CBPF-NF-043/93

Relations Among Solutions for Wave and Klein-Gordon Equations for Different Dimensions

J.J. Giambiagi
Centro Brasileiro de Pesquisas Físicas – CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil
Centro Latino Americano de Física
Av. Wenceslau Braz, 71 – fundos
22290–180 – Rio de Janeiro, RJ – Brasil

Abstract

Relations between solutions of homogeneous wave equations for different dimensions are discussed. Idem for Green functions. A similar discussion is given for the Klein Gordon equations.

Relations with solutions for powers of D'Alembertian and Kleinian are also discussed. Its importance for the understanding of the relations between Analytic and dimensional regularization is briefly mentioned.

Key-words: Field theory; Wave equations; Klein-Gordon equations; Analytical and dimensional regularization.

PACS: 0.350.De; 0.370.+k; 0.520.-y.

I. Introduction

It is very difficult to discuss the wave equation (w.e) or Klein Gordon eq. ($K.G$) and say something new about it. You are never sure that the same thing has not been done many years ago.

In this note we intend to discuss relations between solutions of w.e. (or $K.G$) in different dimensions. The interest in different $n^{\underline{0}}$ of dimensions has increased very much during the last decade so it looks worthy to try to find relations among them. We shall limit this discussion to solutions with spherical symmetry and find very simple relations among them. In particular between solutions in n-dimension and those with $n+2$ (n being $d+1$, space time dimensions).

There are also simple relations with different powers of the D'Alembertian or the Kleinian.

We shall see that the "radiation part" (that surviving at $r = \infty$) implies a multipole of order $\frac{d-3}{2}$. It looks like a space dimension d had, associated with it, an intrinsic multipole of this order. This result was already known from quantum mechanics. These results will prove to be valid for fractional derivations and integrals of the solutions.

In §II we discuss the displacement of the solution with the dimension or with the power of the D'Alembertian; i.e.: with the change of dynamics in the same number of dimensions, when derivations are taken with respect to r^2 in dimension d.

In §III we give some examples for the w.e. and show how the intrinsic multipole appear in the Schroedinger equation.

In §IV we discuss relations among Green functions of w.e.

In §V we discuss similar problems with Klein Gordon equation.

Finally, in the discussion, we show the relevance of these relations in the interrelation of analytic and dimensional regularization.

II. Relating different dimensions

We start with the well known formula

$$I^{\alpha}(\phi) = \frac{1}{\Gamma(\alpha)} \int_u^{\infty} (u' - u)^{\alpha - 1} \phi(u') du' \tag{1}$$

which, when α is an integer > 0 represents the α iterated integration.

According to ref. [1] p.49 the distribution

$$x_+^{\lambda} = \begin{array}{ccc} x^{\lambda} & \text{when} & x > 0 \\ 0 & otherwise & \end{array} \tag{2}$$

has, as analytic function of λ, poles for $\lambda = -k$ with residue

$$(-1)^{k-1} \frac{\delta^{k-1}(x)}{\Gamma(k)} \tag{3}$$

when using this property with form 1 we see that for $\alpha = -n$ (1) gives the n the derivative

$$I^{(-n)}\phi = \frac{d^n\phi(u)}{du^n} \tag{4}$$

For α real > 0 (1) gives then a definition of functional integration or for $\alpha < 0$ fractional differentiations.

Let us prove now that $I^\alpha(\phi)$ gives a solution of w.e. in a dimension $n - 2\alpha$.

For functions depending only on r and t the homogeneous w.e. reads:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2} = 0 \tag{5}$$

If we write it for the variables $u' = r^2$ (5) reads

$$4u'\frac{\partial^2\phi}{\partial u'^2} + 2d\frac{\partial\phi}{\partial u'} - \frac{\partial^2\phi}{\partial t^2} = 0 \tag{6}$$

Multiplying by $(u'-u)^{\alpha-1}\frac{1}{\Gamma(\alpha)}$ and integrating we get

$$\frac{1}{\Gamma(\alpha)}\int_u^\infty 4u'(u'-u)^{\alpha-1}\frac{\partial^2\phi}{\partial u'^2}du' + \frac{2d}{\Gamma(\alpha)}\int_u^\infty du'(u'-u)^{\alpha-1}\frac{\partial\phi}{\partial u'} - \frac{\partial^2}{\partial t^2}I^\alpha(\phi) = 0 \tag{7}$$

Adding and substracting the first term multiplied by u we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_u^\infty du' 4(u'-u)^\alpha\frac{\partial^2\phi}{\partial u'^2} + \frac{4u}{\Gamma(\alpha)}\int_u^\infty (u'-u)^{\alpha-1}\frac{\partial^2\phi}{\partial u'^2}du' + \\ &+\frac{2d}{\Gamma(\alpha)}\int_u^\infty (u'-u)^{\alpha-1}\frac{\partial\phi}{du'}du' - \frac{\partial^2}{\partial t^2}I^\alpha(\phi) = 0\end{aligned} \tag{8}$$

with obvious definitions for $I_1I_2I_3$ we can write (8) as

$$I_1 + I_2 + I_3 - \frac{\partial^2 I^\alpha(\phi)}{\partial t^2} = 0 \tag{9}$$

Comparing the definitions of $I_1I_2I_3$ with (1) we can write:

$$I_1 = -4\alpha\frac{\partial I^\alpha(\phi)}{\partial u}; \quad I_2 = \frac{4u\partial^2 I^\alpha(\phi)}{\partial u^2}; \quad I_3 = 2d\frac{\partial I^\alpha}{\partial u} \tag{10}$$

which can be verified by partial integration and $\alpha > 2$. For other values of α use is made of analytic continuation.

With (10), (8) can be written

$$4u\frac{\partial^2 I^\alpha(\phi)}{\partial u^2} + 2(d-2\alpha)\frac{\partial I^\alpha(\phi)}{\partial u} - \frac{\partial^2 I^{\alpha(\phi)}}{\partial t^2} = 0 \tag{11}$$

to be compared with form (6).

For any α real (11) means that $I^\alpha(\phi)$ gives a solution in dimension $d - 2\alpha$.

For α an integer we have the following results:

A: If $\alpha = n > 0$, this means that n-iterated integration leads to a solution in dimension $d - 2n$.

B: If $\alpha = -n$ n^{th} differentiation shifts to a space of dimension $d + 2n$.

If $\alpha = -\frac{1}{2}$ (non local differentiation) we obtain the solution in $d + 1$.

The same arguments applies if we look for a solution $\psi(\xi)$ with

$$\xi = t^2 - r^2 \tag{12}$$

Then, the w.e. can be written as

$$\frac{d^2\psi}{d\xi^2} + \frac{n}{2\xi}\frac{d\psi}{d\xi} = 0 \quad n = d + 1 = \text{space-time dimension} \tag{13}$$

The solutions for lowest dimensionality are:

$$\begin{array}{ll} n = 2 & \psi = \ell n\xi \\ n = 3 & \frac{1}{\sqrt{\xi}} \\ n = 4 & \frac{1}{\xi} \\ n = 5 & \frac{1}{\xi^{3/2}}, ...etc \end{array} \tag{14}$$

Then, it can be immediately verified that if $\psi_n(\xi)$ is a solution in dimension n, then

$$\psi_{n+2} \cong \frac{d\psi_u}{d\xi} \tag{15}$$

$$\psi_{n+2k} \cong \frac{d^k}{d\xi^k}\psi_u, etc. \tag{16}$$

The same applies for integration

$$\psi_{n-2k} \simeq I^k(\psi) \tag{17}$$

and (17) is generalized (when k is not an integer)

$$\psi_{n-2\alpha}(\xi) \simeq I^\alpha(\psi_u) \tag{18}$$

This is a non local relation, which reduces to a local one when α is a negative integer.

III. Examples

Suppose we start from a solution of the homogeneous, w.e. in 3+1 of the form

$$\phi_3 = \frac{f(t-r)}{r} \tag{19}$$

Taking

$$\frac{d\phi_3}{dr^2} \sim \frac{1}{r}\frac{d\phi_3}{dr} = \phi_5$$

so

$$\phi_5 \cong \frac{1}{r^3} f(t-r) + \frac{f'(t-r)}{r^2} \tag{20}$$

is a solution in 5+1. Again

$$\phi_7 \simeq \frac{3f(t-r)}{r^5} + \frac{3f'(t-r)}{r^4} + \frac{f''(t-r)}{r^3} \tag{21}$$

$$\phi_9 = 15\frac{f(t-r)}{r^7} + 15\frac{f'(t-r)}{r^6} + 6\frac{f''(t-r)}{r^5} + \frac{f'''(t-r)}{r^4} \tag{22}$$

and so on.

In particular, if we choose $f = \delta(t-r)$ then (20), (21) and (22) shows us that a pure outgoing wave i.e., a $\frac{\delta(t-r)}{r}$ is not a solution of w.e. for dimensions greater than 3.

In fact, eq. (20) tells us that for large r in five dimensions, the dominant wave (radiation part) is that of a dipole (21) says that for seven dimensions, the radiation part is a quadrupole, etc.

In general, for dimension d, the radiation wave is given by a multipole

$$\ell = \frac{d-3}{2} \quad (d = odd) \tag{23}$$

It is interesting to observe (as is well known, ref. [2]) that the same intrinsic multipole appears if you consider Schroedinger equation for a potential $V(r)$ in dimension d and look for the ground state (see ref. [3]).

This state has the maximum symmetry and is a function only for r. The Schroedinger equation can be written.

$$\left\{\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + V(r)\right\}\psi = E\psi \tag{24}$$

Making now the usual change to eliminate the first derivative

$$\psi = \frac{\phi}{r^{\frac{d-1}{2}}} \quad \text{we get for the ground state} \tag{25}$$

$$\frac{d^2\phi}{dr^2} + V(r)\phi - \frac{\ell(\ell+1)}{r^2}\phi = E\phi \tag{26}$$

where $\ell = \frac{d-3}{2}$ and represents an intrinsic multipole of dimension d.

This is the same multipole which appears as "radiation multipole in w.e. in dimension n" in form (23) for $f = \delta(t-r)$.

Observe also that if we look for the Green function of the Laplacian in dimension d we get

$$G(d,r) \cong \Gamma\left(\frac{d-2}{2}\right)(r^2)^{\frac{2-d}{2}} \tag{27}$$

By performing:

$$-\frac{dG(d,r)}{dr^2} \simeq \Gamma\left(\frac{d}{2}\right)(r^2)^{-\frac{d}{2}} \cong G(d+2,r) \tag{28}$$

This result is valid for an euclidean theory of dimension d, but it is also valid for the quantum wave equation when we change d by n and $r^2 \to \xi + i0$, which is obtained when doing first a dilatation ($x_4 = \alpha t, \alpha >$) and then performing analytic continuation in $\alpha = i + \varepsilon$ (See ref. [5]).

So we have

$$\frac{d^k \psi_n(\xi)}{d\xi^k} \simeq \psi_{n+2k} \tag{29}$$

The extension of this result for a continuous real k can be immediately obtained by using (see ref. [4], p.284).

$$\int_\xi^\infty \xi'^{-\nu}(\xi' - \xi)^{\mu-1} d\xi' = \frac{\Gamma(\nu-\mu)\Gamma(\mu)}{\Gamma(\nu)}\xi^{\mu-\nu} \tag{30}$$

See ref. [4] p.284.

Observe that the Green function of Δ^λ is (See ref. [1])

$$G(\lambda, d, r) \simeq \Gamma(\frac{d}{2} - \lambda)(r^2)^{\lambda - \frac{d}{2}} \tag{31}$$

A displacement in d, can also be interpreted as a displacement in λ.

IV. Green Functions

In order to discuss relations among Green functions of arbitrary powers (or different dimensions) of $\Box$, let us recall the definition of $\Box^\alpha$ (see ref. [6]).

$$\Box^\alpha_{\substack{R\\A}} = \frac{2^{2\alpha+1}(t^2-r^2)_+^{-\alpha-\frac{u}{2}}\theta(\mp t)}{\pi^{\frac{u}{2}-1}\Gamma\left(1-\alpha-\frac{u}{2}\right)\Gamma(-\alpha)} \tag{32}$$

where

$$(t^2 - r^2)_+ = \begin{matrix} (t^2-r^2) & \text{when} & t^2 > r^2 \\ 0 & \text{otherwise} & \end{matrix} \tag{33}$$

$\theta(t)$ is the usual step function. R and A stands for advanced and retarded. Similar definitions can be given for Feynman D'Alembertians ($\Box_+$) or antiFeynman ($\Box_-$) see ref. [6].

The corresponding Green functions, such that ($*$means convolution)

$$\Box^\alpha_{\substack{R\\A}} * G^\alpha = \delta(x). \quad \text{see ref. [6] from (22)} \tag{34}$$

are

$$G^\alpha_{\substack{R\\A}} = \frac{2^{(1-2\alpha)}(t^2-r^2)_+^{\alpha-\frac{u}{2}}\theta(\mp t)}{\pi^{\frac{u}{2}-1}\Gamma\left(1+\alpha-\frac{u}{2}\right)\Gamma(\alpha)} \tag{35}$$

and a similar one for $\Box_{\pm}$ in which we change $(t^2 - r^2)_+$ by $(t^2 - r^2 + io)$.

Then, it is easily verified

$$\frac{dG^{\alpha}_{\substack{R\\A}}}{d\xi} \simeq G(n+2, \alpha) \tag{36}$$

but as the dynamical part of (34) depends on $\alpha - \frac{n}{2}$ we can also say that increase the dimension by 2 is the same as taking $\alpha - 1$ i.e. $G(u+2, \alpha) \simeq G(n, \alpha - 1)$

And using the same formula (30) we can easily prove

$$I^{\rho}(G^{\alpha}) \simeq G(n+2\rho; \alpha) \simeq G(n, \alpha - \rho) \tag{37}$$

We see that as in the dynamics of the whole thing it appears the combination $\alpha - \frac{n}{2}$ we can say that either n is shifted by 2ρ or α is decreased by ρ. So, we get the Green function of either of them.

V. Klein Gordon Equation

We can extend the previous considerations to the Klein Gordon equation $(K.G)$ considering, just as was the case with the w.e. - only solutions depending on r.t. (no dependence on angles) $K.G$ reads.

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{(d-1)}{r} - \frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial t^2} - m^2 \phi = 0 \tag{38}$$

The arguments that led from eq. (5) to (11) remain unchanged with the extra mass term. So, instead of (11) we have now

$$4u \frac{d^2}{du^2} I^{\alpha}(\phi) + 2(d - 2\alpha) \frac{dI^{\alpha}(\phi)}{du} - \frac{\partial^2 I^{\alpha}(\phi)}{\partial t^2} - m^2 I^2(\phi) = 0 \tag{39}$$

and the conclusions A, B remain the same and are valid for $K.G$ eq. ($K.G$ to the power λ).

We shall exemplify these results with the Green functions (to the power λ). The previous proof is valid only for $\lambda = 1$. But, according to this example, the results are more general.

The Green function satisfies:

$$(\Box + m^2)^{\lambda} G(n, m, \xi, \lambda) = \delta \tag{40}$$

with

$$G(n, m, \xi, \lambda) = \frac{2^{1-\lambda} m^{\frac{n}{2}-\lambda} K_{\frac{n}{2}-\lambda}\left(m\xi^{1/2}\right)}{\xi^{\left(\frac{n}{2}-\lambda\right)\frac{1}{2}}} \tag{41}$$

See ref. [1] p.362, with $\xi = (x_0^2 - x_1^2 - x_2^2 - \dots x_{u-1}^2) + io$. For simplicity we can think everything in euclidean metric. When written in terms of $\xi, KG(\lambda = 1)$ reduces to

$$4\xi \frac{d^2\psi}{d\xi^2} + 2n \frac{d\psi}{d\xi} - m^2 \psi = 0 \tag{42}$$

Exactly as in the case of form 6 we get for ψ the formula 39 for ξ (without the t-term).

If we take the derivative of (41) with respect to $\xi = t^2 - r^2 + io$ and make use of the recurrence relation.

$$z\ K_\nu(z) \pm \nu K_\nu(z) = -z\ K_{\nu\mp1}(z) \tag{43}$$

we obtain:

$$\frac{dG(u)}{d\xi} = -(2)^{-\lambda} m^{\frac{n+2}{2}-\lambda} \frac{K_{\frac{n+2}{2}-\lambda}(m\xi^{\frac{1}{2}})}{\xi^{\frac{1}{2}}(\frac{n+2}{2}-\lambda)} \cong -G(n+2) \tag{44}$$

This formula is valid either for ξ being euclidean or hyperbolic distance.

It is interesting to observe that as the dynamical part of (41) depends on the combination $\frac{n}{2} - \lambda$ we can say that unstead of getting $G(n+2, \lambda)$ we get $G(n, \lambda - 1)$ as the Green functions of both are equal (proportional).

For instance, $K.G$ in $n = 4$ and $\lambda = 1$, ordinary Klein-Gordon equation has the Green function

$$G(4, 1; m^2, r) \simeq \frac{K_1(m\sqrt{t^2 - r^2 + io})}{(\sqrt{t^2 - r^2 + io})^{2-\lambda}} \tag{45}$$

but for $n = 10$ and $\lambda = 4$ we get exactly the same (up to a constant) function.

Observe that if instead of taking the derivative with respect to ξ, we take with respect to m^2 and proceed in parallel way, with the other sign in (43) we get.

$$\frac{dG}{dm^2} = \frac{1}{2m}\frac{dG}{dm} \cong G(n-2) \tag{46}$$

i.e., it decreases the $n^{\underline{o}}$ of dimension by 2 (or increase the value of λ by 1).

These formula can be extended to an arbitrary order of integration or derivation, as

$$\begin{aligned} I^\alpha(G) &= \tfrac{1}{\Gamma(\alpha)} \textstyle\int_\xi^\infty G(\xi')(\xi' - \xi)^{\alpha-1} d\xi' = \\ &= 2^{1-\lambda} m^{\frac{n}{2}-\lambda} \int_\xi^\infty \frac{K_{\frac{n}{2}-\lambda}(m\xi^{\frac{1}{2}})(\xi'-\xi)^{\alpha-1}}{\xi'^{\frac{1}{2}\left(\frac{n}{2}-\lambda\right)}} \end{aligned} \tag{47}$$

From ref. [4] p.209 formula 59 (Bateman Proj.) we get

$$I^\alpha(G) = 2^{1+\alpha-\lambda} m^{\frac{n-2\alpha}{2}-\lambda} \frac{K_{\frac{n-2\alpha}{2}-\lambda}(m\xi^{\frac{1}{2}})}{\xi^{\frac{1}{2}(\frac{n-2\alpha}{2}-\lambda)}} \tag{48}$$

If α is > 0 (integration) we shift to higher dimensions. If $\alpha < 0$ (differentiation), we go over to smaller dimensions.

Or, equivalently, integration with respect to ξ leads to higher values of λ while differentiation takes us to smaller values of λ.

Discussion

We have seen that very simple relations exists between solutions of w.e. and $K.G$ equations in different dimensions or for different powers of the D'Alembertian or the Klenian.

Differentiation or integration on configuration variables ($t^2 - r^2$ or r^2) leads to higher or smaller dimensions, rsp. The opposite occurs if we differentiate $(K.G)$ with respect to m^2.

Relations between dimensions separated by 2 units (or multiple) are local (differentiation with respect to ξ or r^2) while relations between dimensions separated by one unit are non local.

They imply $\frac{1}{2}$ differentiation.

This fact is related to the Huygen's principle (for a complete bibliography on this subject, see ref. [7]).

As was discussed in III and IV, instead of shifting the dimension, we can shift the dynamics and instead of saying that the dimension is increased by two, we can say that λ is diminished by one, as the dynamics depends essentially on the combination $\frac{n}{2} - \lambda$, as we see from the expression of the $K.G$ Green function.

$$G(n, \lambda, m^2, r) = \frac{2^{1-\lambda} m^{\frac{n}{2}-\lambda}}{\Gamma(\lambda)} \left\{ \frac{K_{\frac{n}{2}-\lambda}(m\xi^{\frac{1}{2}})}{\xi^{\left(\frac{n}{2}-\lambda\right)\frac{1}{2}}} \right\}$$

The dynamical part is concentrated in the brackett. It depends only on $\frac{u}{2} - \lambda$. There is a factor independent of the coordinates which is λ dependent.

This formula is intimately connected to the relations between dimensional regularization (analytic continuation in n) and analytic regularization (analytic continuation in λ). Of course, the difference in λ-dependent constant is responsible for the fact that analytic regularization is not gauge invariant before passing to the limit, while dimensional regularization is.

Acknowledgement

The author is indebted to Dr. J. Benites and O. Obregon for encouraging discussions on the subject, and for the first question which started the idea of this paper. He is also indebted to Prof. C. Aragão (PUC-RJ) for a critical reading of the manuscript.

References

[1] Guelfan I.M. & Chilov G.E. Les Distributions. Dunod. Paris 1962.

[2] Feynman R. Private Communication, 1967.

[3] de Amaral Ruben Gurgel Pinto. Scattering in dimension d. PhD. Thesis. Centro Brasileiro de Pesquisas Físicas , Rio de Janeiro, Brasil.

[4] Gradshteyn I.S. & Ryzhik i.m. tABLE OF INTEGRALS, SERIES and products. Academic Press. New York and London 1965.
Bateman Project. Table of Integral Transforms 2. McGrawHill, 1954.

[5] Bollini C.G. & Giambiagi J.J. J. Math. Physics **15**, 125, 1974.

[6] Bollini C.G. & Giambiagi J.J. J. Math. Physics **34**, 610, 1993.
See also Bollini C.G.; Giambiagi J.J. & Obregon O. Modern Physics Letters A**7** 593, 1992.

[7] Schimming R. A review of Huygen's principle for linear hyperbolic differencial equations. Published in "Group Theoretical Methods in Mechanics", Novosibirsk, USSR, 1974.