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ANISOTROPIC BIANCHI II COSMOLOGICAL MODELS WITH MATTER AND ELECTROMAGNETIC FIELDS

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ABSTRACT

A class of solutions of Einstein-Maxwell equations is presented, which corresponds to anisotropic Bianchi II spatially homogeneous cosmological models with perfect fluid and electromagnetic field. A particular model is examined and shown to be unstable, for perturbations of the electromagnetic field strength parameter about a particular value. This value defines a limiar unstable case in which the ratio $\mathcal{E}$, of the fluid density to the e.m. energy density is monotonically increasing with a minimum finite value at the singularity. Beyond this limiar, the model has a matter dominated singularity ($\mathcal{E} \to \infty$), and a characteristic stage appears where $\mathcal{E}$ has a minimum, at a finite time from the singularity. For large times, the models tend to an exact solution for zero electromagnetic field and fluid with $p = \frac{1}{5}\rho$. Some cosmological features of the models are calculated, as the effect of anisotropy on matter density and expansion time scale factors, as compared to the corresponding Friedmann model.

Anisotropic spatially homogeneous cosmological models are of great interest for theoretical cosmology. Besides their great generality among spatially homogeneous models, they are believed to provide a more appropriate description of early stages of our Universe than Friedmann models. In fact the existence of horizons in Friedmann models makes difficult to understand (in these models) the high degree of isotropy of the Universe as observed presently, unless very special initial conditions are assumed[1]. The Universe could have been primarily anisotropic and some processes in its early stages of evolution have rapidly isotropized it. Among these processes, the mechanism of particle creation has been extensively investigated, and it proves to be significant only for anisotropic metrics[2]. Furthermore it has been suggested that the presence of electromagnetic fields could alter the rate of creation of particles in anisotropic models[3]. Finally, anisotropy can also have a significant influence in the expansion time scales of the early stages of the evolution of the Universe, and thus critically affect physical parameters of these early stages (cf. references (1) and (4) for the problem of helium abundance in the Universe).

In this context we present a class of spatially homogeneous cosmological solutions of Einstein-Maxwell equations for a perfect fluid and electromagnetic field, with a Bianchi type II group[5] acting transitively on the sections of homogeneity of the model. The sections are then endowed with the structure of the orbits of the Lie group generated by the Bianchi II Lie algebra

$$[Y_1, Y_2] = 0 \quad , \quad [Y_2, Y_3] = Y_1 \qquad (1)$$
$$[Y_1, Y_3] = 0$$

Taking (r, θ, φ) as local coordinates on these three dimensional manifolds, an invariant basis – dual to (1) – can be given by the 1-forms

$$\omega^1 = dr + 4m^2(\theta)\, d\varphi$$
$$\omega^2 = \frac{1}{\sin\theta}\, d\theta \tag{2}$$
$$\omega^3 = d\varphi$$

where $m(\theta)$ satisfies the equation

$$4 \sin\theta \; m \frac{dm}{d\theta} = \lambda_1 \quad , \tag{3}$$

λ_1 a constant. The line element is assumed to have the form

$$ds^2 = dt^2 - A^2(t)(\omega^1)^2 - B^2(t)\left((\omega^2)^2 + (\omega^3)^2\right) \tag{4}$$

We also express (4) in the form(*)

$$ds^2 = \eta_{AB}\, \theta^A \theta^B \tag{5}$$

where

$$\theta^0 = dt \;, \quad \theta^1 = A\omega^1 \;, \quad \theta^2 = B\omega^2 \;, \quad \theta^3 = B\omega^3 \tag{6}$$

and we use (6) and (2) to define the tetrad matrix $e^{(A)}_{\alpha}$ by $\theta^A = e^{(A)}_{\alpha} dx^{\alpha}$. Ricci rotation coefficients are defined

$$\gamma_{ABC} = - e^{\alpha}_{(A)\|\beta}\; e_{\alpha(B)}\; e^{\beta}_{(C)} \tag{7}$$

(*) Capital Latin indices are tetrad indices and run form 0 to 3; they are raised and lowered with Minkowski matrices η^{AB}, $\eta_{AB} = diag(+1,-1,-1,-1)$. Greek indices are coordinate indices and run from 0 to 3; they are raised and lowered with the metric $g^{\alpha\beta}$, $g_{\alpha\beta}$.

and for (6) and (2) they have the non-null components

$$\begin{aligned}
\gamma^1{}_{01} &= \frac{\dot{A}}{A} & \gamma^1{}_{23} &= \frac{A}{B^2}\lambda_1 \\
\gamma^2{}_{02} &= \frac{\dot{B}}{B} & \gamma^1{}_{32} &= -\frac{A}{B^2}\lambda_1 \\
\gamma^3{}_{03} &= \frac{\dot{B}}{B} & \gamma^3{}_{21} &= \frac{A}{B^2}\lambda_1
\end{aligned} \tag{8}$$

where a dot means t-derivative.

In our models we have the presence of an electromagnetic field. Since we assume that it is not a pure test field but also acts as source of the curvature, it must then be compatible with the symmetries of the space-time. From (4) it is seen that we have a preferred direction in our Universe, determined by ω^1 . We are then led to take both $\vec{E}$ and $\vec{H}$ along this direction. Spatial homogeneity implies that $\vec{E}$ and $\vec{H}$ must be functions of t only. With these restrictions, the electromagnetic tensor F_{AB} has the only components

$$\begin{aligned}
F_{01} &= -F_{10} = E(t) \\
F_{23} &= -F_{32} = H(t)
\end{aligned} \tag{9}$$

in the local inertial frame determined by (6) and (2). In this frame, Maxwell equations are written(*)

$$e^{\rho}_{[(P)} F_{QR]|\rho} + 2 F_{AB} \gamma^{A}{}_{[PQ} \delta^{B}_{R]} = 0 \tag{10a}$$

$$e^{\rho}_{(P)} F^{P}{}_{D|\rho} - F_{AD} \gamma^{AP}{}_{P} - F_{PB} \gamma_{D}{}^{PB} = 0 \tag{10b}$$

(*) Square brackets denote anti-symmetrization and a bar denotes partial derivative.

Using (8), (9) and (6), equations (10) reduce respectively to

$$
\begin{aligned}
&(HB^2)^{\cdot} - 2EA\lambda_1 = 0 \\
&(EB^2)^{\cdot} + 2HA\lambda_1 = 0
\end{aligned}
\tag{11}
$$

Introducing the new variables $\mathcal{E} = EB^2$, $\mathcal{H} = HB^2$ and $d\tilde{t} = \frac{A}{B^2}\,dt$ equations (11) can be rewritten

$$
\begin{aligned}
\frac{d\mathcal{H}}{d\tilde{t}} &= 2\,\mathcal{E}\,\lambda_1 \\
\frac{d\mathcal{E}}{d\tilde{t}} &= -2\,\mathcal{H}\,\lambda_1
\end{aligned}
\tag{12}
$$

with solutions

$$
\begin{aligned}
\mathcal{E} &= \Sigma \cos 2\lambda_1 \tilde{t} \\
\mathcal{H} &= \Sigma \sin 2\lambda_1 \tilde{t}
\end{aligned}
\tag{13}
$$

where Σ is a constant which we call electromagnetic field strength parameter. For (9) and (13), the energy-momentum tensor of the electromagnetic field

$$
T_{AB} = -F_{AC}\,F_B{}^C + \frac{1}{4}\,\eta_{AB}\,F_{CD}\,F^{CD} \tag{14}
$$

has non-null components

$$
T_{00} = -T_{11} = T_{22} = T_{33} = \frac{\Sigma^2}{2B^4} \tag{15}
$$

The matter content of the models is a perfect fluid. In the local inertial frame considered, an observer comoving with the fluid is assumed to have the four velocity

$$
u^A = \delta^A_0 \tag{16}
$$

It corresponds to a matter velocity field $e^{\mu}_{(0)}$ and determines a congruence of time-like geodesics along which matter propagates. Denoting respectively by ρ and p the density of matter-energy and the pressure of the fluid as measured locally by the comoving observer (16), the energy-momentum tensor of the fluid can be expressed

$$T_{AB} = (\rho + p)\, u_A u_B - p\, \eta_{AB} \tag{17}$$

The total energy-momentum tensor for the model is the sum of the energy-momentum tensors of the fluid (17) and of the electro magnetic field (14).

The non-null components of the Ricci tensor for the metric (4) are calculated

$$\begin{aligned}
R_{00} &= -\frac{\ddot{A}}{A} - 2\frac{\ddot{B}}{B} \\
R_{11} &= \frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{B}}{AB} + 2\frac{A^2}{B^4}\lambda_1^2 \\
R_{22} = R_{33} &= \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2 - 2\frac{A^2}{B^4}\lambda_1^2
\end{aligned} \tag{18}$$

Using (14), (17) and (18), Einstein field equations

$$R_{AB} - \frac{1}{2}\eta_{AB} R = \kappa\, T_{AB}\,(total) \tag{19}$$

reduce to the set of three independent equations

$$\begin{aligned}
R_{00} &= \frac{k}{2}(\rho + 3p) + \frac{\kappa \Sigma^2}{2B^4} \\
R_{11} &= \frac{k}{2}(\rho - p) - \frac{\kappa \Sigma^2}{2B^4} \\
R_{11} &+ \frac{\kappa \Sigma^2}{B^4} - R_{22} = 0
\end{aligned} \tag{20a,b,c}$$

We take equations (20a,b) as determining the functional form of ρ and p. Equation (20c) is then one differential equation for the two metric functions A and B. As usual, this arbitrariness can be eliminated by assuming an equation of state $p = p(\rho)$ for the cosmic fluid. We here have instead assumed a relation between A and B, and from ρ and p given by (20a,b) we have immediately the equation of state $p = p(\rho)$ (*). We take

$$A = A_0 B^{1/2} \tag{21}$$

Equation (20c) reduces then to

$$\frac{\ddot{B}}{B} + \frac{3}{2}\left(\frac{\dot{B}}{B}\right)^2 - \frac{\gamma A_0^2 \lambda_1^2}{B^3} - \frac{2k\Sigma^2}{B^4} = 0 \tag{22}$$

In terms of a new coordinate defined by

$$d\tilde{t} = B^{-3/2}\, dt \tag{23}$$

equation (22) assumes the form

$$B'' - \gamma A_0^2 \lambda_1^2 B - 2k\Sigma^2 = 0 \tag{24}$$

where a prime denotes derivative with respect to $\tilde{t}$. Two independent solutions of (24) are given by

(*) for the spatially homogeneous models we are considering, the fluid described by (ρ, p, u^A) has constant entropy.

$$B = -\frac{\kappa\Sigma^2}{4A_0^2\lambda_1^2} + \begin{pmatrix} \cosh\sqrt{8}A_0\lambda_1\tilde{t} \\ \exp.\sqrt{8}A_0\lambda_1\tilde{t} \end{pmatrix} \tag{25}$$

in terms of $\tilde{t}(t)$. In this parametrization, we must have $B(\tilde{t}) > 0$ so that the coordinate t assumes only real values (cf.(23)) and the signature of the metric remains unaltered (cf.(21)). The density ρ and pressure p have the following expressions

$$2\kappa\rho = \left(4\left(\frac{B'}{B}\right)^2 - 2A_0^2\lambda_1^2 - \kappa\Sigma^2 B^{-1}\right) B^{-3} \tag{26}$$

$$2\kappa p = \left(4\left(\frac{B'}{B}\right)^2 - 26A_0^2\lambda_1^2 - 7\kappa\Sigma^2 B^{-1}\right) B^{-3} \tag{27}$$

Let us consider first the solution

$$B = \cosh\sqrt{8}A_0\lambda_1\tilde{t} - \frac{\kappa\Sigma^2}{4A_0^2\lambda_1^2} \tag{28}$$

From (26) and (27) we note that for $\Sigma = 0$ this solution is non-singular but the matter density ρ is negative for an interval of $\tilde{t}$ around $\tilde{t} = 0$. In order to have a physical matter density, we must consider for this solution the presence of the electromagnetic field, the strenght parameter of the field being such that

$$\frac{\kappa\Sigma^2}{4A_0^2\lambda_1^2} \geqslant 1 \tag{29}$$

in order to eliminate the possibility of negative energy densities. The present parametrization of solutions with $\tilde{t}(t)$ (cf.(23)) will be valid only for

$$|\tilde{t}| \geqslant \frac{1}{\sqrt{8}A_0\lambda_1} \operatorname{arc\,cosh} \frac{\kappa\Sigma^2}{4A_0^2\lambda_1^2} \tag{30}$$

In this region ρ is always positive, with a singularity at $|\tilde{t}_0| = \frac{1}{\sqrt{8}A_0\lambda_1} \operatorname{arc\,cosh} \kappa\Sigma^2/4A_0^2\lambda_1^2$. In what follows we restrict the range of $\tilde{t}$ to positive values only, starting from the singularity $\tilde{t}_0$.

The behaviour of the models differs greatly whether the equality or inequality in (29) holds. To see this we consider the ratio of the energy density ρ of the fluid to the electromagnetic energy density, ($\rho_{em} = T_{\mu\nu}(em)\, e^{\mu}_{(0)} e^{\nu}_{(0)}$),

$$\frac{\rho}{\rho_{em}} = \varepsilon \tag{31}$$

where ε is a number whose value depends on $\tilde{t}$. Using (26) and (15) we can express (31) as

$$\varepsilon = \frac{1}{\kappa\Sigma^2}\left(32A_0^2\lambda_1^2 \frac{\sinh^2\sqrt{8}A_0\lambda_1\tilde{t}}{B} - 2A_0^2\lambda_1^2 B - \kappa\Sigma^2\right) \tag{32}$$

For $\kappa\Sigma^2/4A_0^2\lambda_1^2 = 1$ and using (28), we obtain from (32)

$$\varepsilon = \frac{15}{2}(\alpha+1) \tag{33}$$

where $\alpha = \cosh\sqrt{8}A_0\lambda_1\tilde{t}$, $\alpha \geqslant 1$. From (33) we can see that the maximum contribution of the electromagnetic energy density to the total energy density (matter and e.m. field) of the model occurs at the singularity ($\alpha = 1$) and it represents about 6 per cent of the total energy density.at that stage. For larger times, the contribution of the em. energy density becomes more and more negligible.

For the inequality in (29), we consider the typical value $\kappa\Sigma^2/4A_0^2\lambda_1^2 = 2$, which yields from (32),

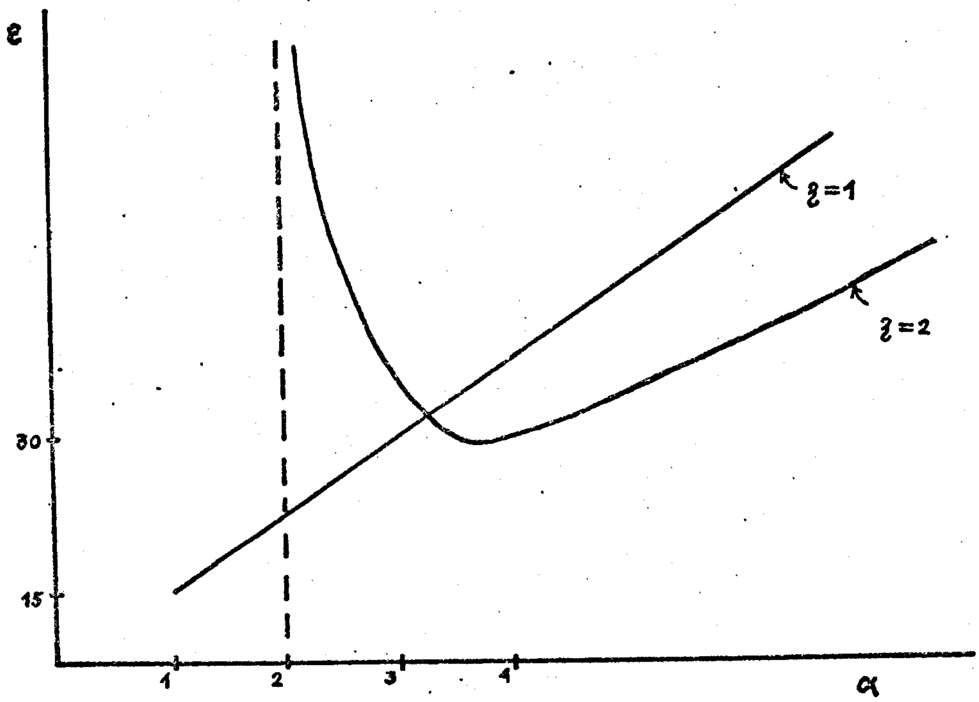
$$\varepsilon = \frac{15\alpha^2 - 12}{4\alpha - 8} \tag{34}$$

where $\alpha = \cosh\sqrt{\varepsilon}\, A_0 \lambda_1 \tilde{t}$, $\alpha \geqslant 2$. From (34) we can distinguish three characteristic stages in the evolution of this model. Sufficiently near the singularity (α = 2) the contribution of the electromagnetic energy density to the total energy density is negligible ($\mathcal{E} \to \infty$) and the singularity is matter dominated. The second characteristic stage occurs for later values of $\tilde{t}$ and it corresponds to a configuration where the electromagnetic energy density has its maximum relative to the total energy density (matter and e.m. field) of the model. $\mathcal{E}(\tilde{t})$ has a minimum at this stage. For (34), $\mathcal{E}(\alpha)$ is calculated to have a minimum at $\alpha \simeq$ 3.8 with $\alpha(3.8) \simeq$ 28.5. Although at $\alpha \simeq$ 3.8 the ratio ρ_{em}/ρ has a maximum, the electromagnetic energy density corresponds to only 3 percent of the total (matter and e.m.) energy density of the model at that instant. The graph for both cases is illustrative.

(figure)

The model is stable for perturbations of the strength parameter Σ , which maintain the inequality in (29), but it is highly unstable for perturbations of Σ around the value $k\Sigma^2/4A_0^2\lambda_1^2 = 1$.

We remark that for $k\Sigma^2/4A_0^2\lambda_1^2 = 1$, sufficiently near the singularity the equation of state of the fluid can be approximated by $p = \frac{3}{5}\rho$. For large $\tilde{t}$, the contribution of the electromagnetic field can be neglected and the present solutions tend to the exact solution for zero electromagnetic field, and fluid



$\alpha = \cosh \sqrt{8}\, A_0 \lambda_1 \tilde{t}$

$\zeta = \dfrac{k \Sigma^2}{4 A_0^2 \lambda_1^2}$

with equation of state $p = \frac{1}{5} \rho$, which we discuss in the following.

For zero electromagnetic field $\Sigma = 0$; we consider the solution (25)

$$B = \exp \sqrt{8}\, A_0 \lambda_1 \tilde{t} \tag{35}$$

which gives for (26) and (27) ,

$$\begin{aligned} 2k\rho &= 30 A_0^2 \lambda_1^2 B^{-3} \\ p &= \frac{1}{5} \rho \end{aligned} \tag{36}$$

We now discuss other cosmological features of these anisotropic models. The fundamental observers are determined by the fluid velocity field (cf. (16))

$$u^\mu = e^\mu_{(A)} u^A = e^\mu_{(0)} \tag{37}$$

and the kinematical quantities(6) discussed here are associated to this velocity field. (37) is geodetic and irrotational since $\gamma_{0A0} = 0$ and $\gamma_{0[AB]} = 0$ (A,B = 1,2,3) respectively. The models expand anisotropically. In fact, from (4) we see that there is a preferred direction of expansion determined by ω^1 and the expansion of (37) along this direction is measured by

$$\theta_{(1)} = - \gamma_0{}^1{}_1 = \frac{\dot{A}}{A}$$

The total averaged (over angles of the observational sphere) expansion of the congruence is given by

$$\theta = u^\alpha{}_{\|\alpha} = \frac{5}{2} \frac{\dot{B}}{B} \tag{38}$$

We than see that along ω^1 the expansion is only 1/5 of the total expansion (38), contrary to an isotropic model in which the expansion along any direction is 1/3 of the total expansion.

The anisotropy of the model can be measured through the distortion of the congruence of comoving observers (37),

$$\sigma_{AB} = \frac{1}{6}\,\frac{\dot{B}}{B}\ \mathrm{diag}\,(0, 2, -1, -1) \tag{39}$$

From (39) we calculate $\sigma^2 = \frac{1}{2}\sigma_{AB}\sigma^{AB} = \frac{1}{12}\left(\frac{\dot{B}}{B}\right)^2$ and we have the result

$$\sigma = \frac{\sqrt{3}}{15}\ \theta \tag{40}$$

As well known from Raychaudhuri's equation[7], the presence of anisotropy can have an important effect on the matter density, and in changing the expansion time scales of the models, as compared to the corresponding isotropic model. For illustration we compare solution (35), (36) to the Friedmann model

$$ds^2 = dt^2 - A^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{41}$$

for fluid with $p = \frac{1}{5}\rho$. In what follows a subscript a or f denotes respectively quantities related to the anisotropic or isotropic model considered. By a convenient choice of integration constants, we can express (35) in terms of the coordinate t (cf.(23)) as

$$B_a = \frac{3\sqrt{8}\,A_0\lambda_1}{2}\quad t^{2/3} \tag{42}$$

Einstein field equations for (41) yield

$$A_f = \left(\frac{t}{t_0}\right)^{5/9} \tag{43}$$

where to is a constant of integration which we use to normalize the volume of the model to unity at a given time. Expressing

$\sigma = \varepsilon\,\theta$, we have for $p = \frac{1}{5}\,\rho$ and the above time-coordinates

$$\rho = \frac{15}{k}\left(\frac{20}{27} - \frac{50}{9}\,\varepsilon^2\right)\, t^{-2} \tag{44}$$

where $\varepsilon = \sqrt{3}/15$, 0 , respectively for the anisotropic or isotropic cases. Concerning time scales, we define the speed up factor s = t_a/t_f (which can be calculated by equating the volume of both models, and has the constant value $s = \frac{\sqrt{2}}{6}\left(\lambda_1 A_0^{8/5} t_0\right)^{-1}$) and from (44) we have

$$\frac{\rho_a}{\rho_f} = 0.9\; s^{-2} \tag{45}$$

Conclusions

From the anisotropic cosmological solutions of Einstein--Maxwell equations presented here, we have examined a particular model which shows some interesting properties, concerning its stability relative to the ratio $k\Sigma^2/4A_0^2\lambda_1^2$, where Σ is the strength parameter of the electromagnetic field ($A_0\lambda_1$ has sometimes been regarded as the gravitational magnetic charge of the space--time[8]). For $k\Sigma^2/4A_0^2\lambda_1^2 < 1$, we have a non-singular solution but negative energy densities appear. For $k\Sigma^2/4A_0^2\lambda_1^2 \geq 1$, the solutions present a singularity at $\tilde{t}_0 = \frac{1}{\sqrt{8}A_0\lambda_1}\,\mathrm{arc\,cosh}\; k\Sigma^2/4A_0^2\lambda_1^2$ and the energy densities are always positive. Nevertheless the behaviour of the solution is rather different, whether the equality or the inequality holds. In the first case the maximum contribution of the electromagnetic energy density to the total energy density (matter and e.m.) of the model occurs at the singularity and it represents about 6 percent of the total energy density (fluid and e.m.) at this stage. Its importance for later times becomes negligible since the ratio $\mathcal{E}$ of the fluid density to the electromagnetic energy density is monotonically increasing, with a minimum finite value at the singularity. But this case is an unstable configuration. In fact, for values of $k\Sigma^2/4A_0^2\lambda_1^2 > 1$ we have a drastic change of the above picture. The singularity is matter dominated since the ratio $\mathcal{E}$ goes to infinity. A characteristic stage appears at a finite time from the singularity where the ratio $\mathcal{E}$ presents a minimum, that is, the electromagnetic energy density has a maximum relative to the fluid energy density. It is at this stage when some processes, depending on the presence of electromagnetic fields, should most probably occur. For large times all models have as limit the exact anisotropic Bianchi II solution, for zero electromagnetic field and fluid with equation of state $p=\frac{1}{5}\rho$. In the anisotropic model with zero electromagnetic field and fluid with $p=\frac{1}{5}\rho$, the effect of anisotropy is to reduce the value of the matter density of the model, relative to the corresponding Friedmann model.

We finally remark that although matter is electrically neutral, our models have the presence of a non-zero electric field which for high values of the matter density (at early times, for instance) could give rise to conduction currents in a magnetohydrodynamic regime(9). In this context our models are a first approximation of more general cosmologies in which a conduction current would appear in the RHS of Maxwell equation (10b). We will discuss this in a future paper.

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$$H_{AB} = A_0 \lambda_1 \left(\frac{\dot{B}}{2B} \right) \mathrm{diag}\,(0, 2, -1, -1).$$

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