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Exceptional Structures in Mathematics and Physics and the Role of the Octonions*

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Abstract

There is a growing interest in the logical possibility that exceptional mathematical structures (exceptional Lie and superLie algebras, the exceptional Jordan algebra, etc.) could be linked to an ultimate "exceptional" formulation for a Theory Of Everything (TOE). The maximal division algebra of the octonions can be held as the mathematical responsible for the existence of the exceptional structures mentioned above. In this context it is quite motivating to systematically investigate the properties of octonionic spinors and the octonionic realizations of supersymmetry. In particular the M-algebra can be consistently defined for two structures only, a real structure, leading to the standard M-algebra, and an octonionic structure. The octonionic version of the M-algebra admits striking properties induced by octonionic p-forms identities.

Key-words: Exceptional Lie algebras; Octonions; M-theory.

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1 Introduction.

The search of an ultimate "Theory Of Everything" (TOE) corresponds to a very ambitious and challenging program. At present the conjectured M-Theory (whose dynamics is yet to unravel), a non-perturbative theory underlying the web of dualities among the five consistent superstring theories and the maximal eleven-dimensional supergravity, is regarded by perhaps the majority of physicists as the most promising candidate for a TOE [1].

While a TOE can be minimally defined as a consistent theory providing the unified dynamics of the known interactions, including the quantum gravity, a more ambitious viewpoint could be advocated, namely that the TOE is *unique*. Features like the dimensionality of the space-time should not be externally imposed to reproduce the known results, rather should arise naturally due to the consistency conditions of the theory. It is quite pleasing that the superstring/M theory program, at least at the Planck scale, satisfies this requirement.

In the last two or three decades several physicists and groups have investigated the possibility that, since the universe is exceptional, it should be better grasped by an "exceptional" ultimate TOE, see [2] and references therein. The word "exceptional" is here used technically, meaning an exceptional mathematical structure, such as the finite and sporadic Monster group, the exceptional Lie and superLie algebras or the exceptional Jordan algebra (which will be briefly discussed in the following).

It would be highly desirable that miraculous features such as the hypercharge balance of the quarks-leptons in each of the three known families, leading to the cancellation of the chiral anomaly and the perturbative consistency of the electroweak standard model, would be "explained" by an underlying exceptional theory [3].

This research program, despite being highly conjectural, has a strong merit. Focusing on exceptional mathematical structures (in contrast with the standard "boring" ones) helps clarifying their role and discovering their mutual, sometimes unsuspected, relations. In the following, several examples will be given.

For what concerns the physical side, various motivations and scattered pieces of evidence for the proposed program can be given. We will list here three of them:

i) At the level of $4D$ GUT, [2, 3] the Georgi-Glashow $SU(5)$ model can be consistently embedded in the group chain $SU(5) \subset SO(10) \subset E_6$, where the last group corresponds to an exceptional group (the "exceptional" viewpoint can explain why $SU(5)$ appears instead of $SU(N)$ for an arbitrary value of N),

ii) The Heterotic String is consistently formulated for the $E_8 \times E_8$ group, which contains as subgroups the above chain of GUT groups (please notice that E_8 is not only exceptional, it is the *maximal* exceptional group),

iii) finally, the phenomenological requirement of dealing with a 4-dimensional $N = 1$ supersymmetric chiral theory implies a Kaluza-Klein compactification from the eleven-dimensional M-theory, based on a seven-dimensional singular manifold with G_2 holonomy [4]. Please notice that G_2 is, once more, an exceptional group. This M-theory/$11D$ SUGRA prescription should be compared with the Calabi-Yau six-dimensional compactifications from perturbative strings admitting $SU(3)$ (a standard group) holonomy.

It is worth to recall very basic mathematical properties of some of the exceptional structures mentioned so far. For what concerns simple Lie algebras, besides the four classical series A_n, B_n, C_n, D_n associated to the special unitary, special orthogonal and symplectic groups, five exceptional Lie algebras appear in the Cartan-Killing classification: G_2, F_4, E_6, E_7 and E_8, the respective numbers denoting their fixed rank. We already encountered three of them, E_6, E_8 and G_2 in the discussion above.The maximal exceptional Lie algebra, E_8, is in some sense the most important of all. All lower-rank Lie algebras, both exceptional and not, can be recovered from E_8 through the coupled procedure of taking subalgebras and eventually folding them to produce the non simply-laced simple Lie algebras [5].

If we extend our considerations to Lie superalgebras, just two exceptional Lie superalgebras, admitting no free parameter, exist. They are denoted as $G(3)$ and $F(4)$ [6].

It is quite remarkable that the existence of all these exceptional algebras and superalgebras is mathematically motivated by their construction through the maximal division algebra, the division algebra of the octonions. The fixed rank of the exceptional Lie (super)algebras is a consequence of the non-associativity of the octonions. In contrast, the classical series of the A_n, B_n, C_n, D_n algebras are related to the associative division algebras of real, complex and quaternionic numbers.

Indeed, G_2 is the 14-component Lie algebra of the group of automorphisms of the octonions [7]. The remaining bosonic exceptional Lie algebras are recovered from the so-called Tits's magic square construction [8], associating a Lie algebra to any given pair of alternative division algebras and Jordan algebras (over a division algebra). By taking the Jordan algebra to be the exceptional Jordan algebra $J_3(\mathbf{O})$ of 3×3 octonionic-valued hermitian matrices, F_4, E_6, E_7 and E_8 are recovered if the alternative division algebra is respectively set equal to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$. A supersymmetric construction based on the octonions also exists for the superalgebras $G(3)$ and $F(4)$, see [9]. We can say that the octonions are at the very core of all these exceptional structures.

Incidentally, we have introduced the exceptional Jordan algebra $J_3(\mathbf{O})$, which can be easily proven admitting F_4 as the Lie algebra of its group of automorphisms. $J_3(\mathbf{O})$ plays a very peculiar and perhaps not yet fully understood role which goes back to the early days of quantum mechanics. The Jordan formulation of quantum mechanics [10], based on the Jordan product between observables, is equivalent to the standard formulation as far as associative structures are taken into account. It was realized by the fathers of quantum mechanics, Jordan himself, von Neumann and Wigner [11], that one and only one notable exception exists, $J_3(\mathbf{O})$, as algebra of observables. F_4 plays here the role of the unitary transformations (i.e. the dynamics), while the states are associated to the non-desarguian Moufang octonionic projective plane $\mathbf{OP}^2$ [12]. So far no physical system has yet been found to be described by this unique octonionic quantum mechanical system. The octonionic quantum mechanics is, at present, just a consistent mathematical curiosity. Interestingly, however, $J_3(\mathbf{O})$ has been recently proposed (by showing it possesses the correct properties) to define an exceptional matrix Chern-Simon theory [13] which could grasp, within the exceptional viewpoint discussed above, the features of a background-independent formulation for quantum gravity.

We are therefore naturally led from the "exceptional considerations" to investigate the division algebra of the octonions, see also [14]. There is another, partially independent, argument justifying the interest in the octonions. Since the octonions are the maximal division algebra (real, complex and quaternionic numbers can be obtained as subalgebras), they can be regarded on the same footing as, let's say, the eleven dimensional maximal supergravity. In a TOE viewpoint it makes sense, as already recalled, investigating the largest algebraic setting available, without imposing any unnecessary external constraint. It is true that the non-associativity of the octonions makes some issues problematics (it is e.g. unclear how to construct octonionic tensor products [15]). Similar issues cannot however be raised as strong objections towards a work which is necessarily still in progress.

It is worth mentioning that in the seventies octonions have been investigated as possible algebraic explanations for the absence of quarks and other colored states of the strong interactions [16]. Analogous ideas, from time to time, resurface again. The unobservability of the extra dimensions (with no need of introducing a Kaluza-Klein compactification or a brane-world scenario) by using an octonionic description, has been suggested as a possibility in [17].

Nowadays the arena where the use of octonions seems most promising concerns supersymmetry. This is of course based on the celebrated connection of supersymmetry with division algebras [18]. Octonionic supersymmetry is a combination of two remarkable ideas and beautiful mathematical structures which, so far, have not yet been detected on physical experiments.

This large introduction was meant to motivate investigating the octonionic supersymmetry [19]. The rest of this paper consists in a quick introduction to the octonionic supersymmetry. The most important result here discussed consists in the octonionic formulation of the M-algebra [20, 21], with its surprising properties derived from octonionic p-form identities [22] leading to the equivalence of the octonionic 5-brane sector ($M5$) with the octonionic 1 and 2-branes sectors ($M1$ and $M2$), symbolically $M5 \equiv M1 + M2$.

The recognition of the importance of the octonionic M-algebra derives from the fact that an 11-dimensional M-algebra can be consistently defined using two and only two structures, i.e. the real ($\mathbf{R}$) structure and the octonionic ($\mathbf{O}$) structure. While the real structure leads to the standard formulation of the M-algebra, the octonionic structure is potentially linked with the exceptional structures discussed above. If we combine together the two assumptions previously discussed, i.e. that

i) the TOE is succesfully described by the M-theory and its related M-algebra and

ii) the TOE is based on an exceptional mathematical structure,

then, the octonionic M-algebra naturally arises as the potential underlying algebra.

2 Octonionic spinors.

The seven imaginary octonions τ_i satisfy the algebraic relation

$$\tau_i \cdot \tau_j \quad = \quad -\delta_{ij} + C_{ijk}\tau_k, \qquad (1)$$

for $i, j, k = 1, \cdots, 7$, where C_{ijk} are the totally antisymmetric octonionic structure constants given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \qquad (2)$$

and vanishing otherwise. The product (1) is non-associative, but satisfies the weaker condition of alternativity [7].

Due to the antisymmetric property of C_{ijk}, the anticommutator between imaginary octonions satisfy

$$\tau_i \cdot \tau_j + \tau_j \cdot \tau_i \quad = \quad -2\delta_{ij}, \tag{3}$$

providing an octonionic realization of the Euclidean Clifford algebra $C(0,7)$ which is of course not equivalent w.r.t. the standard associative realization [19].

Non-associative octonionic realizations of Clifford algebras are available in higher-dimensional space-times as well. This can be easily proven by noticing that two iterative algorithms exist allowing the lifting of a D-dimensional space-time realization of a Clifford algebra into a $(D+2)$-dimensional one. Indeed, if γ_a denote a realization of the $C(p,q)$ Clifford algebra (with p space-like and q time-like directions), then

$$\begin{aligned} \Gamma_r \quad &\equiv \quad \begin{pmatrix} 0 & \gamma_a \\ \gamma_a & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) \quad &\mapsto \quad (p+1, q+1) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \Gamma_r \quad &\equiv \quad \begin{pmatrix} 0 & \gamma_a \\ -\gamma_a & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) \quad &\mapsto \quad (q+2, p), \end{aligned} \tag{5}$$

for $r = 1, \ldots, D+2$, produce a Clifford representation of signature $(p+1, q+1)$ and $(q+2, p)$ respectively.

It turns out in particular that the Clifford algebra associated with the eleven dimensional minkowskian space-time $(10,1)$ admits an octonionic realization in terms of 4×4 Gamma matrices with octonionic entries by applying twice the two algorithms above, e.g. by sending $(0,7) \rightarrow (9,0) \rightarrow (10,1)$ (please notice that each time one of the two algorithms above are applied the size of the new Gamma matrices is doubled).

The 32 real-component spinors entering the $11D$ supergravity or standard M-theory can therefore be replaced by 4 octonionic-component spinors (the total number of real components still being $32 = 4 \times 8$).

It should be noticed that in an octonionic realization the commutators $\Sigma_{rs} = [\Gamma_r, \Gamma_s]$ are no longer the generators of the generalized Lorentz group $SO(p,q)$. They correspond instead to the generators of the coset $SO(p,q)/G_2$, G_2 being, as already recalled, the 14-dimensional group of automorphisms of the octonions. The particular coset associated to the 7-dimensional Euclidean space described by imaginary octonions corresponds to the seven sphere S^7 [23].

3 Octonionic supersymmetry

We have seen in the previous section that one can give an octonionic description for both Clifford algebras and spinors. Since these are the basic ingredients entering supersymmetry, it is quite tempting to apply the previous construction to describe octonionic supersymmetry.

Indeed octonionic supersymmetry can be introduced both at the level of one dimensional octonionic supersymmetric quantum mechanics [19, 24] and at the level of higher-dimensional supersymmetry (even generalized supersymmetries, as it is the case for the octonionic M-algebra discussed in the next section).

For what concerns the $1D$ octonionic supersymmetric quantum mechanics the relation can be understod on the basis of the one-to-one correspondence between irreducible representations of $1D$ N-extended supersymmetries on one side and the special subclass of D-dimensional Clifford Gamma-matrices of Weyl type (namely the ones which are in block-antidiagonal form and can therefore be promoted to be "fermionic" matrices of a superalgebra). The correspondence is obtained by identifying N (the number of extended supersymmetries) with D (the dimensionality of the space-time), i.e. $N = D$ (see [25] for details). The octonionic counterpart of this construction has been discussed in [19, 24], while an application to an $N = 8$ octonionic supersymmetric dynamical system has been introduced in [26] (b.t.w. 8 is the minimal number of $1D$ extended supersymmetries which can be "octonionically" realized).

For what concerns the supersymmetry in higher dimension an interesting application regards the so-called generalized space-time supersymmetries, namely the ones going beyond the standard HŁS scheme [27]. This implies that the bosonic sector of the Poincaré or conformal superalgebra no longer can be expressed as the tensor product structure $B_{geom} \oplus B_{int}$, where B_{geom} describes space-time Poincaré or conformal algebras and the remaining generators spanning B_{int} are Lorentz-scalars.

In the particular case of the Minkowskian $D = 11$ dimensions, where the M-theory should be found, in the standard case the following construction is allowed. As already recalled in this case the spinors are real and have 32 components. By taking the anticommutator of two such spinors the most general expected result consists of a 32×32 symmetric matrix with $32 + \frac{32 \cdot 31}{2} = 528$ components. On the other hand, the standard supertranslation algebra underlying the maximal supergravity contains only the 11 bosonic Poincaré generators and by no means the r.h.s. saturates the total number of 528. The extra generators that should be expected in the right hand side are obtained by taking the totally antisymmetrized product of k Gamma matrices (the total number of such objects is given by the Newton binomial $\left(\begin{array}{c} D \\ k \end{array}\right)$). Imposing on the most general 32×32 matrix the further requirement of being symmetric, the total number of 528 is obtained by summing the $k = 1$, $k = 2$ and $k = 5$ sectors, so that $528 = 11 + 55 + 462$. The most general supersymmetry algebra in $D = 11$ can therefore be presented as

$$\{Q_a, Q_b\} = (A\Gamma_\mu)_{ab} P^\mu + (A\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (A\Gamma_{[\mu_1 \ldots \mu_5]})_{ab} Z^{[\mu_1 \ldots \mu_5]} \qquad (6)$$

(where the real matrix A is equivalent to Γ_0 [22]).

$Z^{[\mu\nu]}$ and $Z^{[\mu_1 \ldots \mu_5]}$ are tensorial central charges, of rank 2 and 5 respectively. These two extra central terms on the right hand side correspond to extended objects [28, 29], the p-branes. The algebra (6) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1 \ldots \mu_5]} \equiv 0$. It is this construction that we are going to analyze in the next section in the case of the octonionic structure, namely for minkowskian eleven-dimensional spinors described by 4 octonionic components.

4 The octonionic M-algebra.

The octonionic counterpart of the eleven dimensional M algebra (6) is given by the expression

$$\{Q_a, {Q_b}^*\} \;=\; \mathcal{Z}_{ab}, \tag{7}$$

where ${Q_a}^*$ denotes the principal conjugation in the given division algebra [7]. The spinors are octonionic 4-component and the r.h.s. is an octonionic-valued 4×4 hermitian matrix $\mathcal{Z}_{ab} = \mathcal{Z}^*_{ba}$, whose total number of independent components, in the real counting, is given by $52 = 4 + \frac{4\times 3}{2}\times 8$.

Just as the real case, the r.h.s. can be expanded in the antisymmetric product of octonionic-valued Gamma-matrices. The number 52, replacing the total number of 528 bosonic components of the real case, represents the maximal number of saturated bosonic components for the octonionic M algebra. In order to understand how this number can be produced and which are the octonionic p-forms entering the r.h.s., let's start with a brief digression concerning the antisymmetrized product of octonionic Gamma matrices. Since the octonionic algebra is non-associative a careful prescription has to be taken in order to correctly define such a product. Remarkably, this prescription turns out to be unique, if two requirements are chosen to be satisfied:

i) that the octonionic p-forms fulfil the Hodge duality and

ii) that the octonionic p-forms admit a well-defined character with respect to the hermiticity condition, being for a given p either *all* hermitian or *all* antihermitian.

It can be easily shown that in an octonionic Gamma matrices realization of a D dimensional Clifford algebra, out of the D Gamma matrices, $D-7$ are purely real, while the remaining 7 are proportional to the imaginary octonions. It is therefore sufficient to correctly define the antisymmetrized products among p imaginary octonions. Since up to $p=2$ the associativity is guaranteed, the only case that we need to explicitly define corresponds to $p=3$ (the case $p=4$, since $4=7-3$, automatically corresponds to the Hodge dual case; similarly $p=5,6,7$ are Hodge-dual of $p=2,1,0$ respectively). Due to the properties of imaginary octonions, only two inequivalent cases have to be examined, either the three imaginary octonions are chosen to produce the associative $su(2)$ subalgebra or not. In the first case, with the (2) choice of the octonionic structure constant, we can take the three τ_i being given by $i=1,2,3$. Their antisymmetric product is proportional to the identity. The second case is non-associative and can be examined by just taking $i=1,2,4$. In order to produce in this case as well an antisymmetric product $[\tau_1\cdot\tau_2\cdot\tau_4]$ proportional to the identity, there is only one prescription which we are forced to take, namely

$$[\tau_1\cdot\tau_2\cdot\tau_4] \;\equiv\; \frac{1}{3!}\sum_{perm.}(-1)^{\epsilon_{i_1 i_2 i_3}}\frac{1}{2}((\tau_{i_1}\cdot\tau_{i_2})\cdot\tau_{i_3}+\tau_{i_1}\cdot(\tau_{i_2}\cdot\tau_{i_3})), \tag{8}$$

for $i_j = 1, 2, 4$. The general formula for the antisymmetrized product of p octonionic Gamma matrices is given in [22]. It should be noticed that the number of (real) components for octonionic p-forms is reduced w.r.t. the associative (real, complex or quaternionic) cases. This is true already for $p = 2$. This case can be easily understood since 14 components are killed by the G_2 automorphisms of the octonions (we have, e.g. $[\tau_i, \tau_j] = 2C_{ijk}\tau_k$).

In D odd dimensional octonionic spacetimes we get the following table, whose columns are labeled by the antisymmetric product of Gamma matrices of rank p and the entries denote the total number of their components (real counting)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$D=7$	$\underline{1}$	7	7	$\underline{1}$	$\underline{1}$	7	7	$\underline{1}$						
$D=9$	$\underline{1}$	$\underline{9}$	22	22	$\underline{10}$	$\underline{10}$	22	22	$\underline{9}$	$\underline{1}$				
$D=11$	1	$\underline{11}$	$\underline{41}$	75	76	$\underline{52}$	$\underline{52}$	76	75	$\underline{41}$	$\underline{11}$	1		
$D=13$	1	13	$\underline{64}$	$\underline{168}$	267	279	$\underline{232}$	$\underline{232}$	279	267	$\underline{168}$	$\underline{64}$	13	1

(9)

The hermitian components are underlined.

Identities relating higher-rank antisymmetric octonionic tensors are expressed in the above table. Let us discuss here the $D = 11$ case, relevant for the octonionic M algebra. The 52 independent components of an octonionic hermitian (4×4) matrix can be expressed either as a rank-5 antisymmetric tensors (simbolically denoted as "$M5$"), or as the combination of the 11 rank-1 ($M1$) and the 41 rank-2 ($M2$) tensors. The relation between $M1+M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into 4 real indices, labeled by $a, b, c, \ldots$ and 7 octonionic indices labeled by $i, j, k, \ldots$. We get, on one side,

$$
\begin{array}{cc}
4 & M1_a \\
7 & M1_i \\
6 & M2_{[ab]} \\
4 \times 7 = 28 & M2_{[ai]} \\
7 & M2_{[ij]} \equiv M2_i
\end{array}
$$

while, on the other side,

$$
\begin{array}{cc}
7 & M5_{[abcdi]} \equiv M5_i \\
4 \times 7 = 28 & M5_{[abcij]} \equiv M5_{[ai]} \\
6 & M5_{[abijk]} \equiv M5_{[ab]} \\
4 & M5_{[aijkl]} \equiv M5_a \\
7 & M5_{[ijklm]} \equiv \widetilde{M5}_i
\end{array}
$$

which shows the equivalence of the two sectors, as far as the tensorial properties are concerned. The correct total number of 52 independent components is recovered

$$52 \;=\; 2 \times 7 + 28 + 6 + 4. \qquad (10)$$

Please notice that the table (9) refers to the antisymmetric product of Gamma matrices, which explains why $M5_i$ and $\widetilde{M5}_i$ are actually different. The octonionic equivalence of different antisymmetric tensors can be symbolically expressed, in odd space-time dimensions, through

$D=7$	$M0 \equiv M3$	
$D=9$	$M0+M1 \equiv M4$	
$D=11$	$M1+M2 \equiv M5$	(11)
$D=13$	$M2+M3 \equiv M6$	
$D=15$	$M3+M4 \equiv M0+M7$	

The octonionic M algebra defined by (7) can therefore be described either through the $11+41$ bosonic generators entering

$$\mathcal{Z}_{ab} = P^{\mu}(A\Gamma_{\mu})_{ab} + Z^{\mu\nu}_{\mathbf{O}}(A\Gamma_{\mu\nu})_{ab}, \tag{12}$$

or through the 52 bosonic generators entering

$$\mathcal{Z}_{ab} = Z^{[\mu_1 \ldots \mu_5]}_{\mathbf{O}}(A\Gamma_{\mu_1 \ldots \mu_5})_{ab}\,. \tag{13}$$

Differently from the real case, the sectors specified by (12) and (13) are not independent[20], leading to an unexpected and far from trivial new structure in the octonionic M-algebra.

5 Conclusions.

In this work we have discussed an approach to the "Theory Of Everything" based on the assumption that it could be described by an exceptional mathematical structure. We have furnished scattered pieces of evidence of the arising of exceptional Lie algebras and groups in the program of the unification of the interactions. We have further mentioned that behind such exceptional structures one can find the division algebra of the octonions. This is the maximal division algebra and in this respect plays a role similar to that of maximal supergravity. In the technical part of this paper we have shown how to link octonions with supersymmetry. We have in particular discussed the fact that the M algebra, which is expected underlying the M theory, can arise in two and only two versions. One is based on real numbers and leads to the standard description of M algebra. The second one, however, is octonionic. As such, it is potentially linked with the "exceptional program" mentioned above. In formula (7) we presented the octonionic generalized supertranslation algebra in the minkowskian $D=11$ spacetime. It is worth mentioning that this algebra admits a superconformal extension [21] which replaces the superalgebra $Osp(1,64|\mathbf{R})$ associated to the standard M theory, with the octonionic-valued superconformal algebra $Osp(1,8|\mathbf{O})$ associated to the octonionic M algebra. Striking new properties in the octonionic case, like the equivalence of different brane sectors, have been thoroughly discussed.

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