On the Octonionic M-superalgebra*

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Abstract

The generalized supersymmetries admitting abelian bosonic tensorial central charges are classified in accordance with their division algebra structure (over **R**, **C**, **H** or **O**). It is shown in particular that in $D = 11$ dimensions, the M-superalgebra admits a consistent octonionic formulation, involving 52 real bosonic generators (in place of the 528 of the standard M-superalgebra). The octonionic $M5$ (super-5-brane) sector coincides with the octonionic $M1$ and $M2$ sectors, while in the standard formulation these sectors are all independent. The octonionic conformal and superconformal M-algebras are explicitly constructed. They are respectively given by the $Sp(8|\mathbf{O})$ $(OSp(1,8|\mathbf{O}))$ (super)algebra of octonionic-valued (super)matrices, whose bosonic subalgebra consists of 232 (and respectively 239) generators.

1 Introduction.

The generalized supersymmetries going beyond the standard HŁS scheme [1] admit the presence of bosonic abelian tensorial central charges associated with the dynamics of extended objects (branes). Classification schemes for generalized supersymmetries are now available [2]. It is worth mentioning that they are based on previous mathematical classifications [3, 4] for spinors and Clifford algebras, in terms of the associative division algebras (**R**, **C**, **H**).

Recently, we investigated [5, 6] the possibility of realizing general supersymmetries in terms of the non-associative division algebra of the octonions. Our work was the first one to suggest a possible octonionic version of the M-theory, with the explicit construction of its corresponding M-superalgebra. In the past, algebras of (super)-matrices with octonionic-valued entries have been introduced in the context of the ten-dimensional superstring theory [7, 8].

Due to the non-associative character of the octonions, the octonionic-valued generalized supersymmetries have peculiar and very surprising features which will be discussed at length in the following. Perhaps the most remarkable and the most unexpected of such features consists in the fact that the different bosonic sectors expressed by the tensorial abelian central charges are no longer independent, as for the standard generalized

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supersymmetries admitting associative realizations, but they are all inter-related. This phenomenon is a peculiar characteristic of the octonionic construction.

It is worth noticing that the Minkowskian 11-dimensional spacetime supports two inequivalent structures, the real structure and the octonionic one. Therefore, besides the standard M-algebra leading to the $OSp(1|32)$ superalgebra [9] (and its $OSp(1|64)$ superconformal algebra), a different M-algebra can be introduced in terms of the octonionic structure and consistently defined as a closed algebra. This is the octonionic M-algebra (it will also sometimes be referred to as M-superalgebra) which will be discussed in this talk. Of course, it is too early to say whether this octonionic M-algebra can be of any relevance for physics. On the other hand, the mere fact that it exists, side by side with the standard M-algebra (not to mention its puzzling features) justifies a thorough investigation of this and its related mathematical structures.

The plan of this talk is as follows. In the next section the classification of Clifford algebras and spinors (i.e. the necessary ingredients to introduce supersymmetry) is recalled. Later, in section 3, the connection of division algebras with the classification of Clifford algebras will be elucidated. In particular the octonionic-valued realizations (which are usually disregarded in the literature) of the Clifford algebras and their corresponding spinors will be introduced. This paves the way for the construction, in Section 4, of the generalized supersymmetries based on the division algebras and, in Section 5, of the octonionic M-algebra. A detailed discussion of its properties will also be given. In particular a table, based on the octonionic structure constants, expressing the equivalence of the different brane sectors in the octonionic description, will be furnished. In Section 6 the octonionic superconformal M-algebra will be introduced. Finally, in the Conclusions, the relation of the octonionic M-algebra with other algebraic structures such as Jordan algebras will be elucidated. Some possible geometrical interpretations underlining the octonionic description will be pointed out and the outline for further future investigations will be given.

2 Classifying Clifford algebras and spinors.

The generalized space-time supersymmetries are the ones going beyond the standard HŁS scheme [1]. This implies that the bosonic sector of the Poincaré or conformal superalgebra no longer can be expressed as the tensor product structure $B_{geom} \oplus B_{int}$, where B_{geom} describes space-time Poincaré or conformal algebras and the remaining generators spanning B_{int} are Lorentz-scalars.

In the particular case of the $D = 11$ dimensions, where the M-theory should be found, the following construction is allowed. In the $D = 11$ Minkowskian spacetime with signature $(10, 1)$ the spinors are real and have 32 components.

By taking the anticommutator of two real spinors the most general result that we can expect consists of a 32×32 symmetric matrix, which admits $32 + \frac{32\cdot 31}{2} = 528$ components.

On the other hand, the standard supertranslation algebra underlining the maximal supergravity contains only the 11 bosonic Poincaré generators and by no means the r.h.s. saturates the number of 528. The extra generators that should be expected in the right hand side are obtained by taking the totally antisymmetrized product of k Gamma matrices (the total number of such objects is given by the Newton binomial $\begin{pmatrix} D \\ k \end{pmatrix}$). The most general 32×32 matrix can be constructed. The further requirement of being symmetric implies that the total number of 528 is obtained by summing the $k = 1$, $k = 2$ and $k = 5$ sectors, so that $528 = 11 + 55 + 462$. The most general supersymmetry algebra in $D = 11$ can therefore be presented as

$$\{Q_a, Q_b\} = (C\Gamma_\mu)_{ab} P^\mu + (C\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1 \ldots \mu_5]})_{ab} Z^{[\mu_1 \ldots \mu_5]} \tag{2.1}$$

(where C is the charge conjugation matrix).

$Z^{[\mu\nu]}$ and $Z^{[\mu_1 \ldots \mu_5]}$ are tensorial central charges, of rank 2 and 5 respectively. These two extra central terms on the right hand side correspond to extended objects [10, 11], the p-branes. The algebra (2.1) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1 \ldots \mu_5]} \equiv 0$.

For the purpose of the classification of generalized supersymmmetries in any given signature space-time, we need at first to have at disposal the mathematical classification of Clifford algebras and spinors (see [3, 4], while a very convenient reference for connection with physics is [12]). In the rest of this section we introduce the fundamental results which will be used in the following. Such results can be very conveniently expressed in terms of the recursive algorithm given below. Two remarks are in order. The first one: despite the fact that a quantum theory is described by complex numbers, without loss of generality (complex numbers can be considered as points in the real plane) it is preferable to work with Clifford algebras expressed by real-valued matrices. The structure of Clifford algebras is much clearer in such a framework (e.g. its connection with division algebras properties). A second comment: the algorithm furnished below permits in individuating a single representative for each irreducible class of representations of Clifford's Gamma matrices.

The construction is as follows. Let us prove at first that a recursive construction of $D + 2$ spacetime dimensional Clifford algebras is available, when assumed known a D dimensional representation. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D = p + q$ spacetime with (p, q) signature (namely, providing a representation for the $C(p, q)$ Clifford algebra) then $2d$-dimensional $D + 2$ Gamma matrices (denoted as Γ_j) of a $D + 2$ spacetime are produced according to either

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p, q) &\mapsto (p + 1, q + 1). \end{aligned} \tag{2.2}$$

or

$$
\begin{aligned}
\Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\
(p,q) &\mapsto (q+2,p). \qquad (2.3)
\end{aligned}
$$

Some remarks are in order. The two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (2.2) or (2.3) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$
\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.4)
$$

All Clifford algebras are obtained by recursively applying the algorithms (2.2) and (2.3) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,3+4m)$ (m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the table below.

Table with the maximal Clifford algebras (up to $d = 256$).

1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
$\underline{(1,0)}$	$\Rightarrow$	(2,1)	$\Rightarrow$	(3,2)	$\Rightarrow$	(4,3)	$\Rightarrow$	(5,4)	$\Rightarrow$	(6,5)	$\Rightarrow$	(7,6)	$\Rightarrow$	(8,7)	$\Rightarrow$	(9,8)	$\Rightarrow$
						(1,4)	$\rightarrow$	(2,5)	$\rightarrow$	(3,6)	$\rightarrow$	(4,7)	$\rightarrow$	(5,8)	$\rightarrow$	(6,9)	$\rightarrow$
					$\nearrow$												
				$\underline{(0,3)}$													
					$\searrow$												
						(5,0)	$\rightarrow$	(6,1)	$\rightarrow$	(7,2)	$\rightarrow$	(8,3)	$\rightarrow$	(9,4)	$\rightarrow$	(10,5)	$\rightarrow$
								(1,8)	$\rightarrow$	(2,9)	$\rightarrow$	(3,10)	$\rightarrow$	(4,11)	$\rightarrow$	(5,12)	$\rightarrow$
							$\nearrow$										
						$\underline{(0,7)}$											
							$\searrow$										
								(9,0)	$\rightarrow$	(10,1)	$\rightarrow$	(11,2)	$\rightarrow$	(12,3)	$\rightarrow$	(13,4)	$\rightarrow$
														(1,12)	$\rightarrow$	(2,13)	$\rightarrow$
													$\nearrow$				
												$\underline{(0,11)}$					
													$\searrow$				
														(13,0)	$\rightarrow$	(14,1)	$\rightarrow$
																(1,16)	$\rightarrow$
															$\nearrow$		
														$\underline{(0,15)}$			
															$\searrow$		
																(17,0)	$\rightarrow$

(2.5)

Concerning the previous table, some remarks are in order. The columns are labeled by the matrix size **d** of the maximal Clifford algebras. Their signature is denoted by the (p, q) pairs. Furthermore, the underlined Clifford algebras in the table are called the "primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are the "maximal descendant Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (2.2) and (2.3). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices. It should be noticed that Clifford algebras in even-dimensional spacetimes are always non-maximal.

Let us discuss concretely a given example, namely the explicit construction of the $D = p + q$ spacetime dimensional Clifford algebras for $D = 11$ (the dimensionality of M-theory). We obtain

(p, q)	*type*	d
(11,0)	$\subset$ (11,2)	64
(10,1)	**M**	32
(9,2)	$\subset$ (11,2)	64
(8,3)	**M**	64
(7,4)	$\subset$ (7,6)	64
(6,5)	**M**	32
(5,6)	$\subset$ (7,6)	64
(4,7)	**M**	64
(3,8)	$\subset$ (3,10)	64
(2,9)	**M**	32
(1,10)	$\subset$ (3,10)	64
(0,11)	**M**	32

where the maximal Clifford algebras are labeled by **M** (the remaining non-maximal algebras are recovered from the maximal ones given on the second column, after deleting a certain number of Γ-matrices). The size of the matrix representation is given by the number on the right (d).

For what concerns the construction of the primitive maximal Clifford algebras of the series $C(0, 3 + 8n)$ (also known as quaternionic series, due to its connection with this division algebra, as we will see later), as well as the octonionic series $C(0, 7 + 8n)$, the answer can be provided with the help of the three Pauli matrices (2.4). We construct at first the 4×4 matrices realizing the Clifford algebra $C(0, 3)$ and the 8×8 matrices realizing the Clifford algebra $C(0, 7)$. They are given, respectively, by

$$C(0,3) \equiv \begin{array}{l} \tau_A \otimes \tau_1, \\ \tau_A \otimes \tau_2, \\ \mathbf{1}_2 \otimes \tau_A. \end{array} \tag{2.6}$$

and

$$C(0,7) \equiv \begin{array}{l} \tau_A \otimes \tau_1 \otimes \mathbf{1}_2, \\ \tau_A \otimes \tau_2 \otimes \mathbf{1}_2, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \\ \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_A \otimes \tau_A \otimes \tau_A. \end{array} \quad (2.7)$$

The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i, = 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\tilde{\tau}_i$, $i = 1,2,\ldots,7$.

In order to construct the remaining Clifford algebras of the series we need at first to apply the (2.2) algorithm to $C(0,7)$ and construct the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2 = \mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2\ldots,8$, with ${\gamma_j}^2 = -\mathbf{1}$). We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$C(0,3+8n) \equiv \begin{array}{lll} \overline{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \ldots \otimes \gamma_9, \\ \mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes \gamma_9 \otimes \gamma_j, \end{array} \quad (2.8)$$

and similarly

$$C(0,7+8n) \equiv \begin{array}{lll} \tilde{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \ldots \otimes \gamma_9, \\ \mathbf{1}_8 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes \gamma_9 \otimes \gamma_j, \end{array} \quad (2.9)$$

Please notice that the tensor product of the 16-dimensional representation is taken n times. The total size of the (2.8) matrix representations is then 4×16^n, while the total size of (2.9) is 8×16^n.

The formulas given above provide quite a practical and efficient tool to operatively construct the irreducible Clifford algebras.

It should be noticed that all Clifford matrices are even-dimensional (power of 2). An important subclass of Clifford Gamma matrices is obtained by the matrices which are decomposable in 2×2 blocks and are non-vanishing only in the anti-diagonal blocks. Such matrices can be named as (generalized) Weyl-type matrices (they can also be regarded of "supersymmetric type" since they can be promoted to be fermionic matrices associated with the representations of the extended supersymmetries, see [13]). An inspection of the previous tables shows that the set of the (generalized) Weyl matrices is found in special signature dimensions. All primitive Clifford algebras are not of (generalized) Weyl type. However, all the derived Clifford algebras, through the two lifting algorithms, are of Weyl-type, once deleted the $\begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$ matrix to produce a non-maximal Clifford algebra.

To give a concrete example, the two-dimensional Euclidean space $(2,0)$ is not of Weyl type, while the two-dimensional Minkowski spacetime $(1,1)$ is of Weyl type. Indeed, the first one is obtained from the $(2,1)$ Clifford algebra by deleting a space-type Gamma matrix, while the second one is obtained from the same $(2,1)$ Clifford algebra by deleting one of the two temporal-type Gamma matrices. Without loss of generality this Gamma matrix can always be chosen to be given by $\begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$.

The importance of the Weyl realization of Clifford algebras is of course related with the possibility of introducing a Weyl projection for Dirac spinors. The commutator between Gamma matrices, $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$, can be regarded as the generator of the Lorentz group corresponding to the given signature space-time. The product of two Gamma matrices, both admitting non-vanishing blocks only in the antidiagonal, correponds to a 2×2 block matrix whose only non-vanishing components are in the diagonal blocks. Since both the Gamma matrices, as well as the Lorentz generators $\Sigma_{\mu\nu}$, act on spinors, the fact that the Lorentz generators are block-diagonals means that we can consistently set, under these conditions, equal to 0 half of the components of the column vector spinors (either the upper half or the lower half), to produce the so-called Weyl spinor, admitting half of the degrees of freedom expected by the original Dirac spinor. This reduction of the components can be operated acting on a Dirac spinor with a projector $P_\pm$ ($P_\pm P_\mp = 0$ and ${P_\pm}^2 = P_\pm$), given by

$$P_\pm = \frac{1}{2}(\mathbf{1} \pm \overline{\Gamma}) \tag{2.10}$$

where $\overline{\Gamma} = \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}$.

In even-dimensional space-times the matrix $\overline{\Gamma}$ is always given by the product of all the other Γ matrices (it corresponds to Γ_5 when we specialize to the standard 4-dimensional Minkowski space-time).

In order to construct lagrangian terms which are scalar under Lorentz transformations and are given by bilinear products of spinors, we need to introduce the notion of barred spinors $\overline{\Psi}$, given by $\Psi^T \cdot A$, where T denotes transposition (remember that in our conventions spinors are without loss of generality assumed to be real) and A is a matrix, given by the product of all temporal Gamma matrices, i.e. the generalization of the Minkowskian 4-dimensional Γ_0 matrix.

3 Division algebras and Clifford algebras.

So far we have shown how to construct the irreducible representations of Clifford algebras, and not yet elucidated their relations with division algebras. Such a relation can be expressed as follows. The three matrices appearing in $C(0,3)$ can also be expressed in terms of the imaginary quaternions τ_i satisfying $\tau_i \cdot \tau_j = -\delta_{ij} + \epsilon_{ijk}\tau_k$. As a consequence, the whole set of maximal primitive Clifford algebras $C(0,3+8n)$, as well as their maximal descendants, can be represented as quaternionic-valued matrices, acting on spinors, which have to be interpreted now as quaternionic-valued column vectors.

Similarly, there exists an alternative realization for the Clifford algebra $C(0,7)$, obtained by identifying the seven generators with the seven imaginary octonions satisfying the algebraic relation

$$\tau_i \cdot \tau_j \quad = \quad -\delta_{ij} + C_{ijk}\tau_k, \tag{3.11}$$

for $i,j,k = 1,\cdots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{3.12}$$

and vanishing otherwise. This octonionic realization of the seven-dimensional Euclidean Clifford algebra will be denoted as $C_{\mathbf{O}}(0,7)$. Due to the non-associative character of the (3.11) octonionic product (the weaker condition of alternativity is satisfied, see [14]), the octonionic realization cannot be represented as an ordinary matrix product and is therefore a distinct and inequivalent realization of this Euclidean Clifford algebra with respect to the one previously considered (2.7). Please notice that, by iteratively applying the two lifting algorithms to $C_{\mathbf{O}}(0,7)$ we obtain matrix realizations (with octonionic-valued entries) for the maximal Clifford algebras of the series $C(n,7+n)$ and $C(8+n,n-1)$, for positive integral values of n ($n=1,2,\ldots$). The dimensionality of the corresponding octonionic-valued matrices is $2^n \times 2^n$. For completeness we should point out that the construction (2.9) leading to the primitive maximal Clifford algebras $C(0,7+8n)$, can be carried on with the help of an octonionic-valued realization of the γ_9 matrix. As a consequence, realizations of $C(0,7+8n)$ and their descendants can be produced acting on column spinors, whose entries are tensor products of octonions. In any case, in the following, we will focus on the single octonionic realizations $C_{\mathbf{O}}(n,7+n)$ and $C_{\mathbf{O}}(9+n,n)$ (here $n=0,1,2,\ldots$) which are of relevance in the context of the M-theory.

One should be aware of the properties of the non-associative realizations of Clifford algebras. In the octonionic case the commutators $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$ are no longer the generators of the Lorentz group. They correspond instead to the generators of the coset $SO(p,q)/G_2$, being G_2 the 14-dimensional exceptional Lie algebra of automorphisms of the octonions. As an example, in the Euclidean 7-dimensional case, these commutators give rise to $7 = 21 - 14$ generators, isomorphic to the imaginary octonions. Indeed

$$[\tau_i, \tau_j] \quad = \quad 2C_{ijk}\tau_k. \tag{3.13}$$

The alternativity property satisfied by the octonions implies that the seven-dimensional commutator algebra among imaginary octonions is not a Lie algebra, the Jacobi identity being replaced by a weaker condition that endorses (3.13) with the structure of a Malcev algebra (see [14]).

Such an algebra admits a nice geometrical interpretation [15, 16]. Indeed the normed 1 unitary octonions $X = x_0 + x_i\tau_i$ (with x_0 and x_i, for $i = 1, \ldots, 7$, real and the summation over repeated indices understood), i.e. restricted by the condition

$$X^\dagger \cdot X \quad = \quad 1, \tag{3.14}$$

describe the seven-sphere S^7. The latter is a parallelizable manifold with a quasi (due to the lack of associativity) group structure. Here $X^\dagger$ denotes the principal conjugation for the octonions, namely $X^\dagger = x_0 - x_i\tau_i$.

On the seven sphere, infinitesimal homogeneous transformations which play the role of the Lorentz algebra can be introduced through

$$\delta X \quad = \quad a \cdot X, \tag{3.15}$$

with a an infinitesimal constant octonion. The requirement of preserving the unitary norm (3.14) implies the vanishing of the a_0 component, so that $a \equiv a_i\tau_i$. Therefore, the above commutator algebra (3.13), generated by the seven τ_i, can be interpreted as the algebra of "quasi" Lorentz transformations acting on the seven sphere S^7. At least in this specific example we discovered a nice geometrical setting underlining the use of the octonionic realization of the $C_{\mathbf{O}}(0,7)$ Clifford algebra. While the associative (2.7) representation of the seven dimensional Clifford algebra is required for describing the Euclidean 7-dimensional flat space, the non-associative realization describes the geometry of S^7.

4 Division algebras and generalized supersymmetries.

It is clear that extra-conditions on the generalized supersymmetries such as (2.1) can be imposed if we assume the fundamental spinors being division-algebra valued (over **C**, **H** or **O**) and not just real. A division algebra analog of the supertranslation algebra can be introduced through the position

$$\begin{aligned} \{Q_a, Q_b\} = \{{Q^\dagger}_a, {Q^\dagger}_b\} = 0, \\ \{Q_a, {Q^\dagger}_b\} = Z_{ab}. \end{aligned} \tag{4.16}$$

where † denotes the principal conjugation in the given division algebra and, as a result, the bosonic abelian algebra on the r.h.s. is constrained to be hermitian

$$Z_{ab} \quad = \quad {Z_{ba}}^\dagger. \tag{4.17}$$

Division-algebra structures for spinors can be consistently imposed only in specific signature space-times. As already recalled, in $D = 11$, either a real or an octonionic structure can be defined for the Minkowskian $C(10,1)$ case, while a quaternionic structure can be imposed for the Euclidean $C(0,11)$ Clifford algebra (from formula (2.8), constructed in terms of the quaternions). The 32 real components spinors can be re-expressed in $(10,1)$ as 4-component octonionic-valued spinors and, for $(0,11)$, as 8-component quaternionic-valued spinors. In the Minkowskian $(10,1)$ case, the hermiticity condition imposed on the 4×4 octonionic-valued hermitian matrix Z_{ab} leaves it with 52 independent components, while in the Euclidean $(0,11)$ case the same hermiticity condition, applied this time on the 8×8 quaternionic-valued Z_{ab} matrix, leaves only 120 surviving bosonic components. Not surprisingly, in both cases this number is less than the total number of 528 independent components obtained from the real structure. This is already a first indication of the constraint produced by the division algebra structures (especially the octonionic one).

It is worth concluding this section with a comment concerning the reconstruction of the division algebra-valued matrix realizations of algebraic structures in terms of their component fields. This is better illustrated with a specific example. Let us discuss the simplest case, the 1-dimensional octonionic $N = 8$ supersymmetry (the considerations below trivially applies to the quaternionic $N = 4$ supersymmetry as well).

Let us specialize (4.16) to the two one-dimensional octonionic-valued fermionic operators $\mathcal{Q}$, $\overline{\mathcal{Q}}$, satisfying the $N = 8$ superalgebra

$$\{\mathcal{Q},\mathcal{Q}\} = \{\overline{\mathcal{Q}},\overline{\mathcal{Q}}\} = 0, \qquad \{\mathcal{Q},\overline{\mathcal{Q}}\} = \mathcal{H}. \tag{4.18}$$

where $\mathcal{H} = H$ is real-valued and represents the hamiltonian.

$\mathcal{Q}$, $\overline{\mathcal{Q}} = \mathcal{Q}^\dagger$ contain a total number of 8 components and we should expect they define an algebra with a total number of $8 + \frac{8\times 7}{2} = 36$ anticommutation relations. On the other hand, the r.h.s. of (4.18) provides us at most of $3 \times 8 = 24$ relations, so that it looks like something is missing. Furthermore, when expanding in terms of components ($i = 1, \ldots, 7$)

$$\begin{aligned} \mathcal{Q} &= Q_0 + \sum_i Q_i t_i, \\ \overline{\mathcal{Q}} &= Q_0 - \sum_i Q_i t_i \end{aligned} \tag{4.19}$$

and taking into account the (3.11) algebraic relations for imaginary octonions, we end up with the following set of relations for the component fields Q_0, Q_i (the convention over repeated indices is understood)

$$\begin{aligned} \{Q_0, Q_0\} - \{Q_i, Q_i\} &= 0, \\ \{Q_0, Q_i\} &= 0, \\ C_{ijk}[Q_j, Q_k] &= 0, \\ \{Q_0, Q_0\} + \{Q_i, Q_i\} &= H, \end{aligned} \tag{4.20}$$

These relations are odd since, in particular, the third one involves a commutator, instead of an anticommutator as we should expect. However, there is nothing wrong with (4.20) and this algebra can be re-expressed in terms of its component fields when correctly interpreted. The correct interpretation for (4.20) consists in setting

$$\begin{aligned} Q_0 &= q, \\ Q_i &= \frac{i\lambda}{\sqrt{7}} \quad for \quad any \quad i = 1, \ldots, 7. \end{aligned} \tag{4.21}$$

In particular this implies that the ordinary component $\mathbf{Q_i}$ fields are not extracted from the Q_i coefficients of t_i, rather they have to be identified with the product λt_i

$$\mathbf{Q_i} \equiv \lambda t_i. \tag{4.22}$$

With the above position the set of "odd" relations (4.20) is now reduced to the set of ordinary relations

$$\begin{aligned} \{q, \lambda\} &= 0, \\ \{q, q\} = -\{\lambda, \lambda\} &= \frac{1}{2}H. \end{aligned} \tag{4.23}$$

It should be noticed that the fermionic operator λ is antihermitian ($\lambda = -\lambda^\dagger$) in order to provide the consistent hermiticity condition on $\mathbf{Q_i}$. It is worth mentioning that all these relations are explicitly realized in the octonionic matricial $N = 8$ supersymmetry algebra which can be recovered from the octonionic 2×2 realization of $C_{\mathbf{O}}(9, 0)$. We recall that this octonionic $N = 8$-supersymmetry [13] is constructed with the set of the hermitian 2×2 octonionic-valued matrices of Weyl type non-vanishing only in the antidiagonal (i.e. the additional Γ_9 matrix in $C_{\mathbf{O}}(9, 0)$ is discarded), given by

$$\begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix} \equiv t_i \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.24}$$

We can identify

$$\begin{aligned} q &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \lambda &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{4.25}$$

which satisfy the correct properties.

The reconstruction of the division-algebra structures in terms of its component fields is more elaborated in the more complicated examples discussed in the following. Nevertheless, even in these cases, it can be performed following the procedure here outlined.

5 The octonionic M-algebra.

The octonionic M-algebra [5] is defined by specializing (4.16) to the $(10,1)$ case. The needed octonionic-valued Clifford matrices are immediately obtained with the help of the lifting algorithm introduced in section **2** (e.g through the procedure $(0,7) \rightarrow (9,0) \rightarrow (10,1)$, while the C matrix introduced below coincides in this case with the unique space-like Gamma matrix). It must be said that two equivalent ways exist of introducing the M-algebra, either in terms of the 4-component $D=11$ spinors, or in terms of the Weyl spinors in $(10,2)$ dimensions (the latter construction leads to the F-algebra interpretation). Here we just limit ourselves to consider the first case.

The only non-vanishing (anti)-commutator of the octonionic M-algebra is given by

$$\{Q_a, {Q^\dagger}_b\} \quad = \quad Z_{ab}, \tag{5.26}$$

where, in this case, the 52 independent components of the hermitian Z_{ab} matrix can be represented either as the $11+41$ bosonic generators entering

$$\mathcal{Z}_{ab} = P^\mu (C\Gamma_\mu)_{ab} + Z^{\mu\nu}_{\mathbf{O}} (C\Gamma_{\mu\nu})_{ab}, \tag{5.27}$$

or as the 52 bosonic generators entering

$$\mathcal{Z}_{ab} = Z^{[\mu_1\ldots\mu_5]}_{\mathbf{O}} (C\Gamma_{\mu_1\ldots\mu_5})_{ab}\,. \tag{5.28}$$

The reason for that lies in the fact that, unlike the real case, the sectors individuated by (5.27) and (5.28) are not independent. This is a consequence of the non-associativity of the octonions. Indeed, one has to point out that, when multiplying antisymmetric products of k octonionic-valued matrices, a certain number of them are redundant. For $k=2$, due to the G_2 automorphisms, 14 such products have to be erased. In the general case [17] a table can be produced. We write it down for D odd-dimensional spacetime octonionic realizations of Clifford algebras. The case suitable for M-theory is recovered for $D=11$. The following table can be easily constructed from the $D=7$ results (which are easily computed), by taking into account that out of the D Gamma matrices, 7 of them are octonionic-valued, while the remaining $D-7$ are purely real.

The following table, up to $D=13$, is easily obtained:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$D=7$	1	7	7	1	1	7	7	1						
$D=9$	1	9	22	22	10	10	22	22	9	1				
$D=11$	1	11	41	75	76	52	52	76	75	41	11	1		
$D=13$	1	13	64	168	267	279	232	232	279	267	168	64	13	1

(5.29)

where the columns are labelled by k, the number of antisymmetrized Gamma matrices.

To reproduce the above formulas one has to be careful in defining the antisymmetric product for $k>2$ octonionic Γ-matrices. Due to the non-associativity of the octonions the order of multiplications must be correctly specified. The correct prescription is the following one. The antisymmetrized product of k octonionic matrices A_i $(i=1,2,\ldots,k)$ is given by

$$[A_1\cdot A_2\cdot\ldots\cdot A_k] \equiv \frac{1}{k!}\sum_{perm.}(-1)^{\epsilon_{i_1\ldots i_k}}(A_{i_1}\cdot A_{i_2}\ldots\cdot A_{i_k}), \tag{5.30}$$

where $(A_1\cdot A_2\ldots\cdot A_k)$ denotes the symmetric product

$$(A_1\cdot A_2\cdot\ldots\cdot A_n) \equiv \frac{1}{2}(.((A_1A_2)A_3\ldots)A_k)+\frac{1}{2}(A_1(A_2(\ldots A_k)).). \tag{5.31}$$

This prescription is consistent and produces the correct result. As an example, in $D=11$ the three-fold antisymmetric product of octonionic Γ-matrices $C[\Gamma_i\cdot\Gamma_j\cdot\Gamma_k]$ furnishes the 75 antihermitian matrices, appearing in the table above, describing together with C an arbitrary 4×4 octonionic antihermitian matrix. The definition (5.30), applied to the five-fold products of the $D=11$ octonionic Γ-matrices, provides their octonionic hermiticity. The explicit computation above shows that precisely 52 independent real tensorial charges describes the five-tensor sector of the octonionic M-algebra, which means that it spans the arbitrary 4×4 octonionic-hermitian matrices. We thus see that one can equivalently write the octonionic M-algebra as (5.27) or as (5.28). In the latter case, out of the 462 real antisymmetric 5-tensorial charges of the standard M-algebra, only 52 are linearly independent, due to the discovered relation

$$[\Gamma_{\mu_1\ldots\mu_5}] = A_{[\mu_1\ldots\mu_5]}{}^{\nu}\Gamma_\nu + A_{[\mu_1\ldots\mu_5]}{}^{[\nu_1\nu_2]}\Gamma_{[\nu_1}\Gamma_{\nu_2]}, \tag{5.32}$$

with constant c-number coefficients $A_{[\mu_1\ldots\mu_5]}{}^{\nu}$, $A_{[\mu_1\ldots\mu_5]}{}^{[\nu_1\nu_2]}$.

The octonionic equivalence of different sectors (which, at least for some spacetimes, can be interpreted as branes sectors) can be simbolically expressed, in different odd space-time dimensions, according to the table

$D=7$	$M0\equiv M3$
$D=9$	$M1+M2\equiv M4$
$D=11$	$M1+M2\equiv M5$
$D=13$	$M2+M3\equiv M6$
$D=15$	$M3+M4\equiv M0+M7$

(5.33)

In $D=11$ dimensions the relation between $M1+M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are splitted into the 4 real indices, labelled by $a,b,c,\ldots$

and the 7 octonionic indices labelled by $i, j, k, \ldots$. We get, on one side,

$$\begin{array}{cc} 4 & M1_a \\ 7 & M1_i \\ 6 & M1_{[ab]} \\ 4\times 7 = 28 & M2_{[ai]} \\ 7 & M2_{[ij]} \equiv M2_i \end{array}$$

while, on the other side,

$$\begin{array}{cc} 7 & M5_{[abcdi]} \equiv M5_i \\ 4\times 7 = 28 & M5_{[abcij]} \equiv M5_{[ai]} \\ 6 & M5_{[abijk]} \equiv M5_{[ab]} \\ 4 & M5_{[aijkl]} \equiv M5_a \\ 7 & M5_{[ijklm]} \equiv \tilde{M}5_i \end{array}$$

which shows the equivalence of the two sectors, as far as the tensorial properties are concerned. Please notice that the correct total number of 52 independent components is recovered

$$52 = 2\times 7 + 28 + 6 + 4. \tag{5.34}$$

6 The octonionic superconformal M-algebra.

In this section the superconformal octonionic M-algebra is introduced following [6].

The conformal algebra of the octonionic M-theory can be introduced adapting to the eleven dimensions the procedure discussed in [8] for the 10 dimensional case. It requires the identification of the conformal algebra of the octonionic $D = 11$ M-algebra with the generalized Lorentz algebra in the $(11, 2)$-dimensional space-time. In such a space-time the octonionic Clifford's Gamma-matrices are 8-dimensional. The basis of the hermitian generators is given by the 64 antisymmetric two-tensors $C\Gamma_{[\mu_1\mu_2]}\mathcal{Z}^{\mu_1\mu_2}$ and the 168 antisymmetric three tensors $C\Gamma_{[\mu_1\mu_2\mu_3]}\mathcal{Z}^{\mu_1\mu_2\mu_3}$ (or, equivalently, by the 232 antisymmetric six-tensors $C\Gamma_{[\mu_1\ldots\mu_6]}\mathcal{Z}^{\mu_1\ldots\mu_6}$). This is already an indication that the total number of generators in the conformal algebra is 232. We will show that this is the case.

According to [8], the conformal algebra can be introduced as the algebra of transformations leaving invariant the inner product of Dirac's spinors. In $(11, 2)$ this is given by $\psi^\dagger C\eta$, where the matrix C, the analogous of the Γ^0, given by the product of the two space-like Clifford's Gamma matrices, is real-valued and totally antisymmetric. Therefore,the conformal transformations are realized by the octonionic-valued 8-dimensional matrices $\mathcal{M}$ leaving C invariant, i.e. satisfying

$$\mathcal{M}^\dagger C + C\mathcal{M} = 0. \tag{6.35}$$

This allows identifying the (quasi)-group of conformal transformations with the (quasi-)group of symplectic transformations. Indeed, under a simple change of variables, C can be recasted to be of the form

$$\Omega = \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix}. \tag{6.36}$$

The most general octonionic-valued matrix leaving invariant Ω can be expressed through

$$\mathbf{M} = \begin{pmatrix} D & B \\ C & -D^\dagger \end{pmatrix}, \tag{6.37}$$

where the 4×4 octonionic matrices B, C are hermitian

$$B = B^\dagger, \qquad\qquad C = C^\dagger. \tag{6.38}$$

It is easily seen that the total number of independent components in (6.37) is precisely 232, as we expected from the previous considerations.

It is worth noticing that the set of matrices $\mathbf{M}$ of (6.37) type forms a closed algebraic structure under the usual matrix commutation. Indeed $[\mathbf{M}, \mathbf{M}] \subset \mathbf{M}$, endowing the structure of $Sp(8|\mathbf{O})$ to $\mathbf{M}$. For what concerns the supersymmetric extension of the superconformal algebra, we have to accommodate the 64 real components (or 8 octonionic) spinors of $(11, 2)$ into a supermatrix enlarging $Sp(8|\mathbf{O})$. This can be achieved as follows. The two 4-column octonionic spinors α and β can be accommodated into a supermatrix of the form

$$\left(\begin{array}{c|cc} 0 & -\beta^\dagger & \alpha^\dagger \\ \hline \alpha & 0 & 0 \\ \beta & 0 & 0 \end{array}\right). \tag{6.39}$$

Under anticommutation, the lower bosonic diagonal block reduces to $Sp(8|\mathbf{O})$. On the other hand, extra seven generators, associated to the 1-dimensional antihermitian matrix A

$$A^\dagger = -A, \tag{6.40}$$

i.e. representing the seven imaginary octonions, are obtained in the upper bosonic diagonal block. Therefore, the generic bosonic element is of the form

$$\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D & B \\ 0 & C & -D^\dagger \end{array}\right), \tag{6.41}$$

with A, B and C satisfying (6.40) and (6.38).

The closed superalgebraic structure, with (6.39) as generic fermionic element and (6.41) as generic bosonic element, will be denoted as $OSp(1, 8|\mathbf{O})$. It is the superconformal algebra of the M-theory and admits a total number of 239 bosonic generators.

7 Conclusions.

The octonions are at the very heart of many exceptional structures in mathematics. It is very well known, e.g., that they can be held responsible for the existence of the 5 exceptional Lie algebras. Indeed, G_2 is the automorphism group of the octonions, while F_4 is the automorphism group of the 3×3 octonionic-valued hermitian matrices realizing the exceptional $J_3(\mathbf{O})$ Jordan algebra. F_4 and the remaining exceptional Lie algebras (E_6, E_7, E_8) are recovered from the so-called "magic square Tit's construction" which associates a Lie algebra to any given pair of division algebras, if one of these algebras coincide with the octonionic algebra [18].

There is a line of thought [19] suggesting that Nature prefers exceptional structures. Following this line of thought, in [20], the already recalled exceptional Jordan algebra $J_3(\mathbf{O})$ was used to define a unique Chern-Simons type of theory in the loop quantum gravity approach. In a different line of research, octonionic structures were investigated in different works [7, 8] in application to the superstring theory.

In this talk we summarized the results recently obtained in a series of works, especially [5, 6] concerning the possibility of introducing an octonionic structure for the M-theory algebra. After briefly recalling the classification of spinors and Clifford algebras in terms of division algebras (and more specifically their octonionic construction) we were able to introduce at first the octonionic M-superalgebra and, later, its superconformal extension presented in formulas (6.39), (6.41).

The features of the octonionic M-superalgebra are puzzling. It is not at all surprising that it contains fewer bosonic generators, 52, w.r.t. the 528 of the standard M-algebra (this is expected, after all the imposition of an extra structure, such as the complex, quaternionic or octonionic structure, puts a constraint on a theory). What is really unexpected is the fact that new conditions, not present in the standard M-theory, are now found. These conditions, which can be symbolically represented in table (5.33), imply that the different brane-sectors are no longer independent. The octonionic 5-brane contains the same degrees of freedom and is equivalent to the $M1$ and the $M2$ sectors. We can write this equivalence, symbolically, as $M5 \equiv M1 + M2$. This result is indeed very intriguing. It implies that quite non-trivial structures are found when investigated the octonionic construction of the M-theory. It also raises some questions, because it is not yet clear how should we interpret it and which is its proper meaning. At least two different viewpoints can be advocated. On one hand, sticking with the original defined octonionic algebra, one should try to investigate its possible quantum-mechanical consistency, understanding whether and to which extent it is possible to adapt the procedure of [21] to the present situation. On the other hand, another possibility can be contemplated. We have discussed at the end of section **3** that the octonionic realization of the 7-dimensional Euclidean Clifford algebra is related with the geometry of the seven sphere S^7. There is a possibility, which deserves being investigated, that the octonionic description of the M-theory would correspond to a particular compactification of the 11-dimensional M-

theory down to $AdS_4 \times S^7$. This compactification corresponds to a natural solutions for the 11-dimensional supergravity [22]. If this would be the case, the relations of equivalence found in the octonionic construction should find a counterpart in the $AdS_4 \times S^7$ special compactification geometry. Needless to say, this possibility is currently under investigation.

We conclude with a last remark that perhaps deserves to be mentioned. We introduced both the conformal and the superconformal extensions of the original M-algebra. They are respectively given by $Sp(8|\mathbf{O})$ and $OSp(1,8|\mathbf{O})$, see formulas (6.39), (6.41). $Sp(8|\mathbf{O})$ is outside the scheme of conformal algebras of a given Jordan algebra (such as $Sp(4|\mathbf{O})$, $Sp(6|\mathbf{O})$, the latter the conformal algebra of $J_3(\mathbf{O})$), usually investigated in the mathematical literature, see [23, 24]. The reason for that is the fact that the bosonic sector of the M-algebra is given by 4×4 octonionic-valued hermitian matrices, and the maximal Jordan algebra of octonionic-valued hermitian matrices is given by 3-dimensional matrices. The construction of the conformal (and superconformal) algebra, however, as we have proven, can be carried on in this case as well and it finally produces the closed and consistent algebraic structures that we mentioned before.

References

[1] R. Haag, J. Łopuszański and M. Sohnius, Nucl.Phys. **B 88**, 257 (1975).

[2] R. D'Auria, S. Ferrara and S. Vaula, Class. Quant. Grav. **18**, 3181 (2001); JHEP 0010 (2000), 013; S. Ferrara, M.A. Lledo, hep-th/0112177.

[3] M.F. Atiyah, R. Bott and A. Shapiro, Topology (Suppl. 1) **3**, 3 (1964).

[4] T. Kugo and P. Townsend, Nucl. Phys. **B 221**, 357 (1983).

[5] J. Lukierski and F. Toppan, Phys. Lett. **B 539**, 266 (2002).

[6] J. Lukierski and F. Toppan, "Octonionic M-theory and $D = 11$ Generalized Conformal and Superconformal Algebras". Work in preparation.

[7] D.B. Fairlie and A.C. Manogue, Phys. Rev. **D 34**, 1832 (1986).

[8] K.W. Chung and A. Sudbery, Phys. Lett. **B 198**, 161 (1987).

[9] E. Bergshoeff and A. Van Proeyen, Class. Quant. Grav. **17**, 3277 (2000).

[10] E. Bergshoeff, E. Sezgin, P, Townsend, Ann. Phys. **18**, 330 (1987).

[11] M.J. Duff. R.R. Khuri, J.X. Lu, Phys. Rep. **259**, 213 (1995).

[12] S.Okubo, J. Math. Phys.**32**, 1657 (1991); *ibid.* **32**, 1669 (1991).

[13] F. Toppan, Nucl. Phys. B (Proc. Suppl.) **102&103**, 270 (2001); H.L. Carrion, M. Rojas and F. Toppan, "Non-Associative Supersymmetries". Work in preparation.

[14] M. Günaydin and S.V. Ketov, Nucl. Phys. **B 467**, 215 (1996).

[15] Z. Hasiewicz and J. Lukierski, Phys. Rev. **145 B**, 65 (1984).

[16] M. Cederwall and C. Preitschopf, Comm. Math. Phys. **167**, 373 (1995).

[17] H.L. Carrion, M. Rojas and F. Toppan, "Quaternionic and Octonionic Spinors". Work in preparation.

[18] C.A. Barton and A. Sudbery, math.RA/0203010.

[19] P. Ramond, "Algebraic Dreams", hep-th/0112261.

[20] L. Smolin, hep-th/0104050.

[21] M. Günaydin, C. Piron and H. Ruegg, Comm. Math. Phys. **61**, 69 (1978).

[22] M.J. Duff and C.N. Pope, "Kaluza-Klein supergravity and the seven-sphere" in Supersymmetry and Supergravity 82, Trieste proceedings; M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Rep. **130**, 1 (1986).

[23] I.L. Kantor, Sov. Math.-Dokl. **14**, 254 (1973).

[24] A. Sudbery, J. Phys. **A 17**, 939 (1984).