs are non-linear non-relativistic systems in $(1 + 1)$-dimensions. However, since the supersymmetry transformations depend on the space coordinate alone, they are classified in agreement with the results for the $1D$ Extended Supersymmetries.

Other dynamical systems admitting invariance under non-associative realizations of the Extended Supersymmetries involve the octonionic spinning particles, as later discussed.

The scheme of this paper is as follows. In the next section we present the classification of the octonionic realizations of the Clifford algebras. This is the necessary ingredient for introducing in section 3 the classification of the octonionic realizations of the Extended Supersymmetries. In section 4 some examples of dynamical systems admitting octonionic realizations of the extended supersymmetries are discussed in some detail. Finally, in the Conclusions, we make further comments on our results, mentioning, among other, the possible relevance of the classification of the one-dimensional octonionic supersymmetries in the dimensional reduction from octonionic string and M-theory.

2 The octonionic Clifford algebras.

The classification of the $1D$ Extended Supersymmetries is based, as recalled in the Introduction, on the classification of Clifford algebras [4, 5]. We summarize here the main results which will be used in following concerning real and octonionic-valued Clifford algebras. A very convenient presentation for them is in terms of the following algorithm, which allows individuating a single representative for each irreducible class of representations of Clifford's Gamma matrices.

Let us prove at first that a recursive construction of $D+2$ spacetime dimensional Clifford algebras is available, when assumed known a D dimensional representation. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D = p + q$ spacetime with (p, q) signature (namely, providing a representation for the $C(p, q)$ Clifford algebra) then $2d$-dimensional $D + 2$ Gamma matrices (denoted as Γ_j) of a $D + 2$ spacetime are produced according to either

$$\begin{aligned}
\Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\
(p, q) &\mapsto (p + 1, q + 1). \qquad (3)
\end{aligned}$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$
$$(p,q) \mapsto (q+2,p). \tag{4}$$

As an example, the two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (3) or (4) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

All Clifford algebras are obtained by recursively applying the algorithms (3) and (4) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,3+4m)$ (m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the table below.

Table with the maximal Clifford algebras (up to $d = 256$).

1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
$\underline{(1,0)}$	⇒	(2,1)	⇒	(3,2)	⇒	(4,3)	⇒	(5,4)	⇒	(6,5)	⇒	(7,6)	⇒	(8,7)	⇒	(9,8)	⇒
						(1,4)	→	(2,5)	→	(3,6)	→	(4,7)	→	(5,8)	→	(6,9)	→
					↗												
				$\underline{(0,3)}$													
					↘												
						(5,0)	→	(6,1)	→	(7,2)	→	(8,3)	→	(9,4)	→	(10,5)	→
								(1,8)	→	(2,9)	→	(3,10)	→	(4,11)	→	(5,12)	→
							↗										
						$\underline{(0,7)}$											
							↘										
								(9,0)	→	(10,1)	→	(11,2)	→	(12,3)	→	(13,4)	→
														(1,12)	→	(2,13)	→
													↗				
												$\underline{(0,11)}$					
													↘				
														(13,0)	→	(14,1)	→
																(1,16)	→
															↗		
														$\underline{(0,15)}$			
															↘		
																(17,0)	→

(6)

Concerning the previous table, some remarks are in order. The columns are labeled by the matrix size d of the maximal Clifford algebras. Their signature is denoted by the (p,q) pairs. Furthermore, the underlined Clifford algebras in the table are called the "primitive maximal Clifford algebras". The remaining maximal Clifford algebras, known as the "maximal descendant Clifford algebras", are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (3) and (4). Any Clifford algebra is said "non-maximal" if obtained by a maximal one by deleting a certain number of Gamma matrices. It should be noticed that Clifford algebras in even-dimensional spacetimes are always non-maximal.

For what concerns the construction of the primitive maximal Clifford algebras of the series $C(0,3+8n)$ (also known as quaternionic series, due to its connection with this division algebra, as we will explain later), as well as the octonionic series $C(0,7+8n)$, the answer can be provided with the help of the three Pauli matrices (5). We construct at first the 4×4 matrices realizing the Clifford algebra $C(0,3)$ and the 8×8 matrices realizing the Clifford algebra $C(0,7)$. They are given, respectively, by

$$C(0,3) \quad \equiv \quad \begin{array}{l} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ \mathbf{1}_2\otimes\tau_A. \end{array} \tag{7}$$

and

$$C(0,7) \quad \equiv \quad \begin{array}{l} \tau_A\otimes\tau_1\otimes\mathbf{1}_2, \\ \tau_A\otimes\tau_2\otimes\mathbf{1}_2, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_1, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_2, \\ \tau_1\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_2\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_A\otimes\tau_A\otimes\tau_A. \end{array} \tag{8}$$

The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i, = 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\tilde{\tau}_i$, $i=1,2,\ldots,7$.

In order to construct the remaining Clifford algebras of the series we need at first to apply the (3) algorithm to $C(0,7)$ and construct the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2=\mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j=1,2\ldots,8$, with ${\gamma_j}^2=-\mathbf{1}$). We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$C(0,3+8n) \quad \equiv \quad \begin{array}{lll} \overline{\tau}_i\otimes\gamma_9\otimes\ldots & \ldots & \ldots\otimes\gamma_9, \\ \mathbf{1}_4\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \mathbf{1}_4\otimes\gamma_9\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \mathbf{1}_4\otimes\gamma_9\otimes\gamma_9\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_4\otimes\gamma_9\otimes\ldots & \ldots & \otimes\gamma_9\otimes\gamma_j, \end{array} \tag{9}$$

and similarly

$$C(0,7+8n) \quad \equiv \quad \begin{array}{lll} \tilde{\tau}_i\otimes\gamma_9\otimes\ldots & \ldots & \ldots\otimes\gamma_9, \\ \mathbf{1}_8\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \mathbf{1}_8\otimes\gamma_9\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \mathbf{1}_8\otimes\gamma_9\otimes\gamma_9\otimes\gamma_j\otimes\mathbf{1}_{16}\otimes\ldots & \ldots & \ldots\otimes\mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_8\otimes\gamma_9\otimes\ldots & \ldots & \otimes\gamma_9\otimes\gamma_j, \end{array} \tag{10}$$

Please notice that the tensor product of the 16-dimensional representation is taken n times. The total size of the (9) matrix representations is then 4×16^n, while the total size of (10) is 8×16^n.

The formulas given above provide quite a practical and efficient tool to operatively construct the irreducible Clifford algebras.

An important subclass of Clifford Gamma matrices is obtained by the matrices which are decomposable in 2×2 blocks and are non-vanishing only in the anti-diagonal blocks. Such matrices can be named as (generalized) Weyl-type matrices (they can also be regarded of "supersymmetric type" since they can be promoted to be fermionic matrices associated with the representations of the extended supersymmetries, see [6]). An inspection of the previous tables shows that sets of (generalized) Weyl matrices are found in special signatures only. All primitive Clifford algebras are not of (generalized) Weyl type. However, all the derived Clifford algebras, through the two lifting algorithms, are of Weyl-type, once deleted the $\left(\begin{array}{cc} \mathbf{1}_d & 0\\ 0 & -\mathbf{1}_d\end{array}\right)$ matrix to produce a non-maximal Clifford algebra.

So far we have shown how to construct the irreducible representations of Clifford algebras, and not yet elucidated their relations with division algebras. Such a relation can be expressed as follows. The three matrices appearing in $C(0,3)$ can also be expressed in terms of the imaginary quaternions τ_i satisfying $\tau_i\cdot\tau_j=-\delta_{ij}+\epsilon_{ijk}\tau_k$. As a consequence, the whole set of maximal primitive Clifford algebras $C(0,3+8n)$, as well as their maximal descendants, can be represented as quaternionic-valued matrices, acting on spinors, which have to be interpreted now as quaternionic-valued column vectors.

Similarly, there exists an alternative realization for the Clifford algebra $C(0,7)$, obtained by identifying the seven generators with the seven imaginary octonions satisfying the algebraic relation

$$\tau_i\cdot\tau_j \quad = \quad -\delta_{ij}+C_{ijk}\tau_k, \qquad (11)$$

for $i,j,k=1,\cdots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$C_{123}=C_{147}=C_{165}=C_{246}=C_{257}=C_{354}=C_{367}=1 \qquad (12)$$

and vanishing otherwise. This octonionic realization of the seven-dimensional Euclidean Clifford algebra will be denoted as $C_{\mathbf{O}}(0,7)$. Due to the non-associative character of the (11) octonionic product (the weaker condition of alternativity is satisfied, see [7]), the octonionic realization cannot be represented as an ordinary matrix product and is therefore a distinct and inequivalent realization of this Euclidean Clifford algebra with respect to the one previously considered (8). Please notice that, by iteratively applying the two lifting algorithms to $C_{\mathbf{O}}(0,7)$, we obtain matrix realizations with octonionic-valued entries for the maximal Clifford algebras of the series $C(m,7+m)$ and $C(8+m,m-1)$, for positive integral values of m ($m=1,2,\ldots$). These realizations are denoted $C_{\mathbf{O}}(m,7+m)$ and $C_{\mathbf{O}}(8+m,m-1)$, respectively. The dimensionality of the corresponding octonionic-valued matrices is $2^m\times 2^m$.

We should point out that the construction (10) leading to the primitive maximal Clifford algebras $C(0,7+8n)$, can be carried on with the help of the octonionic-valued realization $C_{\mathbf{O}}(1,8)$ for the γ_i's and γ_9 matrices. As a consequence, octonionic realizations of $C(0,7+8n)$ and their descendants can be produced acting on column spinors, whose entries are tensor products of octonions. If in the r.h.s. of (10) k octonionic and $n-k$ real realizations are chosen, the maximal Clifford algebras $C(m,7+m+8n)$ and $C(9+8n+m,m)$, for $n\geq 0$ and $m\geq 0$, are realized by matrices with $k+1$-tensorial octonionic entries (the extra 1 being associated to $C_{\mathbf{O}}(0,7)$) and respective size of $2^{4n-3k+m}$ and $2^{4n-3k+m+1}$.

3 The octonionic extended supersymmetries.

We furnish here the classification of the $1D$ octonionic extended supersymmetries. More precisely, we give the list of the octonionic maximal supersymmetries supported by short multiplets of n bosonic and n fermionic fields. This is based both on [1] and the previous section results. The octonionic extensions can be recovered from a suitable restriction of the [1] classification formulas of the real representations of the $1D$ extended supersymmetries. Indeed, the octonionic realizations of the maximal Clifford algebras are obtained by lifting the $C_{\mathbf{O}}(0,7+8n)$ series. No octonionic counterpart exists for the $C(1,0)$ and the $C(0,3+8n)$ series.

The one-to-one correspondence of $1D$ extended supersymmetries and Weyl type real-valued Clifford algebras [1] is obtained by expressing the supersymmetry generators Q_i satisfying the supersymmetry algebra

$$\{Q_i,Q_j\} \;=\; \eta_{ij}H, \tag{13}$$

for the generalized pseudo-Euclidean metric η_{ij}[1] of (p,q) signature, through

$$Q_i \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i H & 0 \end{pmatrix}, \tag{14}$$

where H is the hamiltonian and one can set

$$\Gamma_i \;=\; \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix} \tag{15}$$

satisfying

$$\Gamma_i\Gamma_j+\Gamma_j\Gamma_i \;=\; 2\eta_{ij}. \tag{16}$$

The octonionic realizations are recovered by setting σ_i, $\tilde{\sigma}_i$ as matrices with octonionic-valued entries, instead of being real matrices. From the previous section we know that Weyl type (15) octonionic-valued matrices Γ_i satisfying (16) are recovered from the maximal Clifford algebras derived from the $C_{\mathbf{O}}(0,7+8n)$ series, after deleting the diagonal

[1] the Euclidean case was considered earlier in [8]. Pseudo-Euclidean supersymmetries naturally appear as dynamical invariances for systems like the spinning particles moving in generic space-time target manifolds.

$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ matrix. This leaves us the two series of octonionic maximally extended supersymmetries of (p,q) signature, namely

$$(m, 8+8n+m) \tag{17}$$

and

$$(8+8n+m, m), \tag{18}$$

for integral values $n, m \geq 0$.

In both cases the number of octonionic bosonic, as well as fermionic, components is given by 2^{n+m}. Please notice the equivalence of (17) and (18) under the sign flipping $p \leftrightarrow q$.

This result can be summarized as follows. The inequivalent classes of octonionic irreducible realizations of the maximally extended supersymmetries acting on octonionic multiplets of $d = 2^k$ bosons and equal number of fermions is given by

$$(x + 8\varepsilon(k+1-x), x + 8(1-\varepsilon)(k+1-x)), \tag{19}$$

for integral values $0 \leq x \leq k$ and $\varepsilon = 0, 1$.

At the lowest order of d, the following table can be produced

	(p,q)
$d=1$	$(8,0),(0,8)$
$d=2$	$(16,0),(9,1),(1,9),(0,16)$
$d=4$	$(24,0),(17,1),(10,2),(2,10),(1,17),(0,24)$
$d=8$	$(40,0),(33,1),(26,2),(19,3),(12,4),(4,12),(3,19),(2,26),(1,33),(0,40)$

(20)

Of course, irreducible realizations of non-maximal octonionic extended supersymmetries are recovered from the previous table for the values (p', q'), with $p' \leq p$ and $q' \leq q$, provided that p', q' are not too small. For instance, the irreducible $2+2$ realization of the octonionic $(8,1) \subset (9,1)$ is encountered, while the irreducible $(8,0)$ is directly present in the table and found at $d = 1$.

4 Dynamical systems with octonionic supersymmetry.

In this section we present some examples of dynamical systems admitting invariance under $1D$ octonionic extended supersymmetries.

The first example of a non-associative, octonionic realization of a $1D$ supersymmetry is given by the octonionic $N = 8$. It is associated, according to the previous section results, to $C_{\mathbf{O}}(8,0)$ and is expressed in terms of 2×2 octonionic-valued matrices. Explicitly, the eight supersymmetry generators Q_0, Q_i, for $i = 1, 2, \ldots, 7$, are given by

$$Q_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix}, \qquad Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & t_i \\ -t_i H & 0 \end{pmatrix}, \tag{21}$$

where t_i denote the imaginary octonions and H is the hamiltonian.

The above supersymmetry can also be expressed in an octonionic language. It corresponds to the simplest (for $D=1$) case of a class of higher-dimensional generalized octonionic supersymmetries, investigated in the light of superstring theories, M-theory, etc., see [9]. We can indeed introduce the octonionic supercharge $\mathcal{Q}$ and its octonionic conjugate $\mathcal{Q}^*$ (under octonionic principal conjugation), through

$$\begin{aligned} \mathcal{Q} &= Q_0 + \frac{1}{\sqrt{7}} \sum_{i=1,\ldots 7} Qt_i, \\ \mathcal{Q}^* &= Q_0 - \frac{1}{\sqrt{7}} \sum_{i=1,\ldots 7} Qt_i, \end{aligned} \tag{22}$$

with

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}. \tag{23}$$

As a consequence, the octonionic $N=8$ can be rewritten as

$$\begin{aligned} \{\mathcal{Q},\mathcal{Q}\} = \{\mathcal{Q}^*,\mathcal{Q}^*\} = 2H, \\ \{\mathcal{Q},\mathcal{Q}^*\} = 0. \end{aligned} \tag{24}$$

We already pointed out that the octonionic $N=8$ is an inequivalent realization of the $1D$ $N=8$ supersymmetry with respect to standard $N=8$, obtained by replacing the seven imaginary octonions t_i in (21) with the seven (associative) 8×8 matrices given in (8)[2].

Perhaps the most convenient way of getting ourselves convinced of the inequivalence of the associative-versus-nonassociative realizations of $N=8$, consists in presenting a dynamical system which only admits invariance under the octonionic $N=8$. No counterpart is found with invariance under the associative $N=8$. A nice example of that is given by the $N=8$ KdV. Due to the absence of central extension for N-extended superconformal algebras with $N>4$ [10], superKdV equations only exist for $N\leq 4$. Indeed, the Virasoro central extension is necessary to produce the three-derivative term entering the KdV equation. On the other hand, the mathematical no-go theorem preventing the construction of superKdV equations for $N>4$ can be overcome by noticing that non-Jacobian superconformal algebras like the Non-associative $N=8$ SCA introduced in [3], can present central extension and be regarded as generalized Poisson brackets for a non-associative supersymmetric extension of KdV. In [2] we proved that there exists only one such extension, the $N=8$ KdV, invariant under the global octonionic $N=8$. From the considerations above, it is clear that no $N=8$ superKdV based on the associative $N=8$ can exist, since this is prevented by the no-go theorem.

The $N=8$ superKdV equations are explicitly given by

$$\begin{aligned} \dot{T} &= -T''' - 12T'T - 6Q_a''Q_a + 4J_i''J_i, \\ \dot{Q} &= -Q''' - 6T'Q - 6TQ' - 4Q_i''J_i + 2Q_iJ_i'' - 2Q_i'J_i', \\ \dot{Q}_i &= -Q_i''' - 2QJ_i'' - 6TQ_i' - 6T'Q_i + 2Q'J_i' + 4Q''J_i - \\ &\quad 2C_{ijk}(Q_jJ_k'' - Q_j'J_k' - 2Q_j''J_k), \\ \dot{J}_i &= -J_i''' - 4T'J_i - 4TJ_i' + 2QQ_i' + 2Q'Q_i - C_{ijk}(4J_jJ_k'' + 2Q_jQ_k'). \end{aligned} \tag{25}$$

[2] In the Q_0, Q_i entries 1 is also replaced by $\mathbf{1}_8$.

(the dot and the prime denote, as usual, the time and respectively the space derivative). They involve the eight bosonic fields T, J_i and the eight fermionic fields $Q_0 \equiv Q$, Q_i ($i = 1, \dots, 7$, while $a = 0, 1, \dots, 7$). One should notice the presence of the octonionic structure constants C_{ijk}. The $N = 8$ global supersymmetry transformations leaving the (25) system invariant are generated by $\int dx Q_a(x)$, for $a = 0, 1, \dots, 7$, under the Non-associative $N = 8$ SCA Poisson brackets, see [2] for details. They coincide with the (21) transformations once setting the hamiltonian $H = i\frac{\partial}{\partial x}$. Please notice that, despite the fact that the fields entering the $N = 8$ superKdV are dependent on both the space and time coordinates, the $N = 8$ supersymmetry transformations only depend on the space coordinate x. The time dependence being "frozen", one can directly read these transformations from the $1D$ octonionic $N = 8$ supersymmetry generators given above.

We have clarified the role of the octonionic supersymmetry transformations and presented a first example of a non-trivial system invariant under non-associative supersymmetries. Another example of a class of dynamical systems invariant under octonionic supersymmetry involves the octonionic spinning particles, whose simplest example is again found for $N = 8$. The octonionic generalization of the free real-valued spinning particles is described by the octonionic-valued bosonic and fermionic fields, $x = x_0 + \sum_i x_i \tau_i$ and $\psi = \psi_0 + \psi_i \tau_i$ respectively. The imaginary octonionic super-coordinates x_i, ψ_i can be regarded as super-coordinates associated with the seven sphere S^7 [11] since the latter can be described by unitary octonions. The free kinetic action is given by

$$S = \frac{1}{2}\int dt \cdot tr\{(x^*, \psi^*)\begin{pmatrix} \frac{d^2}{dt^2} & 0 \\ 0 & \frac{d}{dt} \end{pmatrix}\begin{pmatrix} x \\ \psi \end{pmatrix}\}, \qquad (26)$$

where tr denotes both the matrix trace Tr and the projection over the octonionic identity [12], while "$*$" denotes the octonionic principal conjugation (see also formula (22)).

The above free action is invariant under the octonionic $N = 8$ supersymmetry, whose suitably normalized explicitly transformations acting on the component fields are given by

$$\begin{array}{ll} \delta_0 x_0 = \psi_0, & \delta_0 x_i = \psi_i, \\ \delta_0 \psi_0 = \dot{x}_0, & \delta_0 \psi_i = \dot{x}_i \end{array} \qquad (27)$$

and

$$\begin{array}{ll} \delta_i x_0 = -\psi_i, & \delta_i x_j = \delta_{ij}\psi_0 - C_{ijk}\psi_k, \\ \delta_i \psi_0 = \dot{x}_i, & \delta_i \psi_j = -\delta_{ij}\dot{x}_0 + C_{ijk}\dot{x}_k, \end{array} \qquad (28)$$

with $i = 1, \dots, 7$.

The classification of the octonionic spinning particles, invariant under generalized (p, q) supersymmetries, is an immediate consequence of the classification formulas for the octonionic supersymmetries presented in the previous section. The construction of the octonionic spinning particles straightforwardly follows the one here presented for the $N = 8$, i.e. the $(p = 8, q = 0)$, octonionic supersymmetry.

5 Conclusions.

In this work we furnished the classification of the octonionic $1D$ extended supersymmetries acting on small multiplets of n bosonic and n fermionic fields. The key observation allowing us to classify the octonionic supersymmetries consists in noticing that they are in one-to-one correspondence with the class of Weyl type realizations of Clifford algebras, expressed through matrices with octonionic-valued entries. “Weyl type” simply means here the subclass of matrices in a Clifford algebra which can be “promoted” to be fermionic (i.e. odd-graded) elements in a superalgebra.

The classification of the octonionic supersymmetries can therefore be extracted from the classification of the octonionic Clifford algebras. Explicit tables, expressing the number of generalized (p, q) supersymmetries (for p positive and q negative eigenvalues) supported by the $n + n$ field multiplets, are given.

We further mentioned that the octonionic realizations can be put in correspondence with a subclass of the associative representations of the $1D$ extended supersymmetries. Basically, this can be done by replacing the seven imaginary octonions with the seven antisymmetric matrices producing the Euclidean Clifford algebra $C(0, 7)$. Nonassociative and associative realizations of supersymmetry remain, nevertheless, inequivalent. This point is better understood by noticing that $N = 8$-octonionic invariant systems, such as the $N = 8$ KdV, indeed exist, while on the other hand is known (due to a mathematical no-go theorem discussed in the previous section) that no $N = 8$ extension of KdV based on the associative $N = 8$ supersymmetry can be constructed.

We further explicitly discussed another example of a class of dynamical systems invariant under the octonionic supersymmetry, i.e. the one provided by the octonionic spinning particles.

Finally, it is worth mentioning the possible relevance in physical applications of the supersymmetric systems investigated in this paper. While it is well known since the work of [13] that division algebras are associated with extended supersymmetries, quite understandably due to the complications arising from the non-associativity, octonionic realizations received less attention than the associative division algebras. Nevertheless, octonions continued being investigated as, e.g., in [14, 15] in the context of superstring theory. More recently, in [9], the existence of an octonionic version of the M-theory with surprising features, among the others the equivalence of the $M5$ five-brane sectors with the $M1$ and $M2$ sectors, was pointed out. In general higher-dimensional octonionic theories have peculiar features, for instance they are no longer invariant under the full Lorentz group, but under its G_2 coset, since this is the group of the octonionic automorphisms.

For what concerns the specialization to $D = 1$, i.e. the case treated in this paper, we should mention that in the Jordan framework and at least for the restricted class of Jordan algebras, see [16], a consistent octonionic quantum mechanics is available. On the other hand it is clear that higher-dimensional octonionic supersymmetric theories, like the superstring or the M-theory mentioned above, can be dimensionally reduced to $1D$. In this passage we obtain octonionic quantum mechanical systems admitting, as in the standard associative case [17], extended number of supersymmetries. Such systems must be constructed in terms of the octonionic multiplets here classified, leaving room for promising applications of the results and the techniques presented in this paper.

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