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CONSEQUENCES OF TEMPORAL AND AXIAL GAUGE FIXING

R.P. Mondaini

RIO DE JANEIRO
BRASIL

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R.P. Mondaini

Universidade Federal do Rio de Janeiro
Instituto de Física
Av. Brig. Trompowsky, s/nº - 21.931 - R.J.
Brasil

and

Centro Brasileiro de Pesquisas Físicas
Av. Wenceslau Braz, 71 - 22.290 - R.J.
Brasil

ABSTRACT - It is shown the role played by temporal ($A^0_a = 0$) and axial ($A^3_a = 0$) gauges in a non-Abelian gauge theory, within the methods of Dirac's Hamiltonian formulation. The necessary equations of the theory are written and some solutions are presented.

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INTRODUCTION

We intend here to discuss the procedures of axial and temporal gauge fixing in a non-Abelian theory like Yang - Mills, from the viewpoint of its solutions. For simplicity, we have worked with the theory without external sources. In section 1, we analyze the work done in the $A^3_a = 0$ gauge, putting this restriction in the equations of motion, and also, a priori, in the Lagrangian. As a consequence, Coleman's non-Abelian plane wave is revisited. In section 2, we make some observations and discussions about gauge fixing, and compare the work in the temporal and axial gauges. In section 3 and 4, we introduce the fundamentals of the Hamiltonian formulation for the canonical quantization of constrained non-Abelian theories. The Dirac's canonical procedures are used to deduce the equations of motion. In section 5, the particular case of electrodynamics is considered; we observe the possibility for doing trivial quantization in axial gauge. In section 6 we assert the requirements which a solution should satisfy, and we complete the discussions of section 2. In $A^3_a = 0$ gauge Coleman's non-Abelian plane wave is studied again with the methods introduced in sections 3 and 4, and we present a solution in $A^0_a = 0$ gauge. Finally, in section 7, we put emphasis in the difficulties for finding other non-trivial solutions and we sketch the guidelines for future work.

1. AXIAL GAUGE - INTRODUCTORY WORK

Let us consider the equations of Yang-Mills:

$$\frac{\delta S}{\delta A_a^{\mu}} = D_{\mu}^{ab} F_b^{\mu\nu} = 0 \quad [2] \qquad (1)$$

where

$$S = -\frac{1}{4} \int F_{\mu\nu}^{a} F_a^{\mu\nu} d^4x \qquad (2)$$

and D_{μ}^{ab} , the covariant derivatives in the adjoint representation are given by:

$$D_{\mu}^{ab} = \delta^{ab} \partial_{\mu} - g f_{abc} A_{\mu}^{c} \qquad (3)$$

Let us work in the axial gauge $A_a^3 = 0$ with the "Ansatz":

$$A_a^2 = 0 \quad ; \quad A_a^0 = A_a^1 = u_a . \qquad (4)$$

So, for the field strengths which are given by

$$F_a^{\mu\nu} = \partial^{\mu} A_a^{\nu} - \partial^{\nu} A_a^{\mu} + g f_{abc} A_b^{\mu} A_c^{\nu} \qquad (5)$$

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[2] a,b = 1,...,n, and we are using a n-parameter semi - simple Lie group.

We have:

$$F^{01}_a = (\partial_0 + \partial_1) u_a \quad , \quad F^{12}_a = (\partial_2 - \partial_1) u_a$$

$$F^{02}_a = (\partial_0 + \partial_2) u_a \quad , \quad F^{13}_a = \partial_3 u_a \qquad (6)$$

$$F^{03}_a = \partial_3 u_a \quad , \quad F^{23}_a = \partial_3 u_a$$

Our work is done in Minkowski's space-time, with the signature of the metric: (+,-,-,-) as $-g^{00}=g^{11}=g^{22}=g^{33}= -1$.

The equations of motion are:

$$\nu=0: \ (\partial_1^2 + \partial_2^2 + \partial_2^2 + \partial_0 \ \partial_1) \ u_a + g \ f_{abc} \ u_b(\partial_0 + \partial_1) \ u_c=0, \qquad (7)$$

$$\nu=1: \ (\partial_0^2 - \partial_2^2 - \partial_3^2 - \partial_0 \ \partial_1) \ u_a - g \ f_{abc} \ u_b(\partial_0 + \partial_1) \ u_c=0, \qquad (8)$$

$$\nu=2: \ \partial_2(\partial_0 + \partial_1) \ u_a = 0. \qquad (9)$$

The fourth equation[3] need not be considered, since by the choice of the gauge $A^3_a = 0$, we have only A^0_a , A^1_a and A^2_a as independent variables in the initial Lagrangian /1/. A self-evident solution of the above equations is:

$$u_a = h(x^2, x^3, x^0, x^1) \ f_a(x^0 - x^1) \qquad (10)$$

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[3] We mean by "fourth equation" the set of "a" equations of motion for ν=3, Analogously for "first equation".

where

$$h = x^2 \xi(x^0 - x^1) + x^3 \eta(x^0 - x^1) + \zeta(x^0 - x^1). \qquad (11)$$

Equation (10) with (11) is the solution introduced by Coleman/2/. We should observe that Coleman's work does not impose $A_a^3 = 0$, "a priori" in the Lagrangian, so, we have four equations of motion.

Solution (10) satisfies this fourth equation or:

$$\nu = 3: \qquad \partial_3 (\partial_0 + \partial_1) u_a = 0. \qquad (12)$$

In the beginning of section 6 we give a proof that it is necessary for the solution (10) to satisfy:

$$\partial_3 \left[\partial_3 (\partial_0 + \partial_1) u_a \right] = 0 \qquad (13)$$

From (12) and (13), we see that there is no contradiction with the previous work (at least for this solution), if we put $A_a^3 = 0$ in the Lagrangian, acording to Fradkin and Tyiutin's prescription.

We know that the axial gauge is very adequate for the quantization procedures, since in the resulting commutation relations there are no fields in the right-hand side, and so, no ghost loops in the diagrams. It would be very interesting to repeat the deduction of the above solutions with the methods of the Hamiltonian formulation. That is what we do in the sixth section.

2. SOME OBSERVATIONS ABOUT GAUGE FIXING

We have mentioned above, that in the work done in the axial gauge, the fourth equation of motion,

$$\frac{\delta S}{\delta A_a^3} = 0 \tag{14}$$

need not be considered. This is due to the fact that there are only 3n independent variables in the Lagrangian.

The same thing occurs in the temporal $A_a^0 = 0$ gauge. Here, the first equation[4]

$$\frac{\delta S}{\delta A_a^0} = 0 \tag{15}$$

is missing. This equation is Gauss'law. There must be a way of saving gauss'law. Actually, we have to implement Gauss'law by imposing it on the physical states of the quantized theory. This leads to a fundamental difference between the two gauges. We cannot save the fourth equation in $A_a^3 = 0$ gauge, since it cannot be imposed on the physical states. These facts are also inherent to the

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[4] See note on page 3.

Hamiltonian formulation. On the set of equations, "n" of them are always missing, if we put the gauge restrictions $A_a^0 = 0$ or $A_a^3 = 0$ in the Lagrangian. We will come back to this point in section 6.

3. HAMILTONIAN FORMULATION /3,4/

In phase space, the momenta associated with A_a^1, A_a^2, A_a^3, are given by

$$\Pi_k^a = \frac{\delta L}{\delta \dot{A}_a^k} = F_a^{0\,k} \qquad , \; k=1,2,3. \tag{16}$$

where

$$L = -\frac{1}{4} \int F_{\mu\nu}^a F_a^{\mu\nu} d^3x \tag{17}$$

The first component Π_a^0 is a "primary constraint" of the theory

$$\upsilon^a = \Pi_0^a \simeq 0 \tag{18}$$

we mean by this notation, as is known /4/, that the Poisson bracket of Π_0^a with some dynamical variable may be different from zero. This can be seen from

$$\{\Pi_{\mu}^{a}(x),\ A_{b}^{\nu}(x')\}_{t=t'} = -g_{\mu}{}^{\nu}\ \delta_{ab}\ \delta^{3}(x-x'). \tag{19}$$

we are using for the equal time Poisson brackets of two arbitrary functionals A(x), B(x'):

$$\{A(x),\ B(x')\}_{t=t'} = \int\left(\frac{\delta A}{\delta A_{a}^{\mu}(y)}\ \frac{\delta B}{\delta \Pi_{\mu}^{a}(y)} - \frac{\delta A}{\delta \Pi_{\mu}^{a}(y)}\ \frac{\delta B}{\delta A_{a}^{\mu}(y)}\right) d^{3}y \tag{20}$$

The equations of motion must be written as

$$\frac{\delta S}{\delta A_{a}^{\nu}} = \delta^{ab}\ \partial_{\mu}\ F_{b}{}^{\mu\nu} + g\ \ f_{abc}\ A^{b}{}_{\mu}\ F_{c}^{\mu\nu} \simeq 0 \tag{21}$$

We observe that these equations, like equation (18) , are incompatible with the relations (19).

Eq.(21) for $\nu=0$ (Gauss'law) can be also obtained, if we impose that the condition (18) is maintained in time.

$$\upsilon_{2}^{a} = \dot{\Pi}_{0}^{a} = \{\Pi_{0}^{a}(x),\ H\} = -\frac{\delta H}{\delta A_{a}^{0}(x)} \simeq 0, \tag{22}$$

where H is the modified Hamiltonian:

$$H = \int T^{00}\ d^{3}x + \int(\lambda_{1}^{a}\ \upsilon_{1}^{a} + \lambda_{2}^{a}\ \upsilon_{2}^{a})\ d^{3}x\ , \tag{23}$$

λ_{1}^{a}; λ_{2}^{a} are arbitrary functions; surface terms are disregarded. The above Hamiltonian should be consistent with the equations of motion, or:

$$\dot{A}^{\mu}_{a} = \{A^{\mu}_{a}(x), H\} = \frac{\delta H}{\delta \Pi^{a}_{\mu}(x)} \quad , \tag{24}$$

$$\dot{\Pi}^{a}_{\mu} = \{\Pi^{a}_{\mu}(x), H\} = - \frac{\delta H}{\delta A^{\mu}_{a}(x)} \quad . \tag{25}$$

From (24), (25), using (23), we deduce that:

$$\lambda^{a}_{1} = \dot{A}^{0}_{a} \quad ; \quad \lambda^{a}_{2} = 0. \tag{26}$$

4. GAUGE FIXING

Working with the Hamiltonian formulation, let us consider the imposition of a gauge

$$\upsilon^{a}_{3} = A^{3}_{a} \simeq 0. \tag{27}$$

If we require υ^{a}_{3} to be maintained in time, using (24), we have:

$$\upsilon^{a}_{4} = \Pi^{a}_{3} - \partial_{3} A^{0}_{a} \simeq 0. \tag{28}$$

From (28) and (22) we see that the third component of momentum is dependent on the other two.

Equations (18), (22), (27), (28) are the first class constraints of the constrained system studied here.

We should calculate now, according to Dirac's method /3/ the Poisson bracket matrix of the first class constraints:

$$D_{\alpha\beta}(x,x') = \{\upsilon_\alpha(x),\ \upsilon_\beta(x')\}_{t=t'} \quad , \alpha,\beta=1,2,3,4 \tag{29}$$

With (29), we can introduce the Dirac brackets, which for two arbitrary dynamical variables are given by

$$\{A(x),B(x')\}^* = \{A(x),B(x')\} - \int \{A(x),\ \upsilon_\alpha(x'')\}\ D^{-1}_{\alpha\beta}(x'',x''') \ . \ \{\upsilon_\beta(x'''),B(x')\}\ d^3x''\ d^3x''' \tag{30}$$

All the brackets are calculated under the requirement of equal times.

The brackets (30) satisfy: $(i,j = 1,2)$

$$\{\Pi^a_i(x),\ \Pi^b_j(x')\}^* = 0 \ , \tag{29}$$

$$\{A^i_a(x),\ A^j_b(x')\}^* = 0 \ , \tag{30}$$

$$\{\Pi^a_i(x),\ A^j_b(x')\}^* = -g_i{}^j\ \delta_{ab}\ \delta^3(x-x') \ , \tag{31}$$

we have then, for the equations of motion /5/:

$$\dot{\Pi}_i^a = \{\Pi_i^a, H\}^* = D_k^{ab} F_b^{ik} + g f_{abc} u_c \Pi_i^b \quad , \tag{32}$$

$$\dot{A}_a^i = \{A_a^i, H\}^* = \Pi_i^a - D_i^{ab} u_b \quad , \tag{33}$$

together with Gauss'law derived from (22):

$$D_k^{ab} \Pi_k^b = 0. \tag{34}$$

Equations (34) and (28) lead to:

$$u_a = -\int G(x,x') D_i^{ab} \Pi_i^b(x') \, d^3x' \tag{35}$$

where the Green's function G(x,x') satisfies:

$$\partial_3^2 G(x,x') = \delta^3(x-x'). \tag{36}$$

Analogous, working in the temporal $A_a^0 = 0$ gauge, with the same procedure as above, we readily obtain for the equations of motion /5/:

$$\dot{\Pi}_k^a = D_\ell^{ab} F_b^{k\ell}, \qquad k,\ell=1,2,3 \tag{37}$$

$$\dot{A}_a^k = \Pi_k^a \; . \tag{38}$$

5. A PARTICULAR CASE: ELECTRODYNAMICS

A interesting result is deduced in the axial gauge for an Abelian theory like electrodynamics. The limit of electrodynamics is reached when we put g=0, in all equations above.

The equations of motion are, then:

$$\dot{\Pi}_i^a = \partial_k F^{ik}_{\cdot a} , \tag{39}$$

$$\dot{A}_a^i = \Pi_i^a - \partial_i u_a , \tag{40}$$

$$\partial_k \Pi_k^a = 0 , \tag{41}$$

where $u_a \equiv A_a^0$ is given by:

$$u_a = - \int G(x,x') \partial_i' \Pi_i^a(x') d^3x' . \tag{42}$$

Taking the temporal derivative of the last equation, using (39), and integrating successively by parts, we have:

$$\dot{u}_a = - \int \delta^3(x-x') \partial_i' A_a^i d^3x' = - \partial_i A_a^i . \tag{43}$$

From (39) and (40), we have, using (43):

$$\Box A_a^i = 0 . \tag{44}$$

where $\Box \equiv \partial_\mu \partial^\mu$ is the d'Alembertian.

We note that the function $u_a \equiv A^0_a$ also satisfies (44). From (40) we get:

$$\ddot{u}_a = \partial_i \partial_i u_a - \partial_i \Pi^a_i \tag{45}$$

Using now (41) and (28), we obtain from (45):

$$\Box \dot{u}_a = 0 \tag{46}$$

Starting from this equation, we can try to make canonical quantization in the axial gauge.

6. SOLUTIONS IN THE HAMILTONIAN FORMULATION

We are going to present the requirements which a solution should satisfy. Firstly, returning to configuration space, we have from (34) and (32), written in the axial gauge:

$$D^{ab}_k \; F^{0k}_b = 0 \, , \tag{47}$$

$$D^{ab}_\mu \; F^{\mu i}_b = 0 \, , \quad i = 1,2 \tag{48}$$

These are the equations of motion in configuration space in the gauge $A^0_a = 0$.

With a little algebra it is easy to prove that, starting from the identities:

$$D^{ab}_\mu \; D^{bc}_\nu \; F^{\mu\nu}_c = 0 \qquad \mu,\nu = 0,1,2,3 \qquad (49)$$

$$D^{ab}_k \; D^{bc}_\ell \; F^{k\ell}_c = 0 \qquad k,\ell = 1,2,3 \qquad (50)$$

$$D^{ab}_i \; D^{bc}_j \; F^{ij}_c = 0 \qquad i,j = 1,2 \qquad (51)$$

together with (47), (48) we obtain:

$$\partial_3 \left(\frac{\delta S}{\delta A^3_a} \right) = \partial_3 \left(D^{ab}_\mu \; F^{\mu 3}_b \right) = 0 \qquad (52)$$

In temporal gauge $A^0_a = 0$, from (37), (38), we can deduce for the equations of motion in configuration space:

$$D^{ab}_\mu \; F^{k\mu}_b = 0 \qquad k = 1,2,3 \qquad (53)$$

Analogously as in the $A^3_a = 0$ case, starting from (53) and (50), it is easy to show that:

$$\partial_0 \left(\frac{\delta S}{\delta A^0_a} \right) = \partial_0 \left(D^{ab}_\mu \; F^{0\mu}_b \right) = 0 \qquad (54)$$

Equations (52) and (54) are necessary conditions for solutions to satisfy the equations of motion in axial and temporal gauges respectively.

From the aspect of (52) and (54), it is easy to understand the fundamental difference between the work in the two gauges. From (54), we see that $D^{ab}_{\mu} F^{o\mu}_{b}$ can be diagonalized simultaneously with the Hamiltonian, then, we can impose Gauss'law on the physical states, or:

$$D^{ab}_{\mu} F^{o\mu}_{b} |\Psi> = 0 \qquad (55)$$

in $A^{o}_{a} = 0$ gauge.

On the contrary, we cannot impose the fourth equation (14), on the physical states, because its left hand side is not a constant of the motion.

Now let us make some observations about the possible solutions.

In $A^{3}_{a} = 0$ gauge, we have shown that, working with the "Ansatz" (4), the solution was the non-Abelian plane wave. Actually, we can infer from (47), (48) that the solution should have the wave form:

$$u_a = F(x^2, x^3, x^0, x^1) \, f_a(x^0 - x^1) \qquad (56)$$

Its Poynting's vector components are calculated by:

$$T_{0k} = P_k = \varepsilon_{k\ell m} E_a^{\ell} B_a^m \quad , \quad k,\ell,m = 1,2,3 \tag{57}$$

The "magnetic" and "electric" fields are given respectively by:

$$B_k^a = \frac{1}{2} \varepsilon_{k\ell m} F_a^{\ell m} \tag{58}$$

$$E_a^k = - \Pi_k^a = - F_a^{0k} \tag{59}$$

We should have:

$$\vec{E}_a \cdot \vec{B}_a = E_a^k B_k^a = 0 \tag{60}$$

The energy density is given by:

$$T^{00} = \frac{1}{2} (\Pi_k^a)^2 + \frac{1}{4} F_{k\ell}^a F_a^{k\ell} . \tag{61}$$

T^{00} and $T^{0k} \equiv P^k$ should be related by :

$$\partial_0 T^{00} + \partial_k P^k = 0 \tag{62}$$

From (56) we have that (60) and (62) are satisfied trivially, and the further requirement of bounded energy density says that F should be a linear function of x^2, x^3, which is evident from:

$$H_0 = \int f_a^2(x^0 - x^1)\left[(\partial_2 F)^2 + (\partial_3 F)^2\right] d^3x \qquad (63)$$

So, we have:

$$F = \xi\, x^2 + \eta\, x^3 + \zeta \qquad (64)$$

where ξ, η, ζ are bounded functions of $(x^0 - x^1)$.

We have mentioned that this solution satisfies the fourth equation. With the remarks of section 2, we can infer, that it should have solutions in $A_a^3 = 0$ which do not satisfy the fourth equation. Correspondingly, it should have solutions in temporal gauge $A_a^0 = 0$ which do not satisfy the first equation $(\nu = 0)$.

In order to present a solution in temporal gauge which does not satisfy the first equation, let us consider, the "Ansatz":

$$A_a^2 = A_a^3 = A_a^1 = \chi_a \qquad (65)$$

The resulting equations of motion are:

$$(\partial_2^2 + \partial_3^2 - \partial_0^2 - \partial_1(\partial_2 + \partial_3))\chi_a + g\, f_{abc}\, \chi_b\, (\partial_2 + \partial_3 - 2\,\partial_1)\chi_c = 0\,, \qquad (66)$$

$$(\partial_1^2 + \partial_3^2 - \partial_0^2 - \partial_2(\partial_1 + \partial_3))\chi_a + g\, f_{abc}\, \chi_b\, (\partial_1 + \partial_3 - 2\,\partial_2)\chi_c = 0\,, \qquad (67)$$

$$(\partial_1^2+\partial_2^2-\partial_0^2-\partial_3(\partial_1+\partial_2))\chi_a + g\, f_{abc}\, \chi_b\, (\partial_1+\partial_2 - 2\,\partial_3)\, \chi_c = 0. \qquad (68)$$

The equation corresponding to (54) is:

$$\partial_0 \left[\partial_0 (\partial_1 + \partial_2 + \partial_3)\chi_a + 3g\, f_{abc}\, \chi_b\, \partial_0\, \chi_c \right] = 0 \qquad (69)$$

We have for solution:

$$\chi_a = x^0 f_a(x^1 + x^2 + x^3) \qquad (70)$$

where f_a are bounded functions.

Solution (70) satisfies the requirements mentioned for the last case.

7. CONCLUSIONS

We have tried to present a solution which does not satisfy the fourth equation in $A_a^3 = 0$ gauge. In temporal gauge ($A_a^0 = 0$), we have looked for a solution analogous to (56). Working respectively with the "Ansätze":

$$A_a^3 = 0 \;:\; A_a^2 = A_a^1 = A_a^0 = y_a\,, \qquad (71)$$

$$A_a^0 = 0 \quad : \quad A_a^2 = 0, \; A_a^3 = A_a^1 = z_a . \tag{72}$$

we have not found any non-trivial solutions. Particularly (62) seems to be a severe restriction on the existence of such solutions.

We think that a careful consideration of surface terms in the Hamiltonian formulation should be done for the specific cases above /6/. We have to explain also why it was possible to find solutions like (56)and (70) with the "wrong" formulation. This will be the subject of a forthcoming publication.

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