CBPF-NF-041/02

A New Approach of the Stationary Axisymmetric Vacuum S(A) Solutions

J. Gariel[1*], G. Marcilhacy[1] and *N. O. Santos*[1,2,3†]

[1]LERMA/CNRS-FRE 2460, Université Pierre et Marie Curie, ERGA,
Boîte 142, 4 place Jussieu, 75005 Paris Cedex 05, France.
[2]Laboratório Nacional de Computação Científica,
25651-070 Petrópolis RJ, Brazil.
[3]Centro Brasileiro de Pesquisas Físicas,
22290-180 Rio de Janeiro RJ, Brazil.

Abstract

We revisit axisymmetric stationary vacuum solutions of the Einstein equations, like we did for the cylindrical case [1]. We explicitly formulate the simplest hypothesis under which the S(A) solutions, or axisymmetric Lewis solutions can be found and demonstrate that this hypothesis leads to a linear relation between the potentials. We show that the field equations still can be associated to the motion of a classical particle in a central field, where an arbitrary harmonic χ function plays the role of time. Three classes of solutions are obtained without the need of invoking the Papapetrou class. They depend on two real parameters, and the potentials are functions of χ only. The new approach exempts the need of complex parameters. We interpret one of the parameters as related to the vorticity of the source.

*e-mail: gariel@ccr.jussieu.fr
†e-mail: santos@ccr.jussieu.fr and nos@cbpf.br

I Introduction

Axially symmetric stationary vacuum spacetimes in Einstein's theory are important because they can describe the exterior fields of massive rotating astrophysical objects [2, 3]. Here we reexamine the S(A) class of solutions of these spacetimes (see [8] p. 204).

In a preceding paper [1] we have already reexamined the vacuum solutions obtained by Lewis [4], and van Stockum [5], for a stationary cylindrically symmetric spacetime. Lewis established the existence of three classes of solutions in terms of four parameters. One of these classes appeared by the introduction of complex parameters. Through our approach the three classes arised without the need of complexification. We cannot use the Ernst formalism [6, 7] in the cylindrical case since the partial differential equations which link the dragging ω to the twist potential Φ become ill defined. Furthermore, we showed that the structure of the field equations can be associated to the motion of a classical particle in a central field. This association allowed a kinematical interpretation of the parameters, describing the Lewis spacetime without the need of specifying a particular matter source of the field.

Here we extend our analysis to the axisymmetric case. In order to proceed, we formulate the fundamental hypothesis (section III) which allows the employment of our method. By doing this we obtain directly the S(A) solutions without making use of the Papapetrou class [8] as it is usually done. Thus the S(A) solutions arise as a natural extension to the axisymmetric case of the Lewis solutions. Hence these solutions could be appropriately called the axisymmetric Lewis solutions. Then we follow some similar steps of the paper [1] and show that the classification and mechanical interpretation used in the cylindrical case can be extended, also, to the axisymmetric case.

The paper is organized as follows. In section II we recall the system of equations to be solved for the axially symmetric stationary vacuum metrics. We introduce in section III the fundamental hypothesis from which the linear dependence between the potentials is deduced. In section IV, we examine the main consequence of the kinematical role of the arbitrary harmonic functions of these solutions. The solutions and classification are presented in section V and its vorticity is calculated. We end with a brief conclusion.

II Field equations

The general line element for a stationary axisymmetric spacetime, with signature +2, can be written like

$$ds^2 = -fdt^2 + 2kdtd\phi + e^{\mu}(dr^2 + dz^2) + ld\phi^2, \tag{1}$$

where f, l, k and μ are all functions of the Weyl coordinates r and z. Defining for convenience,

$$f = rF(r,z), \quad l = rL(r,z), \quad k = rK(r,z), \tag{2}$$

we obtain from Einstein's vacuum field equations [4, 5],

$$\triangle F = -F\Omega, \tag{3}$$

$$\triangle L = -L\Omega, \tag{4}$$

$$\triangle K = -K\Omega, \tag{5}$$

$$\mu_r = -\frac{1}{2r}[1 + r^2(F_r L_r - F_z L_z + K_r^2 - K_z^2)], \tag{6}$$

$$\mu_z = -\frac{r}{2}(F_r L_z + F_z L_r + 2K_r K_z), \tag{7}$$

with

$$FL + K^2 = 1, \tag{8}$$

where the Laplacian $\triangle$ and Ω are defined by

$$\triangle F = F_{rr} + \frac{1}{r}F_r + F_{zz}, \tag{9}$$

$$\Omega = F_r L_r + K_r^2 + F_z L_z + K_z^2, \tag{10}$$

with the indexes standing for differentiation. The function μ is obtained by quadratures and, thus, we have only to determine F, L and K.

Let us note that the field equations (3)-(5) can also be written in the more symmetric form,

$$F\triangle L = L\triangle F, \tag{11}$$

$$L\triangle K = K\triangle L, \tag{12}$$

$$K\triangle F = F\triangle K. \tag{13}$$

III The fundamental hypothesis on the F, L and K functions

In the cylindrically symmetric case, where in (2) F, L and K, depend only on r, we have demonstrated the existence of a linear dependence between the potentials [1]. However, in the axially symmetric case, when F, L and K are functions of r and z, such a general demonstration is no longer possible. Thus, we have to introduce some further hypothesis to solve the field equations.

Keeping in mind the method used in the cylindrical case [1] we make the hypothesis that there exists a functional relation, different from (8), between F, L and K,

$$\Phi(F, L, K) = 0. \tag{14}$$

Then, from (8) and (14) we can obtain two general relations that can be expressed, for example, as

$$F = F(K), \quad L = L(K). \tag{15}$$

From (15) we have the identities,

$$\nabla F \cdot \nabla L + (\nabla K)^2 \equiv (1 + F_K L_K)(\nabla K)^2, \tag{16}$$
$$\triangle F \equiv F_K \triangle K + F_{KK}(\nabla K)^2, \tag{17}$$
$$\triangle L \equiv L_K \triangle K + L_{KK}(\nabla K)^2, \tag{18}$$

where ∇ is the gradient operator. With (15)-(18), we can rewrite the two first field equations (3) and (4) like

$$(1 + F_K L_K)(K F_K - F) = F_{KK}, \tag{19}$$
$$(1 + F_K L_K)(K L_K - L) = L_{KK}, \tag{20}$$

which is a system of two differential equations permitting to determine the functions (15), as we shall see (equations (41)). Hence, the only partial derivative equation to solve is the third field equation, (5), for the function $K(r, z)$,

$$\triangle K = -K(1 + F_K L_K)(K_r^2 + K_z^2). \tag{21}$$

A kinematical interpretation can be given from (19)-(21). Indeed, considering (19) multiplied by L and (20) by F and subtracting both equations, we obtain,

$$(1 + F_K L_K)K = \frac{(L F_K - F L_K)_K}{L F_K - F L_K}. \tag{22}$$

Without any loss of generality, we can make an arbitrary change of unknown function by putting $K = K(\chi)$, where $\chi(r, z)$ is a new unknown function. Then (21) becomes

$$K_\chi \triangle \chi = \left[f(K) - K_{\chi\chi}\right](\nabla \chi)^2, \tag{23}$$

where

$$f(K) = -K(1 + F_K L_K). \tag{24}$$

Always without loss of generality, we can fix this change of function such that $K(\chi)$ satisfies the differential equation

$$\frac{K_{\chi\chi}}{K_\chi^2} = f(K), \tag{25}$$

implying that χ is an harmonic function.

Let us examine what (25) implies on the two first field equations (19) and (20). Substituting (25) into (22) and integrating we obtain

$$L F_\chi - F L_\chi = C_1, \tag{26}$$

where C_1 is an integration constant. In a similar way, but starting from (4) and (5) with $L = L(F)$ and $K = K(F)$, and considering (3) with $F(\chi)$; and repeating again from (3) and (5) with $F(L)$ and $K(L)$ and considering (4) with $L(\chi)$, we obtain

$$K L_\chi - L K_\chi = C_2, \tag{27}$$
$$F K_\chi - K F_\chi = C_3, \tag{28}$$

respectively, where C_2 and C_3 are also integration constants.

The equations (26)-(28) express the conservation of an *angular momentum* $\vec{C} = (C_1, C_2, C_3)$ in the *space* (F, L, K), like in the cylindrical case [1], but here it is χ which plays the role of *time*, instead of $\tau = \ln r$ in [1]. In section IV we study consequences of this fact. Besides, from (26)-(28), we can immediately deduce a linear relation between the potentials,

$$K = \alpha L + \beta F, \tag{29}$$

where α and β are constants. The relation (29) is the one that we were looking for when we stated (14), and it describes a family of 2 parameters planes in the space (F, L, K). Hence, most of the interpretation in terms of a classical particle in a central field made in [1] holds here again. In particular, the discussion about the nature of the conic, which is the intersection of the surfaces (8) and (29) in the (F, L, K) space, followed in [1] for the cylindrical case, remains the same in the axisymmetric case.

Let us stress that all the results of this section can be obtained in the axisymmetric case only under the hypothesis (14), that we call the *fundamental hypothesis* for the S(A) class, while in the cylindrical case they were general, i.e. valid without any hypothesis. A well known counter example of an axisymmetric solution that does not satisfy this hypothesis is Kerr solution.

The linear dependence between the potentials (29) allow us to write this relation using the well known Papapetrou functions f_P and ω giving

$$f_P = r\left(\omega^2 + \frac{\omega}{\alpha} - \frac{\beta}{\alpha}\right)^{-1/2}. \tag{30}$$

We recognize from (30) the class S(A) (see [8] p. 204) of stationary vacuum solutions, which thus presents itself as the most natural generalization of the cylindrical class of Lewis solutions.

These solutions can also be named the axisymmetric Lewis solutions.

IV Consequences of the kinematical role of the harmonic function χ

In order to analyse these consequences we return to the cylindrically symmetric case. We give now an integration method of the $K(r)$ equation slightly different from the one presented in [1]. By doing this, we want to enlight the common feature of the two types of Lewis solutions, cylindric and axisymmetric, namely the fact that they only depend on a harmonic function. However, this function is imposed in the cylindric case, whereas it is arbitrary in the axial case.

In the cylindrical case, (21) with (29) reduces to

$$K_{rr} + \frac{1}{r}K_r - \frac{\delta K K_r^2}{\Delta} = 0, \tag{31}$$

with

$$\Delta \equiv \delta K^2 - 4\alpha\beta, \quad \delta \equiv 1 + 4\alpha\beta. \tag{32}$$

Changing the unknown function $K = K(\chi)$ in (31) in such a way that

$$\frac{K_{\chi\chi}}{K_\chi} = \frac{\delta K K_\chi}{\Delta} \tag{33}$$

leads to

$$\frac{\chi_{rr}}{\chi_r} = -\frac{1}{r}. \tag{34}$$

Consequently, after integration of (34), we obtain

$$\chi = k_1 \ln\left(\frac{r}{r_0}\right), \tag{35}$$

where k_1 and r_0 are integration constants, and, by integration of (33),

$$\int \frac{dK}{\sqrt{\Delta}} = k_1 \ln \frac{r}{r_0} + k_2, \tag{36}$$

where k_2 is an integration constant. The study of the integral (36) leads to the cylindrical solutions of Lewis [1]. Let us note that all these solutions depend only on the solution of the differential equation (34), i.e.,

$$\triangle\chi = \chi_{rr} + \frac{1}{r}\chi_r = 0, \tag{37}$$

which means that χ is a harmonic function. In this special case, of cylindrical symmetry, the differential equation (37) can be explicitly integrated, giving the only solution (35).

It is no longer the case in the more general axisymmetric situation, for which the corresponding equation (hereafter (40)) is a partial differential equation, even though the line reasoning remains the same. Indeed, coming back to (21), it can be written as

$$\triangle K = f(K)(\nabla K)^2. \tag{38}$$

The standard procedure of changing the unknown function $K = K(\chi)$ used in (23), gives now with (29),

$$\frac{K_{\chi\chi}}{K_\chi} - \frac{\delta K K_\chi}{\Delta} = 0. \tag{39}$$

With (39), (38) reduces to

$$\triangle\chi = 0. \tag{40}$$

We have that (39) is (33) with $\chi(r, z)$ arbitrary harmonic functions in place of the particular harmonic function, $\ln r$, convenient for the cylindrical case.

So, we can obtain from the functional hypothesis (14) the different classes of the Lewis solution by an analysis similar to the one used in the cylindric case [1].

V Three classes of axisymmetric solutions obtained from (29)

The solutions $K(\chi)$ of (39), expressed in terms of an arbitrary harmonic function $\chi(r,z)$ can be classified following the sign of δ, defined in (32), like in the procedure used in the cylindrical case [1].

The corresponding functions $F(\chi)$ and $L(\chi)$ are deduced from the relations

$$F = \frac{K \mp \sqrt{\Delta}}{2\alpha}, \quad L = \frac{K \pm \sqrt{\Delta}}{2\beta}, \tag{41}$$

obtained from (8) and (29). From (6)-(8) and (41) the potential μ obeys the equations

$$\mu_r = -\frac{1}{2r} + \epsilon\frac{r}{2}(\chi_r^2 - \chi_z^2), \tag{42}$$

$$\mu_z = \epsilon r \chi_r \chi_z, \tag{43}$$

with the following values for ϵ,

$$\epsilon = \begin{cases} +1, & \delta > 0 \\ 0, & \delta = 0 \\ -1, & \delta < 0. \end{cases}$$

For this axisymmetric spacetime we can calculate its vorticity vector Ω^α given by

$$\Omega^\alpha = \frac{\epsilon^{\alpha\beta\gamma\delta}}{2\sqrt{g}} u_\beta (u_{[\gamma;\delta]} + u_{[\alpha;\mu} u_{\delta]} u^\mu), \tag{44}$$

where u^α is a time like vector

$$u^\alpha = \frac{1}{\sqrt{-g_{tt}}}\delta_t^\alpha.$$

Calculating the scalar of (44) for (1) we obtain

$$\Omega^2 = g_{\alpha\beta}\Omega^\alpha\Omega^\beta = \frac{(KF_\chi - FK_\chi)^2}{4e^\mu F^2}\left(\chi_r^2 + \chi_z^2\right). \tag{45}$$

Some remarks about the vorticity of the S(A) solutions is presented in the conclusion. Finally, we present the three classes of solutions obtained, which are the following.

V.1 Class I: $\delta > 0$

V.1.1 $\alpha\beta > 0$

$$K = 2\left(\frac{\alpha\beta}{\delta}\right)^{1/2} \cosh\chi, \tag{46}$$

$$F = \left(\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}}\cosh\chi \mp \sinh\chi\right), \tag{47}$$

$$L = \left(\frac{\beta}{\alpha}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}}\cosh\chi \pm \sinh\chi\right). \tag{48}$$

V.1.2 $\alpha\beta < 0$ with $-\alpha\beta < 1/4$

$$K = 2\left(-\frac{\alpha\beta}{\delta}\right)^{1/2} \sinh\chi, \tag{49}$$

$$F = \left(-\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \sinh\chi \mp \cosh\chi\right), \tag{50}$$

$$L = \left(-\frac{\beta}{\alpha}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \sinh\chi \pm \cosh\chi\right). \tag{51}$$

V.1.3 $\alpha\beta = 0$

Here we use (8) and (29), instead of (41).

Case $\alpha = 0$ **and** $\beta \neq 0$

$$K = e^{\chi}, \tag{52}$$

$$F = \frac{1}{\beta} e^{\chi}, \tag{53}$$

$$L = \beta(e^{-\chi} - e^{\chi}). \tag{54}$$

From (45) with (52) and (53) we have $\Omega^2 = 0$.

Case $\alpha \neq 0$ **and** $\beta = 0$

$$K = e^{\chi}, \tag{55}$$

$$F = \alpha(e^{-\chi} - e^{\chi}), \tag{56}$$

$$L = \frac{1}{\alpha} e^{\chi}. \tag{57}$$

From (45) with (55) and (56) we have $\Omega^2 \neq 0$.

Case $\alpha = \beta = 0$

We use (3), (8) and (29) obtaining the Weyl static metric,

$$K = 0, \tag{58}$$

$$F = e^{\chi}, \tag{59}$$

$$L = e^{-\chi}. \tag{60}$$

This solution, without dragging, is an axisymmetric extension of the cylindrical Levi-Civita solution.

V.2 Class II: $\delta < 0$

We remark here, as we did in [1], that there is no need of introducing complex parameters in our approach, as it is usually done in the corresponding cylindrical case [8, 9, 10].

$$K = 2\left(\frac{\alpha\beta}{\delta}\right)^{1/2} \sin\chi, \tag{61}$$

$$F = \left(-\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{-\delta}} \sin\chi \mp \cos\chi\right), \tag{62}$$

$$L = \left(-\frac{\beta}{\alpha}\right)^{1/2} \left(\frac{1}{\sqrt{-\delta}} \sin\chi \pm \cos\chi\right). \tag{63}$$

V.3 Class III: $\delta = 0$ or $\alpha\beta = -1/4$

$$K = \chi, \tag{64}$$

$$F = \frac{1}{2\beta}(\chi \mp 1), \tag{65}$$

$$L = \frac{1}{2\alpha}(\chi \pm 1). \tag{66}$$

Here we can integrate (42) and (43) obtaining $e^{\mu} = c/\sqrt{r}$ where c is an integration constant. This class corresponds to the van Stockum's class [5] (see [8] p. 205).

VI Conclusion

The general solution of the cylindrically symmetric stationary vacuum Einstein's field equations is the Lewis solution. It is no longer the case for the more general equations with axial symmetry. We precised here the most general hypothesis under which we can find the axisymmetric solutions obtained by Lewis [4, 5]. This hypothesis (14) is a functional dependence between the potentials F, L and K different from (8), and allowed us to demonstrate a linear relation between the potentials. This fact implied that the field equations can be interpreted as describing the motion of a classical particle in a central force field, like in the cylindrical symmetric case [1]. We can recognize the solutions as belonging to the S(A) class (see [8] p. 204). We obtained these solutions without recalling to the Papapetrou class, as is usually done. These solutions depend upon an arbitrary harmonic function, and its classification in three classes is similar to the cylindrically symmetric case. Here again, as in [1], we do not need to appeal to complex constants, like in [9, 10]. This harmonic function plays the role of time in the motion of the precedent classical particle interpretation. It is interesting to observe in V.1.3, that for $\alpha = 0$ and $\beta \neq 0$ the vorticity scalar Ω vanishes, while for $\alpha \neq 0$ and $\beta = 0$ it does not. This shows a similarity with the corresponding solutions for the cylindrical case [1] where α is associated to the parameter that produces the vorticity of the source, as showed by [11, 9]. On the other hand, β in spite of being also associated to the stationarity of the source does not produce vorticity, but topological defect as shown in [11] and topological frame dragging demonstrated in [12]. For $\delta = 0$, in V.3, we have $e^{\mu} = c/\sqrt{r}$ which has the same r dependence as in the cylindrical system [11, 9] with energy density per unit length $\sigma = 1/4$. This class of solutions, like in the cylindrical case, is in the frontier between the two other corresponding classes.

Acknowledgment

NOS gratefully acknowledges financial assistance from CNPq, Brazil.

References

[1] Gariel J, Marcilhacy G and Santos N O, *J. Math. Phys.* **41**, 7535 (2000)

[2] Koide S, Shibata K, Kudoh T and Meier D L, *Science* **295**, 1688 (2002)

[3] van Putten M H P M and Levinson A, *Science* **295**, 1874 (2002)

[4] Lewis T, *Proc. R. Soc.* **136**, 176 (1932)

[5] van Stockum W J, *Proc. R. Soc. Edinburgh* **57**, 135 (1937)

[6] Ernst F J, *Phys. Rev.* **167**, 1175 (1968)

[7] Gariel J, Marcilhacy G and Santos N O, *Class. Quantum Grav.* **19**, 2157 (2002)

[8] Kramer D, Stephani H, MacCallum M and Herlt E Exact Solutions of Einstein's Field Equations (Cambridge: Cambridge University Press, 1980)

[9] da Silva M F A, Herrera L, Paiva F M and Santos N O, *Class. Quantum Grav.* **12**, 111 (1995)

[10] MacCallum M A H and Santos N O, *Class. Quantum Grav.* **15**, 1627 (1998)

[11] da Silva M F A, Herrera L, Paiva F M and Santos N O, *Gen. Rel. Grav.* **27**, 859 (1995)

[12] Herrera L and Santos N O, *J. Math. Phys.* **39**, 3817 (1998)