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ON A NEW COMAPCT FORM OF MULTI-INSTANTON SOLUTION

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Abstract: A new compact form of multi-instanton solution is obtained. It has all its singularities located at points different from those of the known solution and related to the latter by a gauge transformation. We calculate the winding number to show how the the two forms together may be used with advantage in any practical calculation involving path integrals.

Classical solutions with finite action of SU(2) Yang- Mills theory have recently drawn much interest. Belavin,Polyakov,Schwartz and Tyupkin[1] constructed a regular one-instanton solution in Euclidean four-space. Multi-instanton solutions with increasing level of generality were subsequently given by Witten[2], t'Hooft[3] and Jackiw,Nohl and Rebbi[4]. The solutions cited possess singularities in the form ~ $\lambda^2/(x-a)^2$ which obscures their topological interpretation which, however, may be restored (see Refs. 7 and 8).

We obtain in this note another form of N-instanton self-dual solution, A'_μ , in a compact form which has its singularities located at different points than those of the above mentioned solutions, say, for definiteness sake, the solution A_μ of Ref. 3. The two solutions, aside from being in compact form, are related by a singular gauge transformation. They may thus be used together with advantage for calculations involving path integrals. We define two overlapping regions R and R' in which A_μ and A'_μ are regular respectively, excluding the pertinent singular points. Following then Wu and Yang [5] prescription, for example, while calculating action or topological quantum number, whenever a particular integration path approaches a singularity of the solution, say A_μ ,we perform the integration by using A'_μ which is regular there., e.g. we perform integration by going from one plane to another.

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The N-instanton self dual solution of t' Hooft is given by

$$A_\mu = 2f(x)\eta_\alpha \bar{\sigma}_{\alpha\mu} \tag{1}$$

where

$$(y_i)_\alpha = (x_\alpha - (a_i)_\alpha) ,$$

$$\eta_\alpha = \sum_{i=1}^{N} \lambda_i^2 (y_i)_\alpha / (y_i^2)^2$$

and

$$f(x) = i \left[1 + \sum_{i=1}^{N} \lambda_i^2 / y_i^2 \right]^{-1} \tag{2}$$

Here $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are the generators [4] of the two SU(2) subgroups of O(4) group. On performing a gauge transformation with $U = \frac{1}{\sqrt{\eta^2}}(\eta_4 + i\vec{\eta}\cdot\vec{\sigma})$ we obtain for the self-dual solution the following form

$$A'_\mu = -2f(x)\eta_\alpha \sigma_{\alpha\mu} + \frac{2i}{\eta^2}\eta_\lambda \eta_{\mu\rho}\sigma_{\lambda\rho} \tag{3}$$

where

$$\eta_{\mu\rho} = \sum_{i=1}^{N} \frac{\lambda_i^2}{(y_i^2)^2}\left[\delta_{\mu\rho} - 4\frac{(y_i)_\mu (y_i)_\rho}{y_i^2}\right] \tag{4}$$

We verify that A'_μ is no more singular at $x = a_i$, $i = 1,2,..N$, and ,in fact, it vanishes there simplifying calculations. The corresponding field strength is shown to be finite, continuous and self-dual at these points. The singularities of A'_μ are all lumped together in the second term of Eq. (3) at the zeros of $\eta^2 = 0$ and at these locations the first term drops out. For $|x| \longrightarrow \infty$ the first term in Eq. (3) which falls as $\sim 1/x^3$ drops out again in comparasion to the second term which vanishes as $\sim 1/x$. The latter moreover takes the form of pure gauge term, e.g. $U \partial_\mu U^{-1}$. We note also that the form A'_μ could not be obtained with the simple ansatz [6] $A_\mu = u_{,\alpha}\bar{\sigma}_{\alpha\mu}$ (or $\sigma_{\alpha\mu}$) and the role of the two forms of SU(2) generators is complementary in this context. For example, the only finite action self dual solution with the ansatz is found to be the one-instanton solution of Ref. 1. It is clear that the case of multi-anti-instanton may be similarly treated by performing gauge transformation with U^{-1}.

For illustration purpose we calculate the winding number using using the form A'_μ. The two alternative forms show that the Pontryagin density $S^* = \frac{1}{2}\mathrm{Tr}(\tilde{F}_{\mu\nu}F_{\mu\nu})$ is regular every where. For the points where the field and potential are regular we have the identity

$$S^* = \partial_\mu I_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\,\partial_\mu \mathrm{Tr}\left[A_\nu F_{\alpha\beta} - \frac{2}{3}A_\nu A_\alpha A_\beta\right] \tag{5}$$

whence making use of Gauss theorem we obtain

$$\int_V S'^* d^4x = \lim_{\Delta j \to 0}\int_{V-\Sigma\Delta_j} S'^* d^4x$$

$$= \left[\lim_{R\to\infty}\int_{S_3} - \sum_{(\eta_\lambda=0)}\lim_{\epsilon_j\to 0}\int_{S'^j_3}\right] I'_\mu\, d\sigma_\mu \tag{6}$$

Here Δ_j indicates an infinitesimal spherical region of radius ϵ_j in E_4 with 3- dimensional surface S'^j_3 enclosing a zero of $\eta^2 = 0$. S_3 indicates a large spherical surface of radius R and $d\sigma_\mu$ is the directed surface element. From the assymptotic behavior of A'_μ discussed above we conclude that we may replace in Eq. (6), I'_μ by I^G_μ, the contribuition coming from the second (pure gauge) term of A'_μ. From the fact that a potential of the form $U\partial_\mu U^{-1}$, $U(x) \in SU(2)$ gives vanishing field strength at the points it is regular we deduce

$$0 = \int_{V - \Sigma \Delta_j - \sum_{i=1}^{N} \delta_i} S^{*G} d^4x$$

$$= \left[\lim_{R \to \infty} \int_{S_3} - \sum_{(\eta_\lambda = 0)} \lim_{\epsilon_j \to 0} \int_{S_3^{\prime j}} - \sum_{i=1}^{N} \lim_{\epsilon_i \to 0} \int_{S_3^i} \right] I_\mu^G \, d\sigma_\mu \tag{7}$$

where δ_i is spherical region with surface S_3^i enclosing point $x = a_i$. From Eqs. (6) and (7) it follows that

$$\int S'^{*} d^4x = \sum_{i=1}^{N} \lim_{\epsilon_i \to 0} \int_{S_3^i} I_\mu^G \, d\sigma_\mu \tag{8}$$

The right hand side is easily calculated to give the winding number q= N. The calculation using the expression in Eq. (1) is also straightforward. Using the arguments as above we find

$$\int S^* d^4x = \left[\lim_{R \to \infty} \int_{S_3} - \sum_{i=1}^{N} \lim_{\epsilon_i \to 0} \int_{S_3^i} \right] I_\mu \, d\sigma_\mu \tag{9}$$

and the right hand side is easily calculated. Essentially similar procedure was used, though in a different way, in the first calculation in Ref. 4.

We mention finally the recent observations made by Giambiagi and Rothe [7] and independently by Sciuto [8]. The fact that, near the singularites, A_μ behaves as pure gauge term, it should be possible to gauge away these singularities. They showed the existence of a regular form for multi-instanton solution obtained by a product of successive gauge transformations. The topological interpretation of winding number is then restored. The resulting form, however, is not compact for practical calculations.

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