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Abstract. The axial anomaly is calculated for any number of space -time dimension, using the dimensional regularization method.

Recently[1-4] some interest have emerged in higher dimensional space-time axial anomalies. The motivation for studying these anomalies comes from consideration that higher dimensional theories may underly unification of gravity with other elementary forces.

We show in this paper that we can calculate the axial anomaly in any space-time dimension using dimensional regularization[5], in a way similar to that already used to calculate the Adler -Bell-Jackiw anomaly[6] with the same method[7].

For simplicity we shall consider anomalies in Abelian theory since the generalization to non-Abelian theory involves only an overall group multiplicative factor[2,8].

Since in odd space-time we have no axial anomaly[2,3] let us consider an even space-time with dimension $n=2j$ for which the relevant diagram[3] is a $j+1$ sided polygon. At one of its vertices a pseudo-vector particle with momentum q enters with coupling $\gamma_k \gamma_{n+1}$. Starting at this point we enumerate the remaining vertices in clockwise direction from 1 to j; each one with a vec-

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tor coupling γ_{μ_ℓ} $(\ell=1,\ldots,j)$. From each of this j vertices a particle emerges with momentum p_ℓ and the fermionic loop carries an integration variable r.

As the anomaly is mass-independent we take for simplicity m=0, then the contribution from the above described diagram and from all others obtained by permutation of the photon four momenta and polarization indices is, in N-dimension (N>n):

$$(1) \qquad R_{k\mu_1\ldots\mu_j} = \frac{j!}{(2\pi)^{2j}}\int d^N r \, \frac{Tr\,\gamma_k\gamma_{n+1}\not{r}\,\gamma_{\mu_1}(\not{r}+\not{p}_1)\gamma_{\mu_2}\ldots\gamma_{\mu_j}(\not{r}+\not{p}_1+\ldots+\not{p}_j)}{r^2(r+p_1)^2\ldots(r+p_1+\ldots+p_j)^2}$$

We use the following prescription for γ_{n+1} (the analog of γ_5, in n-dimension):

$$(2) \qquad \gamma_{n+1} = C\,\tau^{\mu_1\ldots\mu_n}\,\gamma_{\mu_1}\ldots\gamma_{\mu_n}$$

where $\mu_i=0,\ldots,N-1$; C is a normalization factor and $\tau^{\mu_1\ldots\mu_n}$ is a totally antisymmetric tensor of n-th rank which is equal to the totally antisymmetric Levi-Civita tensor when N=n.

Taking the divergence of (1) and noting that $q=\sum_i p_i$, we obtain:

$$(3) \qquad q^K R_{k\mu_1\ldots\mu_j} \equiv A_{\mu_1\ldots\mu_j} = \frac{j!}{(2\pi)^{2j}}\int \frac{d^N r}{D}\; Tr\{(\not{p}_1+\ldots+\not{p}_j)\gamma_{n+1}[\,]\}$$

where

$$[\;] \equiv \not{r}\gamma_{\mu_1}(\not{r}+\not{p}_1)\,\gamma_{\mu_2}\ldots\gamma_{\mu_j}(\not{r}+\not{p}_1+\ldots+\not{p}_j)$$

and

$$D = r^2(r+p_1)^2 \ldots (r+p_1+\ldots+p_j)^2$$

The numerator of the integrand in (3) can be written as:

$$(4) \qquad (\not{r}+\not{p}_1+\ldots+\not{p}_j)\gamma_{n+1}[\,] - \not{r}\gamma_{n+1}[\,] = -\{\gamma_{n+1},\not{r}\}[\,] + \gamma_{n+1}\not{r}[\,] + \\ + (\not{r}+\not{p}_1+\ldots+\not{p}_j)\gamma_{n+1}[\,]$$

The contribution of the last term in (4) to the integral (3) is proportional to:

$$(5) \qquad \int d^N r \, \frac{\mathrm{Tr}\, \gamma_{n+1}\not{r}\gamma_{\mu_1}(\not{r}+\not{p}_1)\gamma_{\mu_2}\ldots\gamma_{\mu_{j-1}}(\not{r}+\not{p}_1+\ldots+\not{p}_{j-1})\gamma_{\mu_j}}{r^2(r+p_1)^2\ldots(r+p_1+\ldots+p_{j-1})^2}$$

note that p_j have disappeared from (5). The fact that $\mathrm{Tr}\,\gamma_{n+1}\gamma_{a_1}\ldots\gamma_{a_n}$ is antisymmetric in a_i indices implies that the integral can only depend linearly on each $p_i (i=1,\ldots,j-1)$, and as the linear term in r disappears after r-integration we see that (5) should be zero. In the same way the second term in the right hand side of (4) can be shown to give null contribution to (3). Then we are left with

$$(6) \qquad A_{\mu_1\ldots\mu_j} = -\frac{j!}{(2\pi)^{2j}} \int \frac{d^N r}{D} \, \mathrm{Tr}\{\gamma_{n+1},\not{r}\}[\,]$$

If we use (see appendix) that $\mathrm{Tr}\{\gamma_{n+1},\gamma_\alpha\}\gamma_{a_1}\ldots\gamma_{a_{n+1}}$ is totally antisymmetric in a_i indices we obtain under a convenient change of variables:

$$A_{\mu_1 \cdots \mu_j} = \frac{-(j!)^2}{(2\pi)^{2j}} p_1^{\alpha_1} \cdots p_j^{\alpha_j} \mathrm{Tr}\{\gamma_{n+1}, \gamma_\beta\} \gamma_{\alpha_0} \gamma_{\mu_1} \gamma_{\alpha_1} \cdots \gamma_{\mu_j} \gamma_{\alpha_j} \int_0^1 dx_1 \ldots dx_{j+1} \delta\Big(\sum_{i=1}^{j+1} x_i - 1\Big)$$

$$\int d^N r \; \frac{r^\beta r^{\alpha_0}}{(r^2 - a)^{j+1}} \tag{7}$$

In order to perform the last r-integral in (7) we pass to Euclidean metric and replace

$$r^\beta r^{\alpha_0} \rightarrow -\frac{r_E^2}{N} \delta^{\beta\alpha_0}$$

Using now

$$\int d^N r \; \frac{r^2}{(r^2+a)^{j+1}} = \frac{\pi^{N/2} N \, \Gamma(\frac{n-N}{2})}{2 \, j! \; a^{\frac{n-N}{2}}}$$

we obtain

$$A_{\mu_1 \cdots \mu_j} = \frac{(-1)^{j+1} i \, \pi^{N/2} j!}{2(2\pi)^{2j}} \Gamma\Big(\frac{n-N}{2}\Big) p_1^{\alpha_1} \cdots p_j^{\alpha_j} \mathrm{Tr}\{\gamma_{n+1}, \gamma_\beta\} \gamma^\beta \gamma_{\mu_1} \gamma_{\alpha_1} \cdots \gamma_{\mu_j} \gamma_{\alpha_j}$$

$$\int_0^1 dx_1 \ldots dx_{j+1} a^{\frac{N-n}{2}} \delta\Big(\sum_{i=1}^{j+1} x_i - 1\Big) \tag{8}$$

From (2) and straightforward γ-algebra, it is easy to prove that

$$\gamma_\alpha \gamma_{n+1} \gamma^\alpha = (N-2n) \gamma_{n+1}$$

so that

$$\{\gamma_{n+1}, \gamma_\alpha\} \gamma^\alpha = 2(N-n) \gamma_{n+1}$$

Then

$$A_{\mu_1\cdots\mu_j} = \frac{(-1)^j\, 2i\, \pi^{N/2}\, j!}{(2\pi)^{2j}} \left(\frac{n-N}{2}\right) \Gamma\left(\frac{n-N}{2}\right) p_1^{\alpha_1}\cdots p_j^{\alpha_j}\, \mathrm{Tr}\, \gamma_{n+1}\gamma_{\mu_1}\gamma_{\alpha_1}\cdots\gamma_{\mu_j}\gamma_{\alpha_j} \int_0^1 dx_1\cdots dx_{j+1}\, \delta\left(\sum_{i=1}^{j+1} x_i - 1\right) a^{\frac{N-n}{2}} \tag{9}$$

As the expression (9) is now finite we can take the limit $N \to n$. Then using

$$\mathrm{Tr}\, \gamma_{n+1}\gamma_{a_1}\cdots\gamma_{a_n} = i\, 2^j\, \epsilon_{a_1\cdots a_n} \tag{10}$$

where we have chosen the constant c in order to obtain $\gamma_{n+1}^2 = 1$, we obtain for the anomaly

$$A_{\mu_1\cdots\mu_j} = \frac{(-1)^{j+1}}{2^{j-1}\, \pi^j}\, p_1^{\alpha_1}\cdots p_j^{\alpha_j}\, \epsilon_{\mu_1\alpha_1\cdots\mu_j\alpha_j} \tag{11}$$

which gives the usual value for n=4 and equal values to those obtained in reference [1] for n=6,8,10. We finally point out that for this deduction (see reference [7]) it is essential to note that γ_{n+1} does not anticommute with γ_α when the space-time dimension is N>n.

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APPENDIX

Antisymmetry of $Tr\{\gamma_{n+1},\gamma_\alpha\}\gamma_{a_1}\cdots\gamma_{a_{n+1}}$

Let us take the following trace:

$$\text{(A-1)} \qquad Tr\{\gamma_{n+1},\gamma_\alpha\}\gamma_{a_1}\cdots\gamma_{a_{n+1}} = Tr\,\gamma_{n+1}\gamma_\alpha\gamma_{a_1}\cdots\gamma_{a_{n+1}} + Tr\,\gamma_{n+1}\gamma_{a_1}\cdots\gamma_{a_{n+1}}\gamma_\alpha$$

If in the first term of the right hand side of (A.1) we pass the γ_α matrix from its position through the γ_{a_i}, as we have an odd number of γ_{a_i} matrices we get zero for the terms with γ_α matrices and we are left with:

$$\text{(A.2)} \qquad Tr\{\gamma_{n+1},\gamma_\alpha\}\gamma_{a_1}\cdots\gamma_{a_{n+1}} = 2Cg_{\alpha a_1}\tau_{a_2\ldots a_{n+1}} - 2Cg_{\alpha a_2}\tau_{a_1 a_3\ldots a_{n+1}} + \ldots + 2Cg_{\alpha a_{n+1}}\tau_{a_1\ldots a_n}$$

Where we have used that the product of γ_{n+1} times a number n of γ_{a_i} matrices has the trace proportional to the totally antisymmetric tensor τ. Then we see from (A.2) that $Tr\{\gamma_{n+1},\gamma_\alpha\}\gamma_{a_1}\cdots\gamma_{a_{n+1}}$ is totally antisymmetric in a_i indices.

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Adendum

After this work has been completed we received two preprints: "Finite-Mode Regularization of the fermion functional integral" -A. Andrianov and L. Borona IFPD 16/83.
"ABJ anomalies in any even dimensions" - L. Borona and P. Pasti IFPD 20/83 in which in the first the authors compute the axial anomaly (to one-loop order) in any space-time dimension with the finite-mode regularization.