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Dually-Charged Mesoatom on the Space of Constant Negative Curvature

by

V.D. Ivashchuk and V.N. Melnikov

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Abstract

The discrete spectrum solutions corresponding to dually-charged mesoatom on the space of constant negative curvature are obtained. The discrete spectrum of energies is finite and vanishes, when the magnetic charge of the nucleus exceeds the critical value.

Key-words: Atom; Gravitation; Spectre.

1 Introduction

The behaviour of atom-like systems in curved backgrounds is of interest due to the problem of stability of atoms and basic standards in external fields. Such systems were studied by many authors (see, for example, [1-10] and references cited there).A lot of papers were devoted to calculations of the curvature-induced energy-level shifts within the framework of the perturbation theory.

In this paper we consider the "motion" of massive charged scalar particle (meson) in the field of static dually-charged nucleus on the space of constant negative curvature. We find the discrete spectrum solutions of the Klein-Gordon equation (see formulas (3.29) and (3.31)). The discrete spectrum of the mesoatom is finite. The largest principle number N_0 (see (3.26)) depends on the radius of curvature a and the magnetic charge g_m. For sufficiently small values of a or large values of g_m the discrete spectrum is empty.

It should be noted that the expression for the energy levels (formula (3.24) of this paper) was obtained earlier in [8]. But the expressions for N_0 and the wave functions in [8] were wrong [10].

2 The model

We consider the space-time $R \times L^3(a)$ with the metric

$$g = c^2 dt \otimes dt - \gamma = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \qquad (2.1)$$

where

$$L^3(a) \equiv \{z | z = (z^0, \vec{z}) \in R^4, \ z^0 > 0, \ (z^0)^2 - (\vec{z})^2 = a^2\} \qquad (2.2)$$

is 3-dimensional space of constant negative curvature (a is radius of curvature) with the canonical metric

$$\gamma = \gamma_{ij}(\vec{x})dx^i \otimes dx^j = a^2[d\chi \otimes d\chi + \sinh^2\chi(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)], \ (2.3)$$

$0 < \chi < +\infty$ ($a \sinh\chi = |\vec{z}|$).

We consider a static dually-charged nucleus with the electric charge $(-Ze)$ and a magnetic charge g_m, placed in the coordinate origin $\chi = 0$. Let $U \subset L^3(a)$ be a domain with the trivial cohomology group $H^2(U, R) = 0$

and $\{\chi = 0\} \notin U$. The electromagnetic 4-potential A_μ on $R \times U$, corresponding to the nucleus has the following form

$$A = A_\mu dx^\mu = (-\frac{Ze}{a})(\coth\chi - 1)dt + \mathcal{A}, \qquad (2.4)$$

where

$$\mathcal{F} = d\mathcal{A} = g_m \sin\theta d\theta \wedge d\varphi \qquad (2.5)$$

is the strength of the electromagnetic field, corresponding to the magnetic charge g_m. The relation (2.5) is correct, since due to $H^2(U, R) = 0$ any closed 2-form on U is exact, i.e. $d\mathcal{F} = 0$ entails the existence of $\mathcal{A}$ such that $d\mathcal{A} = \mathcal{F}$. For $U = U_\pm$, where

$$U_\pm = L^3(a) \setminus \{\theta = \frac{\pi}{2} \pm \frac{\pi}{2}\}, \qquad (2.6)$$

the 1-form on $U = U_\pm$

$$\mathcal{A} = \mathcal{A}^\pm = g_m(\pm 1 - \cos\theta)d\varphi \qquad (2.7)$$

satisfies the relation (2.5).

A massive charged scalar particle (meson), moving in the field of the static dually–charged nucleus, has the following action

$$S[\varphi] = \frac{1}{c} \int_{M_*} d^4x (-\det g_{\mu\nu})^{1/2} \{\hbar^2 g^{\mu\nu} (\overline{D_\mu(A^*)\varphi_*})(D_\nu(A^*)\varphi_*) - m_0^2 c^2 \bar\varphi_* \varphi_*\}, \qquad (2.8)$$

where $D_\mu = D_\mu(A^*) \equiv \nabla_\mu + (ie/\hbar c)A_\mu^*$, ∇_μ is covariant derivative, corresponding to the metric (2.1); the symbol $* = \pm$ and $A = A^\pm$ is a result of substitution of $\mathcal{A}^\pm$ from (2.7) to (2.4); m_0 is mass of the scalar particle and e is its charge (opposite in sign to the nucleus charge). The pair of functions

$$\varphi_\pm : M_\pm = R \times U_\pm \longrightarrow C \qquad (2.9)$$

satisfy the overlapping condition

$$\varphi_+(t, \vec{x}) = \Omega(\vec{x})\varphi_-(t, \vec{x}), \qquad (2.10)$$

$\vec{x} \in U_+ \cup U_-$, where

$$\Omega : U_+ \cup U_- \longrightarrow U(1) \qquad (2.11)$$

is a smooth overlapping function. The scalar particle (meson) wave function is a smooth section of a vector C-bundle with the base $R \times (L^3(a) \setminus \{\chi = 0\})$. This section is defined by the pair of functions (2.9), satisfying the condition (2.10). (The function $\varphi_\pm$ is the representation of the function $\varphi|M_\pm$ in the local trivialization over $M_\pm$).

The action (2.8) is correctly defined, i.e. the right hand side of (2.8) does not depend on the choice of the symbol $* = \pm$ (or equivalently on the choise of local trivialization) if the function Ω (2.11) satisfies the following relation on $U_+ \cup U_-$

$$\mathcal{A}^+ = \mathcal{A}^- + i\frac{\hbar c}{e}\Omega^{-1}d\Omega \qquad (2.12)$$

($\mathcal{A}^\pm$ are defined in (2.7)). It follows from the relations (2.7) and (2.12), that such function does exist if and only if the Dirac quantization condition is satisfied [11]

$$q \equiv eg_m/\hbar c = 0, \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots. \qquad (2.13)$$

In this case

$$\Omega = exp[-2iq(\varphi - \varphi_0)], \qquad (2.14)$$

where $\varphi_0 = const.$

Varying the action (2.8), we obtain the following equation of motion

$$[\hbar^2 g^{\mu\nu}(\overline{D_\mu(\mathcal{A}^*)})(D_\nu(\mathcal{A}^*)) + m_0^2 c^2]\varphi_* = 0. \qquad (2.15)$$

The Lagrangian, corresponding to the action (2.8), has the following form

$$L(\varphi, v) = \int_{U_*} d^3\vec{x}(det\gamma_{ij})^{1/2}\{\frac{\hbar^2}{c^2}|v_* + \frac{ie}{\hbar}V\varphi_*|^2 - $$
$$\hbar^2\gamma^{ij}(\overline{D_i(\mathcal{A}^*)\varphi_*})(D_j(\mathcal{A}^*)\varphi_*) - m_0^2 c^2\bar{\varphi}_*\varphi_*\}, \qquad (2.16)$$

where $V \equiv (-Ze)(coth\,\chi - 1)/a$, $v_+(\vec{x}) = \Omega v_-(\vec{x})$. The Lagrangian (2.16) is a continuous mapping

$$L : H \times H \longrightarrow R, \qquad (2.17)$$

where $H \times H \cong TH$ and TH is tangent vector bundle over the Hilbert space H. This Hilbert space is the configuration space of the Lagrange system. It consists of smooth sections of the monopole vector C- bundle over $L^3(a) \setminus \{\chi = 0\}$ satisfying the restriction

$$\int_{U_*} d^3\vec{x}(det\gamma_{ij})^{1/2}\{\bar{\varphi}_*\varphi_*(1+V^2) + \gamma^{ij}\overline{(D_i(\mathcal{A}^*)\varphi_*)}(D_j(\mathcal{A}^*)\varphi_*)\} < +\infty. \quad (2.18)$$

The scalar product in H is the following

$$(\psi, \varphi) \equiv \int_{U_*} d^3\vec{x} (det \gamma_{ij})^{1/2} \{\bar{\psi}_*\varphi_*(1 + V^2) + \gamma^{ij}\overline{(D_i(\mathcal{A}^*)\psi_*)}(D_j(\mathcal{A}^*)\varphi_*)\}$$
(2.19)

$* = \pm$. Strictly speaking, H is the completion of the pre-Hilbert space (with scalar product (2.19)) of smooth sections with compact support in $U_+ \cup U_-$. (H is the modified Sobolev space.) The field equation (2.15) is equivalent to the Euler-Lagrange equations for the Lagrange system (L, H).

3　The discrete spectrum solutions

We seek solutions of the equation of motion (2.15) in the following form

$$\varphi(t, \vec{x}) = \exp(-iEt/\hbar)F(\vec{x}),$$
(3.1)

where $E \in C$ and $F \in H$. The substitution of (3.1) into (2.15) leads to the following relation

$$\{[\varepsilon + Z\alpha(coth\chi - 1)]^2 + \frac{1}{\sinh^2\chi}\frac{\partial}{\partial\chi}(\sinh^2\chi\frac{\partial}{\partial\chi}) + \frac{1}{\sinh^2\chi}\triangle_q^* - \mu^2\}F_* = 0,$$
(3.2)

where

$$\varepsilon \equiv Ea/\hbar c, \qquad \mu \equiv m_0ac/\hbar, \qquad \alpha \equiv e^2/\hbar c,$$
(3.3)

and

$$\triangle_q^* = \beta^{ij}D_i(\mathcal{A}^*)D_j(\mathcal{A}^*)$$
(3.4)

is the "monopole Laplace operator" [12] on 2-dimensional sphere S^2 (β is the canonical metric on S^2), written in the local trivialization over S_*^2, $* = \pm$, where $S_\pm^2 = S^2 \setminus \{\theta = \frac{\pi}{2} \pm \frac{\pi}{2}\}$. The operator $\triangle_q$ acts on the sections of the monopole vector C-bundle over S^2. For $q = 0$ it coincides with the Laplace operator on S^2. The spectrum of $\triangle_q$ is well-known [12,13], it is discrete

$$\triangle_q Y_{qlm} = [-l(l + 1) + q^2]Y_{qlm},$$
(3.5)

where

$$l = |q|, |q| + 1, \ldots; \qquad m = -l, -l + 1, \ldots, l;$$
(3.6)

and Y_{qlm} are monopole spherical harmonics [13]. For the sake of completeness the explicit expression for Y_{qlm} is presented in the Appendix. The relation (3.5) follows from the representation for $\triangle_q$ [13]

$$-\hbar^2\triangle_q = (\vec{L}_q)^2 - \hbar^2 q^2. \tag{3.7}$$

In (3.7) $\vec{L}_q$ is the modified (monopole) momentum operator [13]

$$(L_q^j)^* = \varepsilon_{jkl}z^k(-i\hbar\frac{\partial}{\partial z^l} + \frac{e}{c}\mathcal{A}_l^*) - \hbar q\frac{z^j}{|z|}, \tag{3.8}$$

$j = 1, 2, 3$; where $\mathcal{A}_i^{\pm}$ are the components of the 1-form (2.7) in z-coordinates (see (2.2))

$$\mathcal{A}^{\pm} = \mathcal{A}_i^{\pm}dz^i = \frac{g_m\varepsilon_{ij3}z^idz^j}{|z|(z^3 \pm |z|)}. \tag{3.9}$$

The components of the operator (3.8) satisfy the commutation relations

$$[L_q^k, L_q^l] = i\hbar\varepsilon_{klj}L_q^j. \tag{3.10}$$

The monopole harmonics Y_{qlm} form a complete orthonormal set (on S^2) of the eigenfunctions of the operators $(\vec{L}_q)^2$ and L_q^3:

$$[(\vec{L}_q)^2 - \hbar^2 l(l + 1)]Y_{qlm} = 0, \tag{3.11}$$

$$[L_q^3 - \hbar m]Y_{qlm} = 0, \tag{3.12}$$

where l and m satisfy (3.6). The equality (3.5) follows from the relations (3.7) and (3.10).

Let F be an eigenfunction of the operators $(\vec{L}_q)^2$ and L_q^3. Then

$$F_*(\chi, \theta, \varphi) = Q(\chi)(Y_{qlm})_*(\theta, \varphi). \tag{3.13}$$

Substituting (3.13) into (3.2) and taking into account (3.5), we get

$$\{[\varepsilon + Z\alpha(coth\chi - 1)]^2 + \frac{1}{\sinh^2\chi}\frac{\partial}{\partial\chi}(\sinh^2\chi\frac{\partial}{\partial\chi})$$

$$-\frac{1}{\sinh^2\chi}[l(l + 1) - q^2] - \mu^2\}Q = 0, \tag{3.14}$$

The inclusion $F \in H$ is equivalent to the convergence of the integral

$$\int_0^\infty d\chi \, \sinh^2 \chi \{|Q|^2(1 + \frac{1}{\sinh^2 \chi}) + |\partial_\chi Q|^2\} < +\infty \qquad (3.15)$$

(this condition follows from (2.18) and (3.13)).

We introduce a new variable x

$$x = 2/(\coth \chi + 1) \qquad (3.16)$$

($0 < x < 1$ for $\chi > 0$). Then eq. (3.14), written in x-variable,

$$\frac{d^2Q}{dx^2} + \frac{2}{x}\frac{dQ}{dx} + \frac{1}{4x^2(1-x)^2}\{[\varepsilon x + 2Z\alpha(1-x)]^2$$
$$-\mu^2 x^2 - 4[l(l+1) - q^2](1-x)]Q = 0 \qquad (3.17)$$

has a generalyzed-hypergeometric form [14]. The standard procedure (see, for example [14]) give the substitution

$$Q = x^{-\frac{1}{2}+\kappa}(1-x)^{\frac{1}{2}+\frac{\lambda}{2}}v, \qquad (3.18)$$

leading to the hypergeometric equation for the function $v = v(x)$

$$x(1-x)\frac{d^2v}{dx^2} + [1 + 2\kappa - (2 + 2\kappa + \lambda)x]\frac{dv}{dx} +$$
$$[Z\alpha\varepsilon - (\kappa + \frac{1}{2})^2 - (Z\alpha)^2 - \lambda(\kappa + \frac{1}{2})]v = 0, \quad (3.19)$$

where

$$\lambda = \sqrt{\mu^2 + 1 - \varepsilon^2}, \qquad \kappa = \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2 - q^2}, \qquad (3.20)$$

and $\sqrt{re^{i\phi}} \equiv r^{1/2}e^{i\phi/2}$, $-\pi < \phi \le \pi$. Here and below we put the following restriction on Z: $Z\alpha < \frac{1}{2}$.

The solution of (3.19) may be expressed in terms of hypergeometric functions

$$v(x) = d_+ F(A_+, B_+, C_+, x) + d_- x^{-2\kappa} F(A_-, B_-, C_-, x), \qquad (3.21)$$

where d_+, d_- are arbitrary constants and

$$A_\pm = \pm\kappa + \frac{1}{2}[\lambda + 1 - \sqrt{\lambda^2 + 4Z\alpha(\varepsilon - Z\alpha)}],$$

$$B_\pm = \pm\kappa + \frac{1}{2}[\lambda + 1 + \sqrt{\lambda^2 + 4Z\alpha(\varepsilon - Z\alpha)}],$$

$$C_\pm = \pm 2\kappa + 1.$$

Using the asymptotic formulas for the hypergeometric functions [14] (for $x \to 0$ and $x \to 1$), we find that the function Q, defined by (3.18) and (3.21), satisfies the restriction (3.15), if and only if $d_- = 0$ and

$$A_+ = -n, \tag{3.22}$$

$n = 0, 1, 2, \ldots$. In this case

$$v(x) = const P_n^{(2\kappa,\lambda)}(1 - 2x), \tag{3.23}$$

where $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial [14] (see also Appendix).

Solving the equation (3.22), we get

$$\varepsilon = Z\alpha + N\frac{[\mu^2 + 1 - N^2 - (Z\alpha)^2]^{1/2}}{[N^2 + (Z\alpha)^2]^{1/2}}, \tag{3.24}$$

where

$$N = n + \kappa + \frac{1}{2} \tag{3.25}$$

is the principal quantum number satisfying the unequality

$$N < N_0 \equiv (Z\alpha)^{1/2}[(\mu^2 + 1)^{1/2} - Z\alpha]^{1/2}. \tag{3.26}$$

Thus, there exists only a finite number of normalizable solutions of the equation of motion (2.15), that have the form (3.1) and are eigenfunctions of the operators $(\vec{L}_q)^2$ and L_q^3. These solutions are the discrete spectrum solutions.

It follows from the definitions (3.20), (3.25) and the unequalitiy (3.26) that the discrete spectrum is absent for $N_0 \leq \frac{1}{2}$. For $N_0 > \frac{1}{2}$ it is also absent if

$$|q| \geq |q|_0 = (N_0)^2 - N_0 + (Z\alpha)^2 \tag{3.27}$$

and exists, if $|q| < |q|_0$. In this case $\varepsilon = \varepsilon(N) = \varepsilon(N(n,l,|q|))$, where the principal quantum number N is defined in (3.25) and

$$l = |q|, \ldots, l_0(|q|), \qquad n = 0, \ldots, n_0(l, |q|), \tag{3.28}$$

In (3.28)

$$l_0(|q|) \equiv max\{l | l - |q| = 0, 1, \ldots; l(l+1) - q^2 < |q|_0\},$$
$$n_0(l, |q|) \equiv max\{n | n = 0, 1, \ldots; n + \kappa + \frac{1}{2} < N_0\}$$

(the relations for l_0 and n_0 follow from the unequality (3.26)).

In the initial notations we have the following expression for the energy spectrum

$$E = \frac{Ze^2}{a} + N\frac{[m_0^2 c^4 + (1 - N^2 - (Z\alpha)^2)(\hbar^2 c^2/a^2)]^{1/2}}{[N^2 + (Z\alpha)^2]^{1/2}}, \tag{3.29}$$

where $N < N_0(a)$, $N_0(a) > 1/2$ and $|q| < |q|_0 = |q|_0(a)$.

Due to (3.29)

$$Ze^2/a < E < m_0 c^2. \tag{3.30}$$

The meson wave function, corresponding to the set of quantum numbers (n, l, m), is

$$\varphi = C\exp(-iEt/\hbar)(\frac{2}{\coth\chi + 1})^{-\frac{1}{2}+\kappa}\exp[-\chi(1+\lambda)]P_n^{(2\kappa,\lambda)}(\frac{\coth\chi - 3}{\coth\chi + 1})Y_{qlm}, \tag{3.31}$$

where C is constant and n, l and m satisfy the restrictions (3.28) and (3.6) correspondingly, $\kappa = \kappa(l, |q|)$ and $\lambda = \lambda(E, a)$ are defined in (3.20).

Now we show that the parameter E is the energy, corresponding to the meson wave function, appropiatly normalized. The energy functional, corresponding to the Lagrangian (2.16), is

$$\mathcal{E}(\varphi, v) = \int_{U_*} d^3\vec{x}(det\gamma_{ij})^{1/2}\{\frac{\hbar^2}{c^2}\bar{v}_* v_* - \frac{e^2}{c^2}V^2\overline{\varphi_*}\varphi_*$$
$$+ \hbar^2\gamma^{ij}\overline{(D_i(A^*)\varphi_*)}(D_j(A^*)\varphi_*) + m_0^2 c^2\overline{\varphi_*}\varphi_*\}. \tag{3.32}$$

The energy is conserved on the solutions of the equation of motion (2.15): $\mathcal{E} = \mathcal{E}(\varphi(t), \dot{\varphi}(t)) = const$. The Lagrangian (2.16) is invariant under the

$U(1)$-transformations: $\varphi \mapsto \varphi^s = \exp(-ise/\hbar)\varphi$. Due to the E. Noether's theorem we have $Q = Q(\varphi(t), \dot\varphi(t)) = const$, where

$$Q(\varphi, v) = \int_{U_*} d^3\tilde{x}(det\gamma_{ij})^{1/2}\{i\hbar(\overline{\varphi_*}v_* - \overline{v_*}\varphi_*) - 2eV\overline{\varphi_*}\varphi_*\} \qquad (3.33)$$

is the charge functional ($Q : H \times H \longrightarrow R$). Using (2.15) and (3.1), we get $\mathcal{E} = EQ/e$. The physical normalization of the wave function $Q = Q(\varphi(t), \dot\varphi(t)) = e$ entails $\mathcal{E} = E$. So, E is the energy of the scalar particle (meson).

Let us consider the flat-space limit:$a \to +\infty$. In this case $|q|_0, N_0 \to +\infty$ and the discrete spectrum (3.29) contains an infinite number of levels for all values of q. For $q = 0$ and $a \to +\infty$ the formulas (3.29) and (3.31) coincide with the well-known relations (see, for example [14]).

For $a \sim 10^{28}cm$ (present cosmological scale), $Z = 1$ and $m = m_{\pi^+}$ (the mass of π^+-meson) we have: $N_0 \sim 10^{20}$ and $|q|_0 \sim 10^{40}$.

From (3.29) it is seen that at present epoch energy shifts due to gravitational cosmological field are small because they are proportional to $(\lambda_c/a) \sim 10^{-40}$, where λ_c is a meson compton wavelength. So, we need not worry about the stability of our basic units like the second or the meter in a cosmological field though some effects may appear in strong local gravitational fields [7].

Appendix

Here we present the explicit expressions for the monopole spherical harmonics Y_{qlm}, $l = |q|, |q| + 1, \dots$; $m = -l, -l + 1, \dots, l$; Y_{qlm} are smooth sections of the monopole vector C-bundle over the sphere S^2. In the local trivialization over $S^2_\pm = S^2 \setminus \{\theta = \frac{\pi}{2} \pm \frac{\pi}{2}\}$ the sections Y_{qlm} are represented by the complex-valued functions on $S^2_\pm$

$$(Y_{qlm})_\pm = M_{qlm}P_n^{(\alpha,\beta)}(\cos\theta)\exp(i(m \pm q)\varphi), \qquad (3.34)$$

where

$$\alpha = -q - m, \qquad \beta = q - m, \qquad n = l + m \qquad (3.35)$$

and $P_n^{(\alpha,\beta)}(x)$ is Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1 - x)^{-\alpha}(1 + x)^{-\beta}\frac{d^n}{dx^n}[(1 - x)^{\alpha+n}(1 + x)^{\beta+n}]$$

$$= \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}F(-n, n + \alpha + \beta + 1, \alpha + 1, (1 - x)/2),$$

(M_{qlm} are constants).

Acknowledgments

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