

# Analytic calculations of trial wave functions of the fractional quantum Hall effect on the sphere

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## Abstract

We present a framework for the analytic calculations of the hierarchical wave functions and the composite fermion wave functions in the fractional quantum Hall effect on the sphere by using projective coordinates. Then we calculate the overlaps between these two wave functions at various fillings and small numbers of electrons. We find that the overlaps are all most equal to one. This gives a further evidence that two theories of the fractional quantum Hall effect, the hierarchical theory and the composite fermion theory, are physically equivalent.

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## I. INTRODUCTION

The fractional quantum Hall effect (FQHE) at the Landau-level (LL) filling fraction  $\nu = 1/m$  with  $m$  an odd integer is very well described by Laughlin's theory.<sup>1,2</sup> The Laughlin wave function is a very good approximation of the exact ground state of the quantum Hall effect (QHE) at  $\nu = 1/m$ . However for the FQHE at  $\nu \neq 1/m$ , there exist *two* well-known theories (notice that we will only consider the case that the electron spins are polarized in this paper). One is the hierarchical theory. The states at  $\nu \neq 1/m$  are formed due to the condensation of the anyonic quasiparticles of Laughlin states.<sup>3-10</sup> The trial wave functions constructed from this theory are called as the hierarchical wave functions. Another theory is based on the composite fermion (CF) approach proposed by Jain,<sup>11</sup> where the FQHE is due to the integer QHE of the composite fermions (CFs) (electrons bounded with an even magnetic flux quanta). The trial wave functions constructed from the CF theory are called as the CF wave functions (or Jain's wave functions). The overlaps of the exact states with the hierarchical wave functions and the CF wave functions are both excellent. It has also been shown that two theories predict the same topological excitations at the same  $\nu$ .<sup>7,8,12</sup> The two theories must be physically equivalent if they both describe correctly the physics of the FQHE. Thus it would be very interesting to study the difference and equivalence of the two theories.

In this paper, we present a framework for the analytic calculations of the two wave functions on the sphere by using projective (or stereographic) coordinates on the sphere. There are several advantages of using spherical geometry. As it is a compact surface, there will be no edge state to be worried if we are only interested in the bulk state. Also the system has rotational invariance symmetries. On the torus, though the system has translational invariance and no boundaries, the hierarchical wave functions are very difficult to calculate and quite complicated due to its nontrivial topology,<sup>13</sup> and we do not even know how to construct the CF wave functions with the correct center coordinate degeneracy on a torus.

Because the states considered in the FQHE are restricted to the lowest Landau level

(LLL), the wave functions are only dependent on holomorphic coordinates (polynomials of the holomorphic coordinates) on the sphere. Therefore it is possible to use only holomorphic coordinates to do all calculations. To compare the two types of hierarchical wave functions is the same to compare the two polynomials of holomorphic coordinates on the sphere. We note that our ultimate goal is to expand those wave functions in polynomials and calculate the overlaps of two wave functions or physical quantities (for example, the density-density correlations) at an *arbitrary number* of electrons by the method (Jack polynomials method) used in studying the Calogero model.<sup>14</sup> We do not know how to do it at the moment, and further progresses on it will enhance our understandings of the theories of the FQHE.

We organize the paper as follows; first we review the Landau level problem on the sphere. A self-contained derivation of eigenstates of an electron on a sphere with a monopole field by using a simple geometric argument and projective coordinates is given in the appendix. Then we show how to classify the many-body eigenstates of the angular momentum in the LLL. We then construct the wave functions based the theory of the hierarchical states and the theory based on the CF picture. The wave functions constructed in this paper are easy to handle in the practical calculation. Finally we calculate the overlaps of the hierarchical wave functions and the CF wave functions at various fillings  $\nu$  and some small numbers of electrons.

## II. QUANTUM MECHANICS ON THE SPHERE

The electrons are constrained to move on the surface of a sphere of radius  $R$  having a magnetic monopole in its center. The total magnetic flux  $4\pi R^2 B$  must be an integer multiple  $\phi = 2S$  of the magnetic flux quantum  $\phi_0 = 2\hbar\pi c/e$  according to the Dirac quantization condition. Therefore, the sphere radius  $R$  is equal to  $S^{1/2}l_0$ , where  $l_0 = (\frac{\hbar c}{eB})^{1/2}$  is the magnetic length. The eigenstates of an electron are given by monopole spherical harmonics.<sup>3,15</sup> First, we briefly review the old method to derive the wave functions of the Landau levels (LLs), then rederive them by using Algebraic Geometry.

For simplicity, we take units  $\hbar$  and  $c$  equal to one in the following formulas. The Hamiltonian of a single electron of mass  $m_e$  is given by  $H = \frac{1}{2m_e}(P + eA)^2$ . However, since the electron is confined on the spherical surface, one shows,

$$H = \frac{1}{2m_e R^2} [r \times (P + eA)]^2 = \frac{\omega_c}{2S} \Lambda^2, \quad (1)$$

where  $\Lambda = r \times (P + eA)$ ,  $\omega_c$  is the cyclotron frequency,  $P = -i\nabla$ ,  $\nabla \times A = B\hat{\Omega}$ , and  $\hat{\Omega} = r/R$ .

The components of  $\Lambda$  obey the commutation relations  $[\Lambda_i, \Lambda_j] = i\epsilon_{ijk}(\Lambda_k - S\Omega_k)$ . The angular momentum operators  $L = \Lambda + S\Omega$ , and their commutation relations are  $[L_i, L_j] = i\epsilon_{ijk}L_k$ . Since  $\Lambda$  is normal to the surface, we have  $\hat{\Omega} \cdot \Lambda = \Lambda \cdot \hat{\Omega} = 0$ , and  $L \cdot \hat{\Omega} = \hat{\Omega} \cdot L = S$ . Using these equations, the relation  $|\Lambda|^2 = |L|^2 - S^2$  can be obtained. Thus the eigenvalues of  $|\Lambda|^2$  can be deduced from the usual angular momentum algebra  $|\Lambda|^2 = |L|^2 - S^2 = L(L+1) - S^2$ ,  $L = S + n$ ,  $n = 0, 1, 2, \dots$ , and the eigenstates of the Hamiltonian are the eigenstates of  $|L|^2$  and  $L_3$ , and they are given by monopole spherical harmonics. We choose a gauge field  $A = -\frac{S}{eR} \frac{(1+\cos\theta)}{\sin\theta} \hat{\varphi}$ , of which the singularity lies on the north pole (we choose a different gauge from the one used in Ref. 3). The wave functions at the LLL are given by

$$u^{S+m} v^{S-m}, \quad (2)$$

where  $m = -S, -S+1, \dots, S$ , and

$$u = \cos\left(\frac{1}{2}\theta\right)e^{i\varphi}, \quad v = \sin\left(\frac{1}{2}\theta\right). \quad (3)$$

All wave functions of the LLs can be derived by this way,<sup>15</sup> and we will not repeat this derivation here. In the following, all eigenstates will be obtained by using projective coordinates,<sup>16,17</sup> and the method developed in Ref. 18.

The projective coordinates are given by  $z = 2R \cot \frac{\theta}{2} e^{i\varphi}$  and its complex conjugate  $\bar{z}$ . We will take  $R = 1/2$  for simplicity. The measure on the sphere is  $\int \frac{dx dy}{(1+z\bar{z})^2}$ . The Hamiltonian of Eq. (1) in projective coordinates is now written by the following formula,<sup>17</sup>

$$H = \frac{2}{m_e} (1 + z\bar{z})^2 (P_z + eA_z)(P_{\bar{z}} + eA_{\bar{z}}), \quad (4)$$

where

$$P_z = -i\frac{\partial}{\partial z}, \quad P_{\bar{z}} = -i\frac{\partial}{\partial \bar{z}}, \quad eA_z = i\frac{\phi}{2} \frac{\bar{z}}{1+z\bar{z}}. \quad (5)$$

and  $\phi$  is the flux (in the unit of the fundamental flux  $\phi_0$ ) out of the surface. Note that the Hamiltonian given by Eq. (5) (we call this Hamiltonian as  $H'$  in the appendix) is different from the one given by Eq. (1) by a constant.

The ground states can be determined from the solutions of the equation  $(P_{\bar{z}} + eA_z)\psi = 0$ , and they are (unnormalized)

$$\psi = \frac{z^l}{(1+z\bar{z})^{\frac{\phi}{2}}}, \quad (6)$$

where  $l = 0, \dots, \phi$ . At any Landau levels, the eigenstates (unnormalized) are given by (from the appendix),

$$\begin{aligned} \psi_{n,l} = & [\partial_z + (\frac{B}{2} + 1)\partial_z \ln g][\partial_z + (\frac{B}{2} + 2)\partial_z \ln g] \dots \\ & \times [\partial_z + (\frac{B}{2} + n - 1)\partial_z \ln g] \psi_{n,l}^{(0)}, \end{aligned} \quad (7)$$

where

$$g = \frac{1}{(1+z\bar{z})^2}, \quad (8a)$$

$$\psi_{n,l}^{(0)} = g^{B/2} \tilde{\psi}_{n,l}^{(0)}, \quad (8b)$$

$$B = \phi/2, \quad (8c)$$

$$\tilde{\psi}_{n,l}^{(0)} = 1, z, \dots, z^l, \dots, z^{\phi+2n}. \quad (8d)$$

Under any finite rotations,  $z$  coordinate is transformed as  $z' = \frac{az+b}{cz+d}$ . The rotation matrix

$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is generated by the rotations along the three Cartesian axes,

$$R_x = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 + \cos \alpha)^{1/2} & i(1 - \cos \alpha)^{1/2} \\ i(1 - \cos \alpha)^{1/2} & (1 + \cos \alpha)^{1/2} \end{pmatrix}, \quad (9a)$$

$$R_y = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 + \cos \beta)^{1/2} & (1 - \cos \beta)^{1/2} \\ -(1 - \cos \beta)^{1/2} & (1 + \cos \beta)^{1/2} \end{pmatrix}, \quad (9b)$$

$$R_z = \begin{pmatrix} \exp(i\gamma/2) & 0 \\ 0 & \exp(-i\gamma/2) \end{pmatrix}. \quad (9c)$$

The rotational invariance of Hamiltonian is shown by the identity:

$$OH(z')O^{-1} = H(z), \quad (10)$$

where

$$O = \left( \frac{cz + d}{\bar{c}\bar{z} + \bar{d}} \right)^{\frac{\phi}{2}}. \quad (11)$$

The wave function is transformed under rotations as

$$\psi' = O\psi\left(\frac{az + b}{cz + d}\right). \quad (12)$$

We list some useful relations when we do a finite rotation on a many-body wave function.

$$d(z_i, z_j) = \frac{z_i - z_j}{\sqrt{1 + z_i\bar{z}_i}\sqrt{1 + z_j\bar{z}_j}}, \quad (13)$$

$z_i - z_j$ , and  $1 + z_i\bar{z}_i$  are transformed under the finite rotation as

$$d(z'_i, z'_j) = \left( \frac{cz_i + d}{\bar{c}\bar{z}_i + \bar{d}} \right)^{\frac{1}{2}} \left( \frac{cz_j + d}{\bar{c}\bar{z}_j + \bar{d}} \right)^{\frac{1}{2}} d(z_i, z_j), \quad (14a)$$

$$z'_i - z'_j = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}, \quad (14b)$$

$$1 + \bar{z}'_i z'_i = \frac{1 + z_i\bar{z}_i}{(cz_i + d)(\bar{c}\bar{z}_i + \bar{d})}. \quad (14c)$$

Finally, the angular momentum operators for  $N$  electrons are

$$\begin{aligned} J_x &= \sum_{i=1}^N J_x(i) \\ &= \frac{1}{2} \sum_{i=1}^N \left[ (1 - z_i^2) \frac{\partial}{\partial z_i} - (1 - \bar{z}_i^2) \frac{\partial}{\partial \bar{z}} + \frac{\phi}{2} (z_i + \bar{z}_i) \right], \end{aligned} \quad (15a)$$

$$\begin{aligned} J_y &= \sum_{i=1}^N J_y(i) \\ &= \frac{i}{2} \sum_{i=1}^N \left[ (1 + z_i^2) \frac{\partial}{\partial z_i} + (1 + \bar{z}_i^2) \frac{\partial}{\partial \bar{z}} + \frac{\phi}{2} (\bar{z}_i - z_i) \right], \end{aligned} \quad (15b)$$

$$\begin{aligned} J_z &= \sum_{i=1}^N J_z(i) \\ &= \sum_{i=1}^N \left( z_i \frac{\partial}{\partial z} - \bar{z}_i \frac{\partial}{\partial \bar{z}} - \frac{\phi}{2} \right). \end{aligned} \quad (15c)$$

### III. PROJECTIONS AND ANGULAR MOMENTUMS IN THE LLL

The FQH state is restricted to the LLL. In this section, we will discuss briefly how to project states to the LLL on the sphere (see Ref. 16, and for the detailed discussions in the case of a plane or a disk, see Ref. 19), and how to find the eigenstates of angular momentums when the particles are restricted to the LLL. Note that the construction of the CF wave functions involves the higher LLs, we need to project the wave functions to the LLL (see Sec. V).

The normalized states with flux  $\phi$  in the LLL are

$$|l\rangle = \left[ \frac{(\phi+1)!}{2\pi l!(\phi-l)!} \right]^{1/2} \frac{z^l}{(1+z\bar{z})^{\phi/2}}, \quad (16)$$

and  $l = 0, 1, 2, \dots, \phi$ . The projection operator to the LLL is  $P = \sum_l |l\rangle\langle l|$ , and it can be written also in the following form,

$$P\psi(z, \bar{z}) = \int \frac{dw d\bar{w}}{(1+w\bar{w})^2} G(z, w) \psi(w, \bar{w}), \quad (17a)$$

$$G(z, w) = \frac{\phi+1}{2\pi} \frac{(1+z\bar{w})^\phi}{(1+z\bar{z})^{\phi/2}(1+w\bar{w})^{\phi/2}}. \quad (17b)$$

For the many-body wave functions,  $P$  (or  $G$ ) is equal to  $\prod_{i=1}^N P_i$  (or  $\prod_{i=1}^N G_i$ ) where  $P_i$  is the projection operator of the  $i$ -th particle and  $N$  is the number of particles.

If the state is not in the LLL, the anti-holomorphic coordinate  $\bar{z}$  will appear. Typically, it appears as

$$\psi = \frac{\bar{z}^i z^{i+l}}{(1+z\bar{z})^{(\phi/2)+j}}, \quad (18)$$

and  $P\psi$  is equal to

$$\frac{(\phi+1)!(l+i)!(\phi+j-l-i)!}{l!(\phi-l)!(\phi+j+1)!} \frac{z^l}{(1+z\bar{z})^{\phi/2}}. \quad (19)$$

On the sphere, if the interactions between electrons are rotationally invariant, the eigenstates of the many-body Hamiltonian should be also the eigenstates of rotational operators  $J^2$  and  $J_z$ . The FQH ground states on the sphere are rotationally invariant and are non-degenerated. In order to find the ground states, we can thus use the rotational invariant

states to diagonalize the Hamiltonian. As the number of all possible rotational invariant states is much less than the number of all possible states, it is thus much easier to find the ground states by using the rotational invariant states to diagonalize the Hamiltonian than by using all possible states. It could be also interesting to find the eigenstates of  $J^2 \neq 0$  (which are not rotationally invariant). The excited states in the FQH are not rotationally invariant. For Fermi-liquid-like systems in a half-filled Landau level, one can have ground states which are not rotationally invariant.<sup>20</sup>

Now we are going to find the many-body wave functions on the LLL which are the eigenstates of  $J^2$  and  $J_z$ . In the LLL, the many-body wave functions  $\Psi$  have the form

$$\Psi = \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2}}} F(z_1, z_2, \dots, z_N), \quad (20)$$

where  $F(z_1, z_2, \dots, z_N)$  is an anti-symmetric holomorphic function. When  $J_+ = J_x + iJ_y$ ,  $J_- = J_x - iJ_y$ ,  $J_z$  act on  $\Psi$ , we have

$$J_- \Psi = \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2}}} \sum_{i=1}^N \frac{\partial}{\partial z_i} F, \quad (21a)$$

$$J_+ \Psi = \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2}}} \sum_{i=1}^N (-z_i^2 \frac{\partial}{\partial z_i} + \phi z_i) F, \quad (21b)$$

$$J_z \Psi = \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2}}} \left[ \left( \sum_{i=1}^N z_i \frac{\partial}{\partial z_i} \right) - \frac{N\phi}{2} \right] F. \quad (21c)$$

Thus the projected  $J$  operators are:

$$J'_- = \sum_{i=1}^N \frac{\partial}{\partial z_i}, \quad (22a)$$

$$J'_+ = \sum_{i=1}^N -z_i^2 \frac{\partial}{\partial z_i} + \phi z_i, \quad (22b)$$

$$J'_z = \left( \sum_{i=1}^N z_i \frac{\partial}{\partial z_i} \right) - \frac{N\phi}{2}, \quad (22c)$$

where they act only on  $F$ . The angular momentum eigenstates of the many-body wave functions restricted to the LLL can be obtained by solving

$$J'_- F(-J) = 0, \quad (23a)$$

$$J'_z F(-J) = -JF(-J), \quad (23b)$$



where  $F(-J)$  is the lowest weight eigenstate with weight  $-J$ . Other states can be obtained by applying  $J'_+$  repeatedly on  $F(-J)$ . Eq. (23) leads to

$$\sum_{i=1}^N z_i \frac{\partial}{\partial z_i} F(-J) = \left(\frac{N\phi}{2} - J\right) F(-J), \quad (24a)$$

$$\sum_{i=1}^N \frac{\partial}{\partial z_i} F(-J) = 0. \quad (24b)$$

The first equation in Eq. (24) means that  $F(-J)$  is a homogeneous polynomial with degree  $\frac{N\phi}{2} - J$ . As  $F(-J)$  is an anti-symmetric function of holomorphic coordinates, it can be factorized as  $F(-J) = \prod_{i<j}^N (z_i - z_j) F'(-J)$ . One can check that

$$J'_- F(-J) = \prod_{i<j}^N (z_i - z_j) J'_- F'(-J), \quad (25a)$$

$$J'_z F(-J) = \frac{N(N-1)}{2} F(-J) + \prod_{i<j}^N (z_i - z_j) J'_z F'(-J). \quad (25b)$$

Thus  $F'(-J)$  is a symmetric function with degree  $L = \frac{N\phi}{2} - J - \frac{N(N-1)}{2}$ , and the power of every coordinate in  $F'(-J)$  shall be less or equal than  $\phi'$  where  $\phi' = \phi - (N-1)$ . By using Eq. (24) and Eq. (25), one finds that  $F'(-J)$  satisfies the conditions:

$$J'_- F'(-J) = 0, \quad (26a)$$

$$J'_z F'(-J) = \left(\frac{N\phi'}{2} - J\right) F'(-J). \quad (26b)$$

Define symmetric polynomials  $\sigma_i$ :

$$P(z_i) = \prod_{i=1}^N (z - z_i) = \sum_{i=0}^N (-1)^i \sigma_i z^{N-i}, \quad (27)$$

where

$$\sigma_0 = 1, \quad \sigma_1 = \sum_{i=1}^N z_i, \quad \dots, \quad \sigma_N = \prod_{i=1}^N z_i. \quad (28)$$

$F'$  can be expanded as

$$\sum_{\mathbf{s}_i} C(\mathbf{s}_i) \prod_{i=1}^N \sigma_i^{s_i}, \quad (29)$$

where  $s_i$  is a non-negative integer. By using Eq. (26), we get equations which  $C(\mathbf{s}_i)$  and  $s_i$  must obey. One of them is

$$\sum_{i=1}^N i s_i = L = \frac{N\phi'}{2} - J. \quad (30)$$

The condition

$$\sum_{i=1}^N s_i \leq \phi' \quad (31)$$

must be satisfied in order that the wave function is normalizable.  $C(s_i)$  shall also satisfy the equation

$$J'_- F'(-J) = \sum_{s'_i} C'(s'_i) \prod_{i=1}^N \sigma_i^{s'_i} = 0, \quad (32)$$

where  $C'(s'_i)$  is a linear combinations of  $C(s_i)$ , and it shall be equal to 0. Because  $[J_z, J_-] = -J_-$ ,  $s'_i$  shall satisfy

$$\sum_{i=1}^N i s'_i = L - 1. \quad (33)$$

Apparently  $\sum_{i=1}^N s'_i$  shall be also less or equal than  $\phi'$ . The number of linear independent solutions for  $C(s_i)$  is equal to the number of solutions of Eq. (30) minus the number of solutions of Eq. (33), and it is also equal to  $M(J, N, \phi')$ , which is the number of states with spin  $J$ .

The generation function for the number of solutions of Eq. (30) or Eq. (33) is

$$G(t, q) = \prod_{i=1}^N \frac{1}{1 - tq^i}. \quad (34)$$

The number of solutions of Eq. (30) is equal to the sum of the coefficient of term  $t^j q^L$  with  $0 \leq j \leq \phi'$  in  $G(t, q)$ . Thus  $M(J, N, \phi')$  is equal to

$$\oint \oint \frac{dt}{2\pi it} \frac{dq}{2\pi i q} G(t, q) \left( \frac{1}{q^L} - \frac{1}{q^{L-1}} \right) \sum_{i=0}^{\phi'} \frac{1}{t^i}. \quad (35)$$

We can also use a generation function of one variable,<sup>21</sup>

$$G(t) = \frac{\prod_{k=1}^{N+\phi'} (1 - t^k)}{\prod_{k=1}^N (1 - t^k) \prod_{k=1}^{\phi'} (1 - t^k)}. \quad (36)$$

The number of solutions of Eq. (30) is then given by the coefficient of  $t^L$  of function  $G(t)$ .

Thus

$$M(J, N, \phi') = \oint \frac{dt}{2\pi it} G(t) \frac{1-t}{t^L}. \quad (37)$$

The asymptotic behavior of  $M(J, N, \phi')$  can be obtained by using the steepest descent method.<sup>22</sup> When  $L$  and  $\phi'$  are both large,  $M(J, N, \phi')$  is equal to  $\exp(S(J, N, \phi'))$  approximately, and  $\exp(S(J, N, \phi'))$  is determined by the following equations,

$$L = -\frac{\exp(-\rho)}{1 - \exp(-\rho)} + \frac{1}{\rho^2} \left( -\int_0^{(N+\phi')\rho} + \int_0^{N\rho} + \int_0^{\phi'\rho} \right) du \frac{u \exp(-\rho u)}{1 - \exp(-\rho u)}, \quad (38a)$$

$$S(J, N, \phi') = L\rho + \ln(1 - \exp(-\rho)) + \frac{1}{\rho} \left( \int_0^{(N+\phi')\rho} - \int_0^{N\rho} + \int_0^{\phi'\rho} \right) du \ln[1 - \exp(-u)]. \quad (38b)$$

We list the number of rotational invariance states at various fillings in Table I.

#### IV. HIERARCHICAL WAVE FUNCTIONS

In this section, we will discuss the construction of the hierarchical wave functions.<sup>2-4</sup> The quasiparticles satisfy fractional statistics, and the condensation of quasiparticles gives rise to the FQH state with  $\nu \neq 1/m$ .

Define

$$\Psi_m = \prod_{i < j}^N (u_i v_j - u_j v_i)^m, \quad (39)$$

where  $m$  is a positive integer. For  $\nu = \frac{1}{m}$  with  $m$  being an odd positive integer, the FQH wave function will be  $\Psi_m$  (the Laughlin wave function). The flux  $\phi$  is equal to  $\phi_m = m(N-1)$ .<sup>3</sup> Or in the projective coordinates, it is

$$\Psi_m = \prod_{i < j}^N d(z_i, z_j)^m. \quad (40)$$

The Laughlin wave function with the presence of quasiparticle excitations is given by acting the quasiparticle excitation operators on the original Laughlin wave function. The quasiparticle excitation operator is given by

$$A^\dagger(\alpha, \beta) = \prod_{i=1}^N (\beta u_i - \alpha v_i), \quad (\text{quasihole}) \quad (41a)$$

$$A(\alpha, \beta) = \prod_{i=1}^N \left( \bar{\beta} \frac{\partial}{\partial u_i} - \bar{\alpha} \frac{\partial}{\partial v_i} \right), \quad (\text{quasielectron}) \quad (41b)$$

where  $\alpha = \cos \frac{\theta}{2} e^{i\varphi}$ , and  $\beta = \sin \frac{\theta}{2}$  are the quasiparticle coordinates. In the projective coordinates, the operators of the quasihole excitation and the quasielectron excitation are given in the following form,

$$A^\dagger(\omega, \bar{\omega}) \Psi_m(z_i) = \prod_{i=1}^N d(z_i, \omega) \Psi_m(z_i), \quad (42a)$$

$$A(\omega, \bar{\omega}) \Psi_m(z_i) = \frac{1}{(1 + \omega \bar{\omega})^{N/2}} \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi_m - 1}{2}}} \\ \times \prod_{i=1}^N [(1 + z_i \bar{\omega}) \partial_{z_i} - \phi_m \bar{\omega}] F_m, \quad (42b)$$

where  $\omega, \bar{\omega}$  is the projective coordinates of the quasiparticle, and  $F_m(z_i) = \prod_{i < j}^N (z_i - z_j)^m$ . The flux  $\phi$  in the presence of a quasielectron (quasihole) is  $\phi_m - 1$  ( $\phi_m + 1$ ).

The slightly entangled appearance of  $A(\omega)$  hides, indeed, a form which is analogous to  $A^\dagger(\omega)$ . To unveil the similarities, one can show that

$$P(\phi_m - 1, z_i) \prod_{i=1}^N d(\bar{z}_i, \bar{\omega}) \Psi_m \quad (43)$$

gives the wave function of the Laughlin state in the presence of a quasihole as that in Eq. (42).  $P(\phi, z_i)$  (here  $\phi = \phi_m - 1$ ) projects the wave function to the LLL with flux  $\phi$  with respect to coordinates  $z_i$ . Thus the construction of the hierarchical wave functions due to the condensation of quasielectrons will *naturally* involve higher Landau levels as in the case of the CF wave functions (see the next section).

Instead using  $A(\omega, \bar{\omega}) \Psi_m(z_i)$ , we can also create a quasielectron excitation using  $\Psi_{m-2} AD$ ,<sup>9,23</sup> where  $D = [\Psi_1(z_i)]^2$ .  $AD$  is here equal to  $P(\phi_2 - 1) \prod_{i=1}^N d(\bar{z}_i, \bar{\omega}) D$ . We call  $(\Psi_1)^{m-2} AD$  as a wave function by the hard core construction.

In the case of many quasiparticle excitations, the operators of excitations are

$$A_{N_q}^\dagger = \prod_{j=1}^{N_q} A^\dagger(\omega_j, \bar{\omega}_j), \quad (44a)$$

$$A_{N_q} = \prod_{j=1}^{N_q} A(\omega_j, \bar{\omega}_j). \quad (44b)$$

When  $A_{N_q}$  acts on  $\Psi_m$ , one can show

$$A_{N_q} \Psi_m = \prod_{j=1}^{N_q} \frac{1}{(1 + \omega_j \bar{\omega}_j)^{N/2}} \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi_m - N_q}{2}}} \times A'_{N_q} F_m(z_i), \quad (45a)$$

where

$$\begin{aligned} A'_{N_q} &= \prod_{j=1}^{N_q} A'(j) \\ &= \prod_{i=1}^N [(1 + z_i \bar{\omega}_{N_q}) \partial_{z_i} - (\phi_m - N_q + 1) \bar{\omega}_{N_q}] \\ &\quad \times \prod_{i=1}^N [(1 + z_i \bar{\omega}_{N_q-1}) \partial_{z_i} \\ &\quad - (\phi_m - N_q + 2) \bar{\omega}_{N_q-1}] \\ &\quad \dots \dots \dots \\ &\quad \times \prod_{i=1}^N [(1 + z_i \bar{\omega}_j) \partial_{z_i} - (\phi_m - j + 1) \bar{\omega}_j] \\ &\quad \dots \dots \dots \\ &\quad \times \prod_{i=1}^N [(1 + z_j \bar{\omega}_1) \partial_{z_j} - \phi_m \bar{\omega}_1]. \end{aligned} \quad (46a)$$

One shall be careful about the ordering of  $A'(j)$  in Eq. (46).  $A'(j)$  in Eq. (46) is defined as

$$A'(j) = \prod_{i=1}^N [(1 + z_i \bar{\omega}_j) \partial_{z_i} - (\phi_m - j + 1) \bar{\omega}_j]. \quad (47)$$

As in the case of a single quasihole excitation, the wave function in the presence of  $N_q$  quasiholes can be also written as

$$P(\phi) \prod_{i=1}^N \prod_{\alpha=1}^{N_q} d(\bar{z}_i, \bar{\omega}_\alpha) \Psi_m, \quad (48)$$

where  $\phi = \Phi_m - N_q$ .

To construct the hierarchical wave functions, we shall normalize the Laughlin wave functions in the presence of quasiparticles. One can show that  $\Psi_{e,q} = [\Psi_1(\omega_\alpha)]^{1/m} A_{N_q}^\dagger \Psi_m$ ,  $\Psi_{e,q} =$

$[\Psi_1(\omega_\alpha)]^{1/m} A_{N_q} \Psi_m$ , or for the hard core constructed wave function,  $[\Psi_1(\omega_\alpha)]^{1/m} \Psi_{m-2} AD$ , are normalized.<sup>24</sup> The hierarchical states are obtained if the quasiparticles are also condensed to the Laughlin states. The wave function of quasiholes is  $\Psi_q = [\Psi_1(\bar{\omega}_\alpha)]^{p_2 + \frac{1}{m}}$ , the wave function of quasielectrons is  $\Psi_q = [\Psi_1(\omega_\alpha)]^{p_2 - \frac{1}{m}}$ , and  $p$  is a positive even integer. Quasiparticles satisfy fractional statistics,<sup>5</sup> and the wave functions here are in singular gauge which shows fractional statistics explicitly. The hierarchical wave function for the electrons is then given by the following formula,

$$\int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\bar{\omega}_\alpha}{(1 + \omega_\alpha \bar{\omega}_\alpha)^2} \Psi_{e,q} \Psi_q, \quad (49)$$

or explicitly,

$$\begin{aligned} \Psi_e(m, p) = & \int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\bar{\omega}_\alpha}{(1 + \omega_\alpha \bar{\omega}_\alpha)^2} \Psi_m(z_i) \prod_{i=1}^N \prod_{\alpha=1}^{N_q} d(z_i, \omega_\alpha) \\ & \times |\Psi_1(\omega_\alpha)|^{2/m} \Psi_{p_2}(\bar{\omega}_\alpha) \end{aligned} \quad (50)$$

is the hierarchical wave function due to the condensation of quasiholes, and the hierarchical wave functions due to the condensation of quasielectrons by the non-hard core construction and the hard core construction are given in the following formulas,

$$\begin{aligned} \Psi_e(m, -p) = & P(\phi, z_i) \int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\bar{\omega}_\alpha}{(1 + \omega_\alpha \bar{\omega}_\alpha)^2} \Psi_m(z_i) \\ & \times \prod_{i=1}^N \prod_{\alpha=1}^{N_q} d(\bar{z}_i, \bar{\omega}_\alpha) \Psi_{p_2}(\omega_\alpha), \end{aligned} \quad (51a)$$

$$\begin{aligned} \Psi_{e,\text{hard}}(m, -p) = & \Psi_{m-2}(z_i) P(\phi_2 - N_q, z_i) \int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\bar{\omega}_\alpha}{(1 + \omega_\alpha \bar{\omega}_\alpha)^2} \\ & \times \Psi_2(z_i) \prod_{i=1}^N \prod_{\alpha=1}^{N_q} d(\bar{z}_i, \bar{\omega}_\alpha) \Psi_{p_2}(\omega_\alpha). \end{aligned} \quad (51b)$$

We also require the wave functions above being rotationally invariant. This requirement leads to

$$m(N-1) + \xi_2 N_q = \phi, \quad (52a)$$

$$p_2(N_q - 1) = N. \quad (52b)$$

$\xi_2 = \pm 1$  in the case of the condensation of quasiholes and quasielectrons respectively. And the Landau level filling fraction  $\nu$  is equal to

$$\frac{1}{m + \frac{1}{\xi_2 p_2}}. \quad (53)$$

For  $m = 1$  and  $\xi_2 = 1$ , the filling  $\nu = \frac{1}{m + \frac{1}{p_2}}$  is equal to the filling of the charge conjugate state,  $1 - \frac{1}{p_2 + 1}$ . Actually, the wave function  $\Psi_e(m, p)$  is also the charge conjugate of the Laughlin wave function at filling  $\nu = \frac{1}{p_2 + 1}$ , and this shows that the construction of the wave function is consistent with physical picture. When  $m \neq 1$ , we notice that, in the formula for  $\Psi_e(m, p)$ , we can not do the integration exactly due to the term  $|\Psi_1(\omega_\alpha)|^{2/m}$ . We can approximate the trial wave function  $\Psi_e(m, p)$  by omitting  $|\Psi_1(\omega_\alpha)|^{2/m}$ , and it becomes

$$\begin{aligned} \Psi_e(m, p) \approx & \int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\bar{\omega}_\alpha}{(1 + \omega_\alpha \bar{\omega}_\alpha)^2} \Psi_m(z_i) \\ & \times \prod_{i=1}^N \prod_{\alpha=1}^{N_q} d(z_i, \omega_\alpha) \Psi_{p_2}(\bar{\omega}_\alpha). \end{aligned} \quad (54)$$

The wave function written in Eq. (54) is still rotationally invariant,<sup>9,17</sup> and we are able to integrate it. When  $m = 1$ , the formula for  $\Psi_e(m, p)$  in Eq. (50) is integrable. When  $m = 1$ , we find that the overlap between the wave functions given in Eq. (50) and Eq. (54) is excellent for a small number of electrons. In Ref. 9, it was also found that the overlapping of the wave functions given by Eq. (54) with the exact ground state of the FQH is all most equal to one for a small number of electrons. We note that the wave functions calculated in Sec. VI are based on the formulas written in Eq. (52) and Eq. (54).

In the formula for  $\Psi_e(m, -p)$  or  $\Psi_{e,\text{hard}}(m, -p)$ , we note that one can do the integration first, and then the projection, or vice versa. In Sec. VI, the overlap between these two wave functions will be calculated for a small number of electrons and it is found that the overlap is all most equal to one.

We will call the above hierarchical states as the *2nd*-level hierarchical states, and the Laughlin states as the *1st*-level hierarchical states. The higher-level hierarchical states can be built in the similar way.<sup>7,8,17,9</sup> We denote the *k*'th-level hierarchical states by  $(p_1, \xi_2 p_2, \xi_3 p_3, \dots, \xi_k p_k)$ , where  $p_1$  is an old positive integer,  $p_i, i \neq 1$  are even positive

integers, and  $\xi_i = \pm$  indicate the quasihole condensation and quasielectron condensation from parent states.

For the higher-level hierarchical wave functions involving the condensation of quasielectrons, we can make a *further* simplification. We take  $(p_1, -p_2, -p_3)$  as an example. The wave function for this state is

$$\begin{aligned} \Psi_e = & P(\phi, z_i) \int \prod_{\alpha=1}^{N_2} \frac{d\omega_\alpha^1 d\bar{\omega}_\alpha^1}{(1 + \omega_\alpha^1 \bar{\omega}_\alpha^1)^2} \Psi_m(z_i) \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i, \bar{\omega}_\alpha^1) \\ & \times P(N_1, \omega_\alpha^1) \int \prod_{\alpha=1}^{N_3} \frac{d\omega_\alpha^2 d\bar{\omega}_\alpha^2}{(1 + \omega_\alpha^2 \bar{\omega}_\alpha^2)^2} \Psi_{p_2}(\omega_\alpha^1) \\ & \times \prod_{\alpha=1}^{N_2} \prod_{\beta=1}^{N_3} d(\bar{\omega}_\alpha^1, \bar{\omega}_\beta^2) \Psi_{p_3}(\omega_\alpha^2), \end{aligned} \quad (55)$$

where  $N_1$  is the number of electrons,  $N_2$  is the number of quasielectrons of the Laughlin state  $(p_1)$ ,  $N_3$  is the number of quasielectrons of the hierarchical state  $(p_1, -p_2)$ ,  $\omega_\alpha^1$  and  $\omega_\alpha^2$  are the coordinates of quasiparticles of the two types respectively. We can prove that  $P(N_1, \omega_\alpha^1)$  can be *drooped* inside the formula. Thus the wave function can be written as

$$\begin{aligned} \Psi_e = & P(\phi, z_i) \int \prod_{\alpha=1}^{N_2} \prod_{\beta=1}^{N_3} \frac{d\omega_\alpha^1 d\bar{\omega}_\alpha^1}{(1 + \omega_\alpha^1 \bar{\omega}_\alpha^1)^2} \frac{d\omega_\beta^2 d\bar{\omega}_\beta^2}{(1 + \omega_\beta^2 \bar{\omega}_\beta^2)^2} \\ & \times \Psi_m(z_i) \Psi_{p_2}(\omega_\alpha^1) \Psi_{p_3}(\omega_\alpha^2) \\ & \times \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} d(\bar{z}_i, \bar{\omega}_\alpha^1) \prod_{\alpha=1}^{N_2} \prod_{\beta=1}^{N_3} d(\bar{\omega}_\alpha^1, \bar{\omega}_\beta^2). \end{aligned} \quad (56)$$

The wave function in Eq. (56) is quite similar to the wave function constructed in Ref. 7,

$$\begin{aligned} \Psi_e = & P(\phi, z_i) \int \prod_{\alpha=1}^{N_2} \prod_{\beta=1}^{N_3} \frac{d\omega_\alpha^1 d\bar{\omega}_\alpha^1}{(1 + \omega_\alpha^1 \bar{\omega}_\alpha^1)^2} \frac{d\omega_\beta^2 d\bar{\omega}_\beta^2}{(1 + \omega_\beta^2 \bar{\omega}_\beta^2)^2} \\ & \times \Psi_m(z_i) \Psi_{p_2}(\omega_\alpha^1) \Psi_{p_3}(\omega_\alpha^2) \\ & \times \prod_{i=1}^{N_1} \prod_{\alpha=1}^{N_2} \frac{1}{d(z_i, \omega_\alpha^1)} \prod_{\alpha=1}^{N_2} \prod_{\beta=1}^{N_3} \frac{1}{d(\omega_\alpha^1, \omega_\beta^2)}. \end{aligned} \quad (57)$$

However it is difficult to handle Eq. (57) in the practical calculation due to the singularities.

Finally by requiring the rotational invariance of the wave function (56) or (57), one gets

$$\begin{aligned} p_1(N_1 - 1) - N_2 &= \phi, \\ N_1 - p_2(N_2 - 1) + N_3 &= 0, \\ N_2 - p_3(N_3 - 1) &= 0, \end{aligned} \quad (58)$$



and Eq. (58) implies that the filling of the FQH state is equal to

$$\frac{1}{p_1 + \frac{1}{p_2 + \frac{1}{p_3}}}. \quad (59)$$

We point out that the wave function proposed in Ref. 7 had been also constructed on the torus.<sup>13</sup> It would be very interesting if we can generalize the construction of the wave function (56) to the torus.

## V. COMPOSITE FERMION WAVE FUNCTIONS

The CF theory of the FQHE has significantly advanced the understanding of the FQHE recently.<sup>11</sup> The FQHE is due to the integer QHE of the CFs, where a CF is the bound state of an electron and an even number of vortices. We will discuss in this section how to calculate the CF wave functions in our framework.

Jain proposed that all trial wave functions of the FQHE (note again in this paper the spin is polarized) can be obtained by using two operations,  $D$  and  $C$ , respectively composite fermionization and charge conjugation, on the wave functions of the integer QHE of the CFs. For example, the trial wave function of electrons at  $\nu = n/(2n+1)$  can be written as  $PD\chi_n$ , where  $\chi_n$  is the wave function of the CFs which fill completely the first  $n$  Landau levels with flux  $\phi^*$  ( $P$  is the projection operator to the LLL as in the previous sections). The flux of the state  $PD\chi_n$  is equal to  $2(N-1) + \phi^*$  where  $\phi^* = \frac{N}{n} - n$ . We can also use  $\Psi_1 P(\phi - N + 1) \Psi_1 \chi_n$  as the trial wave function and we call this wave function as the wave function by the hard core construction. The charge conjugation of  $PD\chi_n$  (or  $\Psi_1 P(\phi - N + 1) \Psi_1 \chi_n$ ) is then trial wave function at  $\nu = 1 - \frac{n}{2n+1} = \frac{n+1}{2n+1}$ . The trial wave function at other fillings can be obtained by acting repeatedly  $D$  and  $C$  on  $PD\chi_n$  ( $\Psi_1 P(\phi - N + 1) \Psi_1 \chi_n$ ) (each state can be obtained only in a unique way in this picture except the ordering of operator  $P$ ).

$\chi_n$  is given by the determinant  $\chi_n = \det(\psi_{s,k}(z_i))$ , where  $s = 0, 1, \dots, n-1$ ,  $k = \phi^* + 2s + 1$ ,  $i = 1, 2, \dots, N$ ,  $N = n\phi^* + n^2$ .  $\det(\psi_{s,k}(z_i))$  can be simplified and it is given by the following formula,

$$\chi_n = \begin{vmatrix}
1 & 1 & \dots & 1 \\
z_1 & z_2 & \dots & z_N \\
\bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\
\vdots & \vdots & \vdots \vdots \vdots & \vdots \\
z_1^{N'-1} & z_2^{N'-1} & \dots & z_N^{N'-1} \\
\bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\
\bar{z}_1 z_1 & \bar{z}_2 z_2 & \dots & \bar{z}_N z_N \\
\vdots & \vdots & \vdots \vdots \vdots & \vdots \\
\bar{z}_1 z_1^{N'-1} & \bar{z}_2 z_2^{N'-1} & \dots & \bar{z}_N z_N^{N'-1} \\
\vdots & \vdots & \vdots \vdots \vdots & \vdots \\
\bar{z}_1^{n-1} & \bar{z}_2^{n-1} & \dots & \bar{z}_N^{n-1} \\
\bar{z}_1^{n-1} z_1 & \bar{z}_2^{n-1} z_2 & \dots & \bar{z}_N^{n-1} z_N \\
\vdots & \vdots & \vdots \vdots \vdots & \vdots \\
\bar{z}_1^{n-1} z_1^{N'-1} & \bar{z}_2^{n-1} z_2^{N'-1} & \dots & \bar{z}_N^{n-1} z_N^{N'-1}
\end{vmatrix}
\times \prod_{j=1}^N \frac{1}{(1 + z_j \bar{z}_j)^{\frac{\phi^*}{2} + n - 1}}, \quad (60)$$

where  $N' = N/n = \phi^* + n$ . We divide  $N$  electrons into  $n$  groups. The set of the original coordinates  $z_i$  can be mapped to  $z_{s,k}$  with  $s = 0, 1, \dots, n-1$ ,  $k = 1, 2, \dots, N'$ . The determinant is proportional to

$$\chi_n = \text{AN} \prod_{s=0}^{n-1} [e_s^{N'}]^s \Psi_{s,1}
\times \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{n-1}{2}}}, \quad (61)$$

where

$$e_s^{N'} = \prod_{k=1}^{N'} \bar{z}_{s,k}, \quad (62a)$$

$$\Psi_{s,1} = \prod_{k_1 < k_2}^{N'} d(z_{s,k_1} - z_{s,k_2}), \quad (62b)$$

and AN is the anti-symmetrizing operator on all coordinates  $z_{s,k}$ . The wave functions  $\Psi = P(\phi)D\chi_n$  and  $\Psi_{\text{hard}} = \Psi_1 P(\phi - N + 1)\Psi_1 \chi_n$  can be written in the following form,

$$\Psi = \mathbf{A} \mathbf{N} P \prod_{s=0}^{n-1} [e_s^{N'}]^s \Psi_{s,2} \prod_{i<j}^N (z_i - z_j)^2$$

$$\times \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2} + n - 1}}, \quad (63a)$$

$$\Psi_{\text{hard}} = \Psi_1 \mathbf{S} \mathbf{Y} P(\phi - N + 1) \prod_s [e_s^{N'}]^s \Psi_{s,2}$$

$$\times \prod_{i<j}^N (z_i - z_j) \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi - N + 1}{2} + n - 1}}, \quad (63b)$$

where  $\mathbf{S} \mathbf{Y}$  is the symmetrizing operator on the electron coordinates, and

$$\Psi_{s,2} = \prod_{k_1 < k_2}^{N'} (z_{s,k_1} - z_{s,k_2}). \quad (64)$$

Before doing the anti-symmetrizing or the symmetrizing in the formulas above, it appears that there are  $n$  different groups of electrons and there are correlations between different groups. The generic terms before doing the projection, for example, in the formula of  $\Psi$ , are

$$\frac{\bar{z}_{s,k}^s z_{s,k}^l}{(1 + z_{s,k} \bar{z}_{s,k})^{\frac{\phi}{2} + n - 1}}. \quad (65)$$

It will be projected to

$$\frac{(\phi + 1)!! (\phi + n - 1 - l)!}{(\phi + n)! (l - s)! (\phi - l + s)!} \frac{z_{s,k}^{l-s}}{(1 + z_{s,k} \bar{z}_{s,k})^{\frac{\phi}{2}}}. \quad (66)$$

As  $\frac{(\phi+1)!}{(\phi+n)!}$  is a constant and is not dependent on  $s, l$ , we can discard it in the process of the projection. Thus  $P$  will act in the following way (discarding constant  $\frac{(\phi+1)!}{(\phi+n)!}$ ),

$$P \frac{\bar{z}_{s,k}^s F(z_{s,k})}{(1 + z_{s,k} \bar{z}_{s,k})^{\frac{\phi}{2} + n - 1}} = \frac{1}{(1 + z_{s,k} \bar{z}_{s,k})^{\frac{\phi}{2}}} \frac{1}{(\phi - z_{s,k} \partial_{z_{s,k}})!}$$

$$\times \partial_{z_{s,k}}^s (\phi + n - 1 - z_{s,k} \partial_{z_{s,k}})! \times F(z_{s,k}). \quad (67)$$

For example, by applying this formula to  $\Psi = P D \chi_2$ , the wave function is then given by

$$\Psi = \mathbf{A} \mathbf{N} P \prod_{i=1}^N \frac{1}{(1 + z_i \bar{z}_i)^{\frac{\phi}{2}}} \prod_{k=1}^{N/2} (\phi + 1 - z_{0,k} \partial_{z_{0,k}})$$

$$\times \partial_{z_{1,k}} \prod_{s=0}^1 (\Psi_{s,2})^3 \prod_{k_1=1}^{N/2} \prod_{k_2=1}^{N/2} (z_{0,k_1} - z_{1,k_2})^2. \quad (68)$$

The trial wave function  $\Psi_c$  for filling  $1 - \nu$  is related to the trial wave function  $\Psi$  at filling  $\nu$  by charge conjugation,

$$\Psi_c = \int \prod_i^M \frac{dz_{N+i} d\bar{z}_{N+i}}{(1 + z_{N+i} \bar{z}_{N+i})^2} \bar{\Psi}(z_{N+i} \dots z_{N+M}) \Psi_1(z_1 \dots z_{N+M}) \quad (69)$$

where  $M$  is the number of particles in the state  $\Psi$ ,  $N$  is the number of electrons in  $\Psi_c$ ,  $N + M = \phi + 1$ , and  $\bar{\Psi}$  is the complex conjugate of  $\Psi$ . Note again, If we use  $PD\chi_n$  as the trial wave function  $\Psi$ , the projection operator  $P$  can be dropped in Eq. (69). However if one uses  $\Psi_{\text{hard}} = \Psi_1 P(\phi - M + 1) \Psi_1 \chi_n$  in Eq. (69), then the operator  $P$  can **not** be dropped in Eq. (69).

One can also act  $D$  on  $\Psi$  and we will get another trial wave function of the FQH state at filling  $\frac{1}{2+\nu}$ , where  $\nu$  is the filling of the state  $\Psi$ . Repeatedly acting  $D$  and  $C$  on  $PD\chi_n$ , we can get the trial wave functions at all observable fillings.

## VI. THE OVERLAPS BETWEEN HIERARCHICAL WAVE FUNCTIONS AND CF WAVE FUNCTIONS

We perform the calculations of the wave functions symbolically by using Maple. The overlaps between the hierarchical wave function and the CF wave functions are calculated, Some overlaps between the wave functions with or without the hard core construction are also calculated. The formula of the trial wave functions for the FQHE in the previous sections need to be normalized before we calculate the overlaps. Table II lists some overlaps at some fillings for a small number of electrons.  $E\Psi$  means a new state formed by the condensation of quasielectrons of parent state  $\Psi$ , and  $H\Psi$  means a new state formed by the condensation of quasiholes of parent state  $\Psi$ . In all cases listed in the table,  $p_i$  is equal to 2 for  $i > 1$  in the constructions of the hierarchical states. The wave functions which involve  $D, P, C$  operations are the CF wave functions.

When  $N = 3, 4$ ,  $\phi = 6$ , there is only one rotational invariant state, which must also be the ground state. This explains why some of overlaps in the table are equal to one exactly.

Because of limited CPU we are allowed to use, we are only able to calculate some hierarchical wave functions up to 6 electrons, and some CF wave functions up to 10 electrons. The detailed calculations can be found in Ref. 25. In the future, we will calculate the wave functions with more numbers of electrons.

From the calculations, we conclude that, the hierarchical wave functions and the CF wave functions are all most the same in the case of a small number of electrons.

## VII. CONCLUSIONS

In this paper, we present a detailed discussion about the calculation of the trial wave functions on the sphere. The projective coordinates are used in performing the calculations. A self-contained derivation of the LLs on the sphere (or any surfaces with a constant curvature) using geometrical method is also given in the paper. The many-body wave function in the LLL are studied and classified in the angular momentum bases. We also simplify the formulas for the hierarchical wave functions and the CF wave functions.

There are many interesting things which we want to study in the future. We shall use theories of polynomials to study those wave functions.<sup>26,27</sup> It would be very interesting if we can obtain the polynomials explicitly for the wave functions at an arbitrary number of electrons.

There is a mapping between a trial wave function in the FQHE and a wave function in an one-dimension space.<sup>28</sup> Because of the existence of the mapping, one may apply the method used to study the Calogero model to study the trial wave functions in the FQHE, and then it may be possible to calculate some physical quantities from the trial wave functions at an arbitrary number of electrons.

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## APPENDIX: LANDAU LEVELS ON COMPACT CLOSED SURFACES

In this appendix, we will study the LLs on general compact closed surfaces, and work out the LLs on the sphere as an example.

If the magnetic field and the curvature are constant, the spectrum, the wave functions and the degeneracy of Landau levels (LLs) can be obtained by using a very simple geometric argument.<sup>18</sup> A self-contained presentation of the idea based on Ref. 18 will be found in this appendix, and some examples will be included.

In the case that the surface is a plane, a sphere, or a torus, the spectrum and eigenfunctions of the LLs can be exactly solved.<sup>29</sup> For example, the LLs on a sphere with a Dirac-monopole on the origin, were solved by Dirac long time ago. The problem in the case of the surface being an open up-half hyperbolic plane with a constant negative curvature was solved completely where there exist a discrete spectrum (this is the spectrum of the LLs) in the low-energy sector and a continuous spectrum in the high-energy sector.<sup>29</sup>

Ref. 18 studied the problem of the LLs on the compact closed Riemann surfaces with Poincaré metric, and obtained the discrete low-energy eigenvalues (or LLs), their multiplicity and wave functions. Previous to Ref. 18, similar problem also was studied and discrete low-energy eigenvalues and their multiplicity was obtained by using the results from the mathematical literature, for example by using Selberg trace formula (see the references quoted in Ref. 18).

Why the problem of the LLs in all those surfaces mentioned above can be solved completely? By closely following the observation in Ref. 18, it is quite clear that the method developed in Ref. 18 can be easily generalized to the case of any *constant* curvature surface

with a *constant* magnetic field applied on the surface (the surface can be a compact and closed surface, or an open surface, for example, an open up-half hyperbolic plane), and thus the problem of the LLs can be solved exactly in such cases.

We will show that, if the curvature and magnetic field are constant, we can get many informations about the spectrum and the degeneracy of the LLs without solving the wave functions of the LLs explicitly by using a simple geometric argument,<sup>18</sup> even though the surface can be a very complicated one. If the magnetic field is constant, the wave functions of the ground states turn out to be a holomorphic line bundle defined on the surface. If the curvature of the surfaces is constant too, for the high LL, the wave functions of the LLs are obtained by repeatedly applying covariant derivatives on some holomorphic line bundles (which will be specified later). The spectrum is obtained without solving the wave functions explicitly and the degeneracy of the LLs can be obtained by the Riemann-Roch theorem. If the sections of some holomorphic line bundles can be obtained, the wave functions of the LLs can be obtained explicitly.

We use two simple examples to demonstrate how to use this geometric approach to solve the LLs. The examples are the LLs on the sphere and the open up-half hyperbolic plane.

### 1. Ground States

We will show here that, when the magnetic field is constant, the ground states satisfies a first-order holomorphic (or anti-holomorphic) differential equation and the ground states belong to the sections of a holomorphic line bundle.

We consider a particle on a surface interacting with a magnetic field. In complex coordinates, the metric is  $ds^2 = g_{z\bar{z}}dzd\bar{z}$  and the volume form is  $dv = [ig_{z\bar{z}}/2]dz \wedge d\bar{z} = g_{z\bar{z}}dx \wedge dy$ . The natural definition of the constant magnetic field to the high genus Riemann surface is

$$F = Bdv = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)dz \wedge d\bar{z}, \quad (\text{A1})$$

where  $B$  is a constant. Thus we have  $\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = ig_{z\bar{z}}B/2$ . If the surface is closed, the magnetic field is then called “monopole” field and subjected to the Dirac quantization

condition. The flux  $\phi$  ( $\phi$  must be an integer) is given by  $2\pi\phi = \int F = BV$ , where  $V$  is the area of the surface and we assume here  $B > 0$  ( $\phi > 0$ ) for simplicity. The Hamiltonian of a particle on the surface is given by the following equation,

$$\begin{aligned} H &= \frac{1}{2m\sqrt{g}}(P_\mu - A_\mu)g^{\mu\nu}\sqrt{g}(P_\nu - A_\nu) \\ &= \frac{g^{z\bar{z}}}{m}[(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + (P_{\bar{z}} - A_{\bar{z}})(P_z - A_z)] \\ &= \frac{2g^{z\bar{z}}}{m}(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + \frac{B}{2m}, \end{aligned} \quad (\text{A2})$$

where  $g^{z\bar{z}} = 1/g_{z\bar{z}}$ ,  $P_z = -i\partial_z$ ,  $P_{\bar{z}} = -i\partial_{\bar{z}}$ ,  $\partial_z = (\partial_x - i\partial_y)/2$ , and  $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$ . We define the inner product between two wave functions as  $\langle \psi_1 | \psi_2 \rangle = \int dv \bar{\psi}_1 \times \psi_2$ .

Define  $H' = \frac{2g^{z\bar{z}}}{m}(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}})$ .  $H'$  is a positive definite hermitian operator because  $\langle \psi | H' | \psi \rangle \geq 0$  for any  $\psi$ . If  $H'\psi = 0$ ,  $\psi$  must satisfy  $(P_{\bar{z}} - A_{\bar{z}})\psi = 0$ . The solutions of this equation are the ground states of the Hamiltonian  $H$  or  $H'$ . In the case of closed compact surface, the existence of the solutions of this equation is guaranteed by the Riemann-Roch theorem.<sup>30,31</sup> The solutions belong to the sections of the holomorphic line bundle with the connection given by the gauge field. The Riemann-Roch theorem tells us that

$$h^0(L) - h^1(L) = \text{deg}(L) - h + 1, \quad (\text{A3})$$

where  $h$  is the genus of the surface,  $h^0(L)$  is the dimension of the sections of the holomorphic line bundle or the degeneracy of the ground states of the Hamiltonian  $H$ ,  $h^1(L)$  is the dimension of the holomorphic differential ( $L^{-1} \times K$ ), where  $K$  is the canonical bundle, and  $\text{deg}(L)$  is the degree of the line bundle, which is equal to the first Chern number of the gauge field, or the magnetic flux out of the surface,  $\phi$ . When  $\text{deg}(L) > 2h - 2$ ,  $h^1(L)$  is equal to zero,<sup>30</sup> thus  $h^0(L) = \phi - h + 1$ . One finds that  $h^0(L)$  indeed gives the right degeneracy of the ground states in the case of a particle on a sphere or a torus interacting with a magnetic-monopole field.

In the case of non-compact surfaces, for example an infinite plane or an up-half hyperbolic plane, the flux out of the surfaces are infinite, and the degeneracy is infinite too. The



degeneracy of the LLs turns out to be infinite. Thus Eq. (A3) also gives correctly the degeneracy, as when the flux is infinite, the equation implies that  $h^0(L)$  becomes infinite. When the surface has a boundary, for example a disc, one would expect that Eq. (A3) is replaced by a new index relation given by the boundary index theory. Note that, when the flux is much bigger than one, the degeneracy of the ground states are approximately equal to the flux  $\phi$  out of the surface.

## 2. Higher Landau Levels

We study the higher LLs in the case of the curvature of the surface being constant. When the curvature is constant,  $g^{z\bar{z}}\partial\bar{\partial}\ln g_{z\bar{z}} = C$ , a Liouville-like integrable equation. For the flat surface,  $C = 0$ , as in the case of a plan or a torus, the spectrum and the wave functions of the LLs can be completely solved. When the surface is flat, the higher LLs are obtained by applying successively a first order differential operator to the states in the LLL. Now we shall generalize such construction of the LLs in the case of a flat surface to the case of a curved surface.

We consider here the *closed* and *curved* surface with constant (non-zero) curvatures. It is easy to generalize to the case of an open surface with a constant curvature and we will demonstrate it in an example in the end of the appendix. When  $C$  is not equal to zero, one has  $g_{z\bar{z}} = (1/C)\partial\bar{\partial}\ln g_{z\bar{z}}$ . As the magnetic field is constant, we can fix the gauge field as  $A_z = -iB'\partial(\ln g_{z\bar{z}})/2$ , and the magnetic field  $F$  is equal to  $Bdv$ , where  $B = 2B'C$ . For example, in the case of the Poincaré metric,  $ds^2 = y^{-2}(dx^2 + dy^2)$ ,  $g_{z\bar{z}} = y^{-2}$ , and  $C = 1/2$ , thus  $B = B'$ . For a closed surface, by Gauss theorem, the flux  $\phi$  out of the surface is equal to  $\phi = B(h - 1)/c = 2B'(h - 1)$ .  $B'$  must be a rational number as  $\phi$  is an integer. For the negative curvature closed surface, according to Gauss theorem, we should have  $h \geq 2$ . On the other hand, for the positive constant curvature surface,  $h$  must be equal to zero, and thus the surface is topologically equivalent to a sphere. Without losing any generalities, we assume in the following discussions that  $B$  is a positive number. For a negative  $B$ , the wave

functions are the complex conjugate of the wave functions in the case of a positive  $B$ .

For any eigenfunctions of the Hamiltonian, they satisfy

$$H\psi = E\psi. \quad (\text{A4})$$

If the domain of  $\tilde{z}$  intersects non-trivially the domain of  $z$ ,  $g_{z\tilde{z}}dzd\tilde{z}$  is invariant under coordinate changes, or

$$g_{z\tilde{z}}dzd\tilde{z} = g_{\tilde{z}\tilde{z}}d\tilde{z}d\tilde{\tilde{z}} \quad (\text{A5})$$

on the intersection of the domains of  $z$  and  $\tilde{z}$ . Define

$$D = \partial - (B'/2)\partial \ln g_{z\tilde{z}}, \quad \bar{D} = \bar{\partial} + (B'/2)\bar{\partial} \ln g_{z\tilde{z}}. \quad (\text{A6})$$

$D$  and  $\bar{D}$  are transformed as

$$\tilde{D} = (dz/d\tilde{z})U^{-1}DU, \quad \tilde{\bar{D}} = (d\tilde{z}/d\tilde{\tilde{z}})U^{-1}\bar{D}U \quad (\text{A7})$$

where  $U(z, \tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^{-B'/2} \left(\frac{d\tilde{z}}{d\tilde{\tilde{z}}}\right)^{B'/2}$ .

We take  $m = 2$  in Eq. (A2) for the simplicity. The Hamiltonian can be written in the following form,

$$H = -g^{z\tilde{z}}D\bar{D} + (B/4). \quad (\text{A8})$$

Thus the Hamiltonian in the domain  $z$  is transformed to the Hamiltonian in the domain  $\tilde{z}$  as

$$\tilde{H} = U^{-1}HU, \quad (\text{A9})$$

and the wave function is transformed as

$$\tilde{\psi} = U^{-1}\psi. \quad (\text{A10})$$

Therefore  $\psi(dz)^{B'/2}(d\tilde{z})^{-B'/2}$  is invariant under the transformation, and it implies that  $\psi$  is a differential form of type  $T_{B'/2}^{B'/2}$ , where we use the following notation: if  $F(z, \tilde{z})(dz)^X(d\tilde{z})^Y$  is invariant under the transformation, then  $F(z, \tilde{z})$  is a differential form of type  $T_X^{-Y}$ .

The ground states are given by the solutions of the equation  $\bar{D}\psi = 0$ . When the curvature is negative,  $C, \phi, B'$  are positive numbers. If  $\phi > 2h - 2$ , or  $B' > 1$ , then the number of the solutions is  $\phi - h + 1$  according to the previous discussions. For smaller  $\phi$ , some discussions can be found in Ref. 18. In the case of compact and closed Riemann surfaces with the Poincaré metric, the wave functions in LLL were constructed by calculating the determinant of holomorphic sections of some bundle.<sup>18</sup>

When the curvature is positive,  $C$  and  $B'$  are negative numbers, and  $h = 0$  as shown in the previous discussions. Now  $\phi$  is equal to  $\phi = -2B'$ . The LLL states are again given by the solutions of the equation  $\bar{D}\psi = 0$ . As  $|\phi| > 2h - 2 = -2$  ( $h = 0$  in this case), the number of the solutions is equal to  $|\phi| - h + 1 = |\phi| + 1$ .

To obtain the spectrum and wave functions of the higher LLs, we introduce the covariant derivative,<sup>18</sup>  $\nabla_z$ , and its Hermitian conjugate  $(\nabla_z)^\dagger = -\nabla^z$ ,

$$\nabla_z : T_k^l \rightarrow T_{k+1}^l, \nabla_z = g^k \partial g^{-k}, \quad (\text{A11a})$$

$$(\nabla_z)^\dagger : T_k^l \rightarrow T_{k-1}^l, (\nabla_z)^\dagger = -g^{-l-1} \bar{\partial} g^l, \quad (\text{A11b})$$

where we call  $g = g_{z\bar{z}}$  for short. Note that  $D$  is the covariant operator  $\nabla_z$  acting on  $T_{B'/2}^{\bar{B}'/2}$ , and  $\bar{D} = g\nabla^z$  where  $\nabla^z$  acts on  $T_{B'/2}^{\bar{B}'/2}$ . The Hamiltonian can be written by using the covariant operators,

$$H - B/4 = -\nabla_z \nabla^z. \quad (\text{A12})$$

One can verify the commutation relation,

$$[\nabla^z \nabla_z] T_n^m = -(m+n)C. \quad (\text{A13})$$

Assume that  $\psi_1$  is a state in the higher LLs and an eigenfunction of  $H$  with eigenvalue  $E_1$ , then  $\psi_1 = -\frac{1}{\epsilon_1} \nabla_z \nabla^z \psi_1$ , where  $\epsilon_1 = E_1 - B/4 > 0$ . Therefore one can write  $\psi_1 = \nabla_z \Phi(1)$ , where  $\Phi(1)$  is a differential form of type  $T_{B'/2-1}^{\bar{B}'/2}$ . More explicitly, we have

$$\psi_1 = (\partial - (B'/2 - 1)\partial \ln g)\Phi(1). \quad (\text{A14})$$

Using the relation  $\partial\bar{\partial}\ln g = gC$ , one can show that,

$$-\nabla_z\nabla^z\psi_1 = (B' - 1)C\psi_1 + \nabla_z[-\nabla_z\nabla^z\Phi(1)]. \quad (\text{A15})$$

We first discuss the case of a negative curvature surfaces. If  $B' \geq 1$ , one can show that  $\langle \psi_1 | \nabla_z[-\nabla_z\nabla^z\Phi(1)] \rangle \geq 0$ . Thus one can conclude that the states of the lowest excited level are obtained, if there exist  $\Phi$  such that  $\nabla_z\nabla^z\Phi(1) = 0$ .  $\nabla_z\nabla^z\Phi(1) = 0$  leads to  $\bar{D}\Phi(1) = 0$ . The solution of  $\bar{D}\Phi(1) = 0$  is  $\Phi(1) = g^{-B'/2}\tilde{\Phi}(1)$  with  $\bar{\partial}\tilde{\Phi}(1) = 0$ , where  $\tilde{\Phi}(1)$  is of the form  $T_{B'-1}$ . By the Riemann-Roch theorem, there exist solutions of the equation  $\bar{\partial}\tilde{\Phi}(1) = 0$  for  $B' \geq 1$ , and the number of the solutions or the degeneracy of this Landau level is the dimension of the sections of the holomorphic bundle  $T_{B'-1}$ , which is equal to  $(2B' - 3)(h - 1)$  if  $B' > 2$ . The energy of this LL or the lowest excited states is

$$E_1 = \frac{3}{2}B'C - C. \quad (\text{A16})$$

If  $B' < 1$ , there is only the *zero'th* "Landau level" or the LLL (there exists a continue spectrum in the high-energy sector, and the states in the continue spectrum are not called as the states in the LLs).

We can generalize the above discussion to higher LLs. The wave function of the *k'th* LL is given by

$$\begin{aligned} \psi_k &= (\nabla_z)^k\Phi(k) \\ &= (\partial - (B'/2 - 1)\partial\ln g)(\partial - (B'/2 - 2)\partial\ln g) \\ &\quad \times \cdots (\partial - (B'/2 - k)\partial\ln g)\Phi(k), \end{aligned} \quad (\text{A17})$$

with  $\Phi(k) = g^{-\frac{B'}{2}}\tilde{\Phi}(k)$  and  $\bar{\partial}\tilde{\Phi}(k) = 0$ .  $\tilde{\Phi}(k)$  is a differential form of the type  $T_{B'-k}$ . Notice that this construction generalizes the standard construction for the harmonic oscillator. The difference for the constructions of the high LLs between the case of the flat surfaces and the case of the curved surfaces is clear now. In the case of the surface being a plan or a torus, the high LLs are obtained by applying successively a first order differential operator to the ground states. However the situation is different when the surface is curved.  $\Phi(k)$  for  $k \neq 0$  is *not* the ground state of the Hamiltonian  $H$ .

Using Eq. (A13), we calculate the eigenvalue of the corresponding wave function  $\psi_k$ , and it is equal to

$$E_k = CB'(k + \frac{1}{2}) - \frac{k(k+1)C}{2}. \quad (\text{A18})$$

The degeneracy of the  $k$ 'th LL is given by the dimension of the sections of the holomorphic bundle of the type  $T_{B'-k}$ , which is equal to  $(2B' - 2k - 1)(h - 1)$  when  $B' - k > 1$ . Because the dimension of  $T_n$  is zero when  $n$  is negative,  $k$  must not be greater than  $B'$ . Hence there is only a *finite* number of "Landau levels".

When  $B'$  is an integer,  $k$  can take value from 0 to  $B'$ . When  $k = B'$ , the corresponding  $\tilde{\Phi}(k)$  is the differential form of the type  $T_0$ .  $T_0$  is a constant function on the surface and the degeneracy of this LL is equal to one. For the twisted boundary conditions, which would physically correspond to the presence of some magnetic flux through the handles. There does not exist a non-zero constant function which satisfies the twisted boundary condition, thus the dimension of  $T_0$  is zero, and the degeneracy of this LL is equal to zero or there does not exist this LL ( $B' - th$  LL). When  $k = B' - 1$ , the degeneracy of this LL is the dimension of the canonical bundle  $T_1$ , which is equal to  $h$  for the non-twisted boundary condition and is equal to  $h - 1$  for the twisted boundary condition (this result can be obtained by the Riemann-Roch theorem).  $B$  could be also an half-integer. Then  $k$  can take value from 0 to  $B' - (1/2)$ . When  $k = B' - (1/2)$ , the degeneracy of this LL is the dimension of the spin bundle  $T_{1/2}$ . The dimension of the holomorphic sections of the spin bundle generically is zero for the even-spin structures and one for the odd ones (or for twisted ones). It is possible that  $B'$  is fractional assuming that  $2B'(h - 1)$  is an integer, and  $k$  can take value from 0 to  $[B']$  where  $[B']$  is the bigger integer which is smaller than  $B'$ . When  $k = [B']$ , the degeneracy of this LL is the dimension of bundle  $T_{B'-[B']}$ .  $B' - [B']$  is a fractional number between 0 and 1 and the discussions of such case can be found in Ref. 18. Beyond those LLs, little is known about the continue spectrum in the case of the complicated negative curvature surfaces.

To normalize  $\psi_k$ , we calculate the inner product  $\langle \psi_k | \psi_k \rangle$ . By using Eq. (A13). It is

given by the following equation,

$$\begin{aligned}
\langle \psi_k | \psi_k \rangle &= \langle \nabla_z^k \Phi(k) | \nabla_z^k \Phi(k) \rangle \\
&= \langle \Phi(k) | (\nabla_z^k)^\dagger \nabla_z^k \Phi(k) \rangle \\
&= \langle \Phi(k) | \Phi(k) \rangle C^k 2^{-k} k! \\
&\quad \times \prod_{i=1}^k (2B' - k - i),
\end{aligned} \tag{A19}$$

where the inner product  $\langle \Phi(k) | \Phi(k) \rangle$  is defined as

$$\langle \Phi(k) | \Phi(k) \rangle = \int dv g^k \bar{\Phi}(k) \times \Phi(k). \tag{A20}$$

The definition of the inner product between two  $\Phi(k)$  given in Eq. (A20) is quite natural because  $\Phi(k)$  is a differential form of the type  $T_{(B'/2)-k}^{\bar{B}'/2}$ .

If  $\Phi(k)$  is normalized to one, then

$$\frac{\psi_k}{[C^k k! 2^{-k} \prod_{i=1}^k (2B' - k - i)]^{1/2}} \tag{A21}$$

is also normalized to one.

Now we come to the case of a closed surface with a positive curvature, which is a little bit different from the case of a surface with a negative curvature. Now we have only  $h = 0$  according to the previous discussion. The wave function  $\psi$  is a differential form of type  $T_{B'/2}^{\bar{B}'/2}$  with  $B'$  being a negative number. In the formula  $-\nabla_z \nabla^z \psi_1 = (B' - 1)C\psi_1 + \nabla_z [-\nabla_z \nabla^z \Phi(1)]$ , one can show that  $\langle \psi_1 | \nabla_z (-\nabla_z \nabla^z \Phi) \rangle \geq 0$  for any negative  $B'$ . By using Riemann-Roch theorem, one finds that there *always* exists  $\Phi(1)$  such that  $\nabla_z \nabla^z \Phi(1) = 0$ , which leads to  $\bar{D}\Phi = 0$ . Therefor for any  $B'$ , there exists a higher LL. One can repeat the argument to obtain the states in the higher LLs and obtains the *full* spectrum and wave functions.

The wave functions of the states in the  $k'$ th LL are again given by Eq. (A17), with  $\tilde{\Phi}(k) = g^{B'/2} \Phi(k)$  and  $\bar{\partial} \tilde{\Phi}(k) = 0$ .  $\tilde{\Phi}(k)$  is a differential form of the type  $T_{B'-k}$ . The degeneracy of the  $k'$ th LL is equal to the dimension of the holomorphic line bundle  $T_{B'-k}$ , which is equal to  $2(B' - k)(h - 1) - h + 1 = -2(B' - k) + 1$  as  $h$  is equal to zero.

The energy is again given by Eq. (A18). However, a higher LL has a *higher* degeneracy and the number of the LLs is *infinite* in such case. Instead, in the case of a surface with a negative curvature, a higher LL has a *smaller* degeneracy. and the number of the LLs is *finite*. From Eq. (A18), one notices that, in the case of a positive curvature surface, the energy gap in the neighboring LLs *increases* when the level increases, and in the case of a negative curvature surface, the energy gap in the neighboring LLs *decreases* when the level increases.

It is easy to generalize the above discussions to non-compact surfaces, and we will work out an example in the following discussion.

### 3. Examples

#### a. Upper half hyperbolic surface

We consider that the surface is a upper half hyperbolic surface (also see Comtet and Dunne in Ref. 29). In the projective coordinates. the metric  $g$  is written as  $\frac{1}{(1-z\bar{z})^2}$ , where  $|z| \leq 1$ , The other quantities are,  $C = 2$ ,  $B = 4B'$ , and  $A_z = -i\frac{B'\bar{z}}{(1-z\bar{z})^2}$ . The wave functions are given by Eq. (A17). As the wave functions of the LLs shall be normalizable (opposite to the wave function of a state inside the continue spectrum),  $\langle \psi_k | \psi_k \rangle$  shall be normalizable. A normalizable  $\langle \psi_k | \psi_k \rangle$  is equivalent to a normalizable  $\langle \Phi(k) | \Phi(k) \rangle$ . A normalizable  $\langle \Phi(k) | \Phi(k) \rangle$  leads the condition  $B' - (1/2) > k \geq 0$ .  $\Phi(k)$  is given by function  $g^{-B'/2} z^l$  where  $l$  is a non-negative integer. Thus the degeneracy is infinite for every LL. This is consistent with the Riemann-Roch theorem as the flux out of the surface is infinite. Finally, the energy is given by Eq. (A18).

#### b. Sphere

Another example is that the surface is a sphere. In the projective coordinates. the metric  $g$  is written as  $g = \frac{1}{(1+z\bar{z})^2}$ . The other quantities are,  $A_z = i\frac{B'\bar{z}}{(1+z\bar{z})^2}$ ,  $C = -2$ , and

$B = 2B'C = -4B'$ . Thus the flux  $\phi = -2B'$  is a non-negative integer according to the Dirac quantization condition (note we always assume  $B > 0$  in this paper). The wave functions are again given by Eq. (A17), the energies are given by Eq. (A18), and  $\Phi(k)$  is given by  $g^{\phi/4} z^l$ . The normalizable condition leads to  $l = 0, 1, \dots, 2k + \phi$ . Thus the degeneracy of the  $k'$ th LL is equal to  $2k + \phi + 1$ . The degeneracy can be also obtained by the Riemann-Roch theorem and the result is consistent with the result obtained by requiring the wave functions being normalizable. In this way, we obtain the full spectrum and all wave functions on the sphere.

The wave functions at the  $n'$ th Landau level ( $n = 0$  is the lowest Landau level) are given by Eq. (7).

From previous discussions, we can easily find the inner product  $\langle \psi_{n,l} | \psi_{n,l} \rangle$  is equal to  $\pi \frac{l!(\phi+2n-l)!}{(\phi+2n+1)(\phi+n)!}$ . The inner product is as previously defined,  $\langle \psi_1 | \psi_2 \rangle = \int dv \bar{\psi}_1 \times \psi_2$ , where  $dv = \int \frac{dx dy}{(1+z\bar{z})^2}$ .

However inside the paper, The definition of the inner product is *different* from the definition in the appendix. The inner product in the paper is *defined* as

$$\langle \psi_1 | \psi_2 \rangle = \int \frac{dz d\bar{z}}{(1+z\bar{z})^2} \bar{\psi}_1 \times \psi_2. \quad (\text{A22})$$

As  $dz d\bar{z} = 2dx dy$ , thus  $\langle \psi_{n,l} | \psi_{n,l} \rangle$  is given by the following formula,

$$\langle \psi_{n,l} | \psi_{n,l} \rangle = 2\pi \frac{l!(\phi + 2n - l)!}{(\phi + 2n + 1)(\phi + n)!}. \quad (\text{A23})$$

This formula is used in the paper.



## TABLES

TABLE I. In this table, we list the number of rotational invariant states at various  $\nu$  and a small number of electrons.  $N_t$  is the dimension of the total Hilbert space (in the LLL) and  $N_r$  is the number of the rotational invariant states.

$\nu$	$N$	$N_t$	$N_r$	$\phi$ (formula)	$\phi$	$\frac{N\phi}{2}$
$\frac{2}{5}$	4	5	1	$\frac{5}{2}N - 4$	6	12
	6	58	3		11	33
	8	910	8		16	64
$\frac{2}{7}$	4	43	2	$\frac{7}{2}N - 2$	12	24
	6	1.242	10		19	57
	8	46.029	80		26	104
$\frac{2}{9}$	4	43	2	$\frac{9}{2}N - 6$	12	24
	6	2.137	13		21	63
	8	139.143	164		30	120
$\frac{2}{11}$	4	150	3	$\frac{11}{2}N - 4$	18	36
	6	11.963	29		29	87
	8	1.229.093	702		40	160
$\frac{2}{13}$	4	150	3	$\frac{13}{2}N - 8$	18	36
	6	17.002	34		31	93
	8	2.502.617	1.137		44	176
$\frac{3}{7}$	9	910	8	$\frac{7}{3}N - 5$	16	27
$\frac{3}{11}$	6	2.137	13	$\frac{11}{3}N - 1$	21	63
	9	610.358	506		32	144
$\frac{3}{17}$	6	17.002	34	$\frac{17}{3}N - 3$	31	93
$\frac{5}{17}$	4	33	2	$\frac{17}{5}N - \frac{13}{5}$	11	22
	9	184.717	217		28	126

TABLE II. The overlaps between the hierarchical wave functions and the CF wave functions at some fillings for a small number of electrons.

$\nu$	$N$	overlap			
$\frac{2}{5}$		$\langle PD\chi_2   E\Psi_3 \rangle$	$\langle \Psi_1 P\Psi_1 \chi_2   E\Psi_3 \rangle$	$\langle PD\chi_2   \Psi_1 E\Psi_1^2 \rangle$	$\langle \Psi_1 P\Psi_1 \chi_2   \Psi_1 E\Psi_1^2 \rangle$
	6	.9993234149	.9993615971	.9998331523	.9999456457
	4	1	1	1	1

$\nu$	$N$	overlap		$\nu$	$N$	overlap
$\frac{2}{5}$		$\langle E\Psi_3   \Psi_1 E\Psi_1^2 \rangle$	$\langle PD\chi_2   \Psi_1 P\Psi_1 \chi_2 \rangle$	$\frac{2}{7}$		$\langle H\Psi_3   DC\Psi_3 \rangle$
	6	.9996479001	.9999288987		6	.9993762574
	4	1	1		4	1

$\nu$	$N$	overlap		
$\frac{2}{9}$		$\langle E\Psi_5   DPD\chi_2 \rangle$	$\langle E\Psi_5   PD^2\chi_2 \rangle$	$\langle DPD\chi_2   PD^2\chi_2 \rangle$
	4	.9999614869	.9999614869	1

$\nu$	$N$	overlap	$\nu$	$N$	overlap
$\frac{3}{11}$		$\langle EH\Psi_3   DCPD\chi_2 \rangle$	$\frac{2}{11}$		$\langle H\Psi_5   D^2C\Psi_3 \rangle$
	6	.9996522383		4	1

$\nu$	$N$	overlap	$\nu$	$N$	overlap
$\frac{2}{13}$		$\langle E\Psi_7   DPD^2\chi_2 \rangle$	$\frac{5}{17}$		$\langle HH\Psi_3   DCDC\Psi_3 \rangle$
	4	.9999218859		4	.9999999997

## REFERENCES

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- <sup>1</sup> R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
- <sup>2</sup> R.E. Prange and S.M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, New York, 1990); R. Laughlin, in *Fractional Statistics and Anyon Superconductivity*, edited by F. Wilczek (World Scientific, Singapore, 1990).
- <sup>3</sup> F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
- <sup>4</sup> B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
- <sup>5</sup> D. Arovas, R. Schrieffer and F. Wilczek, Phys. Rev. Lett **53**, 722 (1984).
- <sup>6</sup> S. Girvin, Phys. Rev. B **29**, 6012 (1984); A.H. MacDonald and D.B. Murry, *ibid.* **32**, 2707 (1985); M.P.A. Fisher and D.H. Lee, Phys. Rev. Lett. **63**, 903 (1989).
- <sup>7</sup> N. Read, Phys. Rev. Lett. **65**, 1502 (1990).
- <sup>8</sup> B. Blok and X.G. Wen, Phys. Rev. B **42**, 8133 (1990); **42**, 8145 (1990); **43**, 8337 (1991).
- <sup>9</sup> M. Greiter, Phys. Lett. B **48**, 3336 (1994).
- <sup>10</sup> J. Yang, Phys. Rev. B **50**, 11196 (1994).
- <sup>11</sup> J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **41**, 7653 (1990); Adv. Phys. **41**, 105 (1992).
- <sup>12</sup> X.G. Wen, preprint, cond-mat/9506066, and references therein.
- <sup>13</sup> F.D.M. Haldane and E.H. Rezayi, Phys. Rev. B **33**, 3844 (1986); D. Li, Int. J. Mod. Phys. B **7**, 2779 (1993); **7**, 2655 (1993); E. Keski-Vakkuri and X.G. Wen, *ibid.* **7**, 4227 (1993).
- <sup>14</sup> F. Lesage, V. Pasquier and D. Serban, Nucl. Phys. B **435**, 585 (1995); Z.N.C. Ha, *ibid.*

- 435, 604 (1995).
- <sup>15</sup> T.T. Wu and C.N. Yang, Nucl. Phys. B **107**, 365 (1976).
- <sup>16</sup> G. Fano, F. Ortolani and E. Colombo, Phys. Rev. B **34**, 2670 (1986).
- <sup>17</sup> D. Li, Nucl. Phys. B(FS) **396**, 411 (1993); R. Iengo e K. Lechner, Phys. Reports **213**, 179 (1992).
- <sup>18</sup> R. Iengo and D. Li, Nucl. Phys. B(FS) **413**, 735 (1994); and references therein.
- <sup>19</sup> S.M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984).
- <sup>20</sup> B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1994); E. Reazayi and N. Read, Phys. Rev. Lett. **72**, 900 (1994); N. Read, preprint, cond-mat/9501090, and references therein.
- <sup>21</sup> G.E. Andrews, *The Theory of Partitions* (Addison-Wesley Publishing Company, Massachusetts, 1976).
- <sup>22</sup> C. Itzykson, In *Quantum Field Theory and Quantum Statistics*, Vol. 1, edited by I. A. Batalin, *et al.* (Adam Hilger, Bristol, 1987); B. Derrida and J.P. Nadal, J. Phys. Lett. (Paris) **45**, 701 (1984).
- <sup>23</sup> J.K. Jain and X.G. Wu, preprint, cond-mat/9312091.
- <sup>24</sup> D. Arovas, in *Geometric Phase in Physics*, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989)
- <sup>25</sup> Carmem Lucia de Souza Batista, Master thesis, CBPF, Rio de Janeiro, 1996.
- <sup>26</sup> P. Di Francesco, M. Gaudin, C. Itzykson and F. Lesage, Int. J. Mod. Phys. A **9**, 4257 (1994); G. Dunne, Int. J. Mod. Phys. B **7**, 4783 (1993).
- <sup>27</sup> I.G. MacDonald, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxford, 1979).

- <sup>28</sup> H. Azuma and S. Ito, Phys. Lett. B **331**, 107 (1994); P.K. Panigrahi and M. Sivakumar, Phys. Rev. B **52**, 13742 (1995).
- <sup>29</sup> P. Dirac, Proc. R. Soc. London A **133**, 60 (1931); I. Tamm, Z. Phys. **71**, 141 (1931); M. Fierz, Helv. Phys. Acta. **17**, 27 (1944); Ref. 15; A. Comtet, Ann. Phys. **173**, 185 (1987), and references therein; G.V. Dunne, Ann. Phys. **215**, 233 (1992), and references therein; J. Zak, Phys. Rev. **134**, A1607 (1994).
- <sup>30</sup> E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Vol. 1 (Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985).
- <sup>31</sup> H.M. Farkas and I. Kra, *Riemann Surfaces* (Springer-Verlag, Berlin, 1980).