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# Parametrization of Kerr Solution

*J. Gariel*[1*], *G. Marcilhacy*[1] and *N.O. Santos*[1,2,3†]

[1]Université Pierre et Marie Curie - CNRS/FRE 2460,
LERMA/ERGA, Tour 22-12, 4éme étage, Boîte 142, 4 place Jussieu,
75252 Paris Cedex 05, France.
[2]Laboratório Nacional de Computação Científica,
25651-070 Petrópolis RJ, Brazil.
[3]Centro Brasileiro de Pesquisas Físicas
22290-180 Rio de Janeiro RJ, Brazil.

**Abstract**

Using Ehlers and unitary transformations, from Kerr solution of the Ernst equation, we build a parametrized Kerr solution depending on three parameters.

*e-mail: gariel@ccr.jussieu.fr
†e-mail:santos@ccr.jussieu.fr and nos@cbpf.br

# 1 Introduction

In a preceding paper [1] we have presented a new three parameters axisymmetric stationary solution of Ernst equation by using Ehlers and unitary transformations on the Bonanos [2] solution. The parameters are associated with the total mass $M$ of the source and its angular momentum per unit mass $a$. The third parameter produces a topological deformation of the ergosphere.

Here we apply these transformations to the Kerr metric and we obtain again a solution with three parameters. The two parameters, $M$ and $a$, are present here too, and we call the third parameter $m_1$. We show that with a simple transformation the solution reduces to that of Kerr again, but notwithstanding, $m_1$ has a peculiar role. It allows to classify the three topological families of Kerr solutions obtained for the three cases $M > a$, $M = a$ and $M < a$, which until now were discussed separately ([3] see p 375). Varying the parameter $m_1$ we can pass continuously from one family to the other. We illustrate clearly this passage by drawing the ergospheres and horizons for different values of $m_1$.

# 2 Method of solution of Ernst equation

The element of a general axisymmetric stationary spacetime is the so called Papapetrou metric, which in cylindrical coordinates, $\rho$, $z$ and $\phi$, reads

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{1}$$

where the gravitational potentials, $f$, $\omega$ and $\gamma$, are functions of $\rho$ and $z$ only. The canonical coordinates of Weyl, $\rho$ and $z$, can be given in terms of prolate spheroidal coordinates, $\lambda$ and $\mu$, by the relations

$$\rho = k(\lambda^2 - 1)^{1/2}(-\mu^2)^{1/2}, \quad z = k\lambda\mu, \tag{2}$$

where $k > 0$ is a constant, $\lambda$ a radial coordinate and $-1 \leq \mu \leq 1$ is an angular coordinate. The metric (1) with relations (2) can be rewritten like

$$ds^2 = f(dt - \omega d\phi)^2$$
$$-\frac{k^2}{f}\left[e^{2\gamma}(\lambda^2 - \mu^2)\left(\frac{d\lambda^2}{\lambda^2 - 1} + \frac{d\mu^2}{1 - \mu^2}\right) + (\lambda^2 - 1)(1 - \mu^2)d\phi^2\right], \tag{3}$$

where the potentials are now functions of $\lambda$ and $\mu$. Einstein vacuum field equations reduce to the Ernst equation [4],

$$(\xi\bar{\xi} - 1)\nabla^2\xi = 2\bar{\xi}\nabla\xi \cdot \nabla\xi, \tag{4}$$

where $\nabla$ and $\nabla^2$ are the gradient and the three-dimensional Laplacian operators respectively, $\bar{\xi}$ is the conjugated complex potential of $\xi$, and in general its solution can be expressed as

$$\xi(\lambda, \mu) = P(\lambda, \mu) + iQ(\lambda, \mu), \tag{5}$$

where $P$ and $Q$ are real functions of $\lambda$ and $\mu$. To determine the potentials $f$, $\omega$ and $\gamma$ of the metric (3), the method consists to use the following relation between $f$, the twist potential $\Phi$ and $\xi$,

$$f + i\Phi = \frac{\xi - 1}{\xi + 1}, \tag{6}$$

which implies, with (5),

$$f = \frac{P^2 + Q^2 - 1}{R^2}, \quad \Phi = \frac{2Q}{R^2}, \tag{7}$$

where

$$R^2 = (P+1)^2 + Q^2. \tag{8}$$

In (6), $\Phi$ is a twist potential defined up to a constant and related to the dragging $\omega$ by the following differential equations,

$$\frac{\partial \omega}{\partial \lambda} = \frac{k(1-\mu^2)}{f^2}\frac{\partial \Phi}{\partial \mu}, \quad \frac{\partial \omega}{\partial \mu} = -\frac{k(\lambda^2-1)}{f^2}\frac{\partial \Phi}{\partial \lambda}. \tag{9}$$

The potential $\omega$ is obtained by integration of (9), and $\gamma$ is determined by quadratures. Any solution of Ernst equation is a solution of Einstein equations.

# 3 Parametrized Kerr solution

We start with the Kerr solution ([5] see p 382),

$$\xi_K = P_K + iQ_K, \tag{10}$$

with

$$P_K = p\lambda, \quad Q_K = q\mu, \tag{11}$$

where $p$ and $q$ are real constants satisfying

$$p^2 + q^2 = 1. \tag{12}$$

By means of the following particular Ehlers transformation on (10),

$$\xi_1 = \frac{c_1 \xi_K + d_1}{\bar{d}_1 \xi_K + \bar{c}_1}, \tag{13}$$

where $c_1$ and $d_1$ are complex constants satisfying

$$\begin{pmatrix} c_1 & d_1 \\ \bar{d}_1 & \bar{c}_1 \end{pmatrix} \in SU(1,1), \quad |c_1|^2 - |d_1|^2 = 1. \tag{14}$$

We choose for $c_1$ and $d_1$

$$c_1 = 1 + i\alpha_1, \quad d_1 = i\alpha_1, \tag{15}$$

being $\alpha_1$ a real constant. Then, a second step consists to perform an unitary transformation on $\xi_1$,

$$\xi_2 = -e^{i\theta_0}\xi_1 = (m+in)\xi_1, \quad m^2+n^2=1, \tag{16}$$

with $\theta_0$ an arbitrary real constant, and $m$ and $n$ real constants. Now we find with (10,13,16)

$$\xi_2 = \frac{A+iB}{C+iD}, \tag{17}$$

where

$$\begin{aligned} A &= P_K(m-\alpha_1 n) - Q_K(\alpha_1 m+n) - \alpha_1 n, \\ B &= P_K(\alpha_1 m+n) + Q_K(m-\alpha_1 n) + \alpha_1 m, \\ C &= -\alpha_1 Q_K - 1, \\ D &= \alpha_1(P_K+1). \end{aligned} \tag{18}$$

Choosing

$$\alpha_1 = -\frac{n}{2(1+m)} \tag{19}$$

and applying the method recalled in section 2, we find the potentials corresponding to the solution (1), with (7), of the Ernst equation,

$$f = \frac{p^2\lambda^2+q^2\mu^2-1}{(p\lambda-1)^2+q^2\mu^2}\cos^{-2}\frac{\theta_0}{2}, \tag{20}$$

$$\Phi = -\frac{2p\mu}{(p\lambda-1)^2+q^2\mu^2}\cos^{-2}\frac{\theta_0}{2}, \tag{21}$$

$$\omega = \frac{2kq(p\lambda-1)(1-\mu^2)}{p(p^2\lambda^2+q^2\mu^2-1)}\cos^{2}\frac{\theta_0}{2}. \tag{22}$$

We observe that for the calculation of $\Phi$ in (21) an adding constant that can be transformed away since has no role for the calculation of $\omega$ in (22). The factor $\cos^{-2}(\theta_0/2)$ in (20) can be absorbed by a rescaling process, such as $ds_2^2 = \cos^{-2}(\theta_0/2)ds^2$. Introducing the coordinates $r$ and $\theta$ through the Boyer-Lindquist transformation

$$\lambda = \frac{r-M}{k}, \quad \mu = \cos\theta, \tag{23}$$

we obtain asymptotically $r \to \infty$ for (20) and (22)

$$f \approx 1 + \frac{2k}{p}\frac{1}{r} + O\left(\frac{1}{r^2}\right), \tag{24}$$

$$\omega \approx 2\left(\frac{k}{p}\right)^2 \frac{\sin^2\theta}{r}\cos^2\frac{\theta_0}{2} + O\left(\frac{1}{r^2}\right), \tag{25}$$

which shows that the solution is asymptotically flat. From (24,25), differently from Kerr metric, we have

$$p = -\frac{k}{M}, \quad q = \frac{a}{M}\cos^{-2}\frac{\theta_0}{2} \tag{26}$$

and from (12) we have,

$$k^2 = M^2 - m_1^2 a^2, \tag{27}$$

where

$$m_1 = \cos^{-2}\frac{\theta_0}{2}. \tag{28}$$

with $m_1 \in [1, \infty)$.

Finally, with the above choice for the constants we can rewrite the potentials like

$$f = 1 - \frac{2Mr}{r^2 + m_1^2 a^2 \cos^2\theta}, \tag{29}$$

$$\omega = \frac{2aMr\sin^2\theta}{r^2 - 2Mr + m_1^2 a^2 \cos^2\theta}, \tag{30}$$

$$e^{2\gamma} = \frac{r^2 - 2Mr + m_1^2 a^2 \cos^2\theta}{r^2 - 2Mr + m_1^2 a^2 \cos^2\theta + M^2 \sin^2\theta}. \tag{31}$$

We see from (29-31) that when $m_1 = 1$ the Kerr metric is reobtained. Asymptotically, $r \to \infty$, (29,30) now becomes

$$f \approx 1 - \frac{2M}{r} + \frac{2m_1^2 a^2 M \cos^2\theta}{r^3} + O\left(\frac{1}{r^5}\right), \tag{32}$$

$$\omega \approx \frac{2aM\sin^2\theta}{r} + \frac{6aM^2\sin^2\theta}{r^2}$$
$$+ \frac{2aM\sin^2\theta(4M^2 - m_1^2 a^2 \cos^2\theta)}{r^3} + O\left(\frac{1}{r^4}\right). \tag{33}$$

Now writing the metric (29-31) in the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \tag{34}$$

we obtain for the metric coefficients,

$$g_{tt} = 1 - \frac{2Mr}{r^2 + m_1^2 a^2 \cos^2\theta}, \tag{35}$$

$$g_{t\phi} = -\frac{2Mar\sin^2\theta}{r^2 + m_1^2 a^2 \cos^2\theta}, \tag{36}$$

$$g_{rr} = -\frac{r^2 + m_1^2 a^2 \cos^2\theta}{r^2 - 2Mr + m_1^2 a^2}, \tag{37}$$

$$g_{\theta\theta} = -r^2 - m_1^2 a^2 \cos^2\theta, \tag{38}$$

$$g_{\phi\phi} = -\left\{ r^2 + m_1^2 a^2 + \frac{2Ma^2 r\sin^2\theta}{r^2 + m_1^2 a^2 \cos^2\theta} \right.$$
$$\left. \times \left[ m_1^2 + \frac{(m_1^2 - 1)2Mr}{r^2 - 2Mr + m_1^2 a^2 \cos^2\theta} \right] \right\} \sin^2\theta. \tag{39}$$

On (29-31) we can make the transformation

$$M_1 = \frac{M}{m_1}, \quad r_1 = \frac{r}{m_1} \tag{40}$$

producing

$$f = 1 - \frac{2M_1 r_1}{r_1^2 + a^2\cos^2\theta}, \tag{41}$$

$$\omega = \frac{2aM_1 r_1 \sin^2\theta}{r_1^2 - 2M_1 r_1 + a^2\cos^2\theta}, \tag{42}$$

$$e^{2\gamma} = \frac{r_1^2 - M_1 r_1 + a^2\cos^2\theta}{r_1^2 - 2M_1 r_1 + a^2\cos^2\theta + M_1^2\sin\theta}, \tag{43}$$

which shows that solution (29-31) corresponds to a Kerr solution with parameters $M_1$ and $a$. We see too from (41-43) that this solution is valid too for $0 < m_1 < 1$, hence the solution (29-31) is valid for

$$m_1 \in ]0, \infty[. \tag{44}$$

The mapping $m_1 = 0$ is not allowed since then (40) has no meaning.

From (32,33) we see that the difference between classic Kerr solution and (29,30) lies in the third order asymptotical behaviour for $f$ and for $\omega$. The parameters $M$ and $a$ can be deduced from observations implying the potentials $f$ and $\omega$ up to second order in $1/r$. Once these parameters are determined, we can find $m_1$ if the observations are refined up to third order in $1/r$ for $f$ and $\omega$, which is the postpost Newtonian approximation.

For given $M$ and $a$, $m_1 \neq 1$ changes the geometry of spacetime, so we can classify the solutions like,

$$i)\ \ m_1 > \frac{M}{a}, \quad \text{Kerr black hole,}$$
$$ii)\ \ m_1 = \frac{M}{a}, \quad \text{extreme Kerr black hole,}$$
$$iii)\ \ m_1 < \frac{M}{a}, \quad \text{Kerr without event horizon.}$$

In figure 1 we have plotted the ergospheres and event horizons, when they are defined, for different values of $m_1$. We obtain the following interesting features.

- The topology of the exterior ergospheres change according to the values of $m_1$: a) For $0 < m_1 < M/a$ (curves $a$ and $b$) we have the usual spherical topology of Kerr. b) For $m_1 = M/a$ (curve $c$) we have the topology of an extreme black hole of Kerr. c) For $m_1 > M/a$ (curves $d$, $e$ and $f$) the topology of the ergospheres is toroidal. When $m_1 \to \infty$ the ergosphere tends to a disc in the equatorial plane.

- The radius of the event horizon decreases and the radius of the Cauchy horizon increases when $m_1 (< M/a)$ increases, until its limiting value when $m_1 = M/a$. For $m_1 > M/a$ the horizons are no more defined.

- The interior ergosphere, for $m_1 < M/a$, increases with increasing $m_1$ and is disconnected of the exterior ergosphere (curves $a$ and $a$). When $m_1$ attains the value $m_1 = M/a$, the interior ergosphere joins the exterior ergosphere continuously (curve

$c$). For $m_1 > M/a$, the interior and exterior ergospheres are connected forming one ergosphere (curves $d$, $e$ and $f$).

- For $m_1 > M/a$, from the fact that both ergospheres are connected and the horizons disappear, the singularity, forming a ring, becomes naked for certain values of the angle aperture centered at the axis $z$. The angle increases with increasing values of $m_1$. This configuration is identical to Kerr spacetime for $M < a$ ([3] see p 375).

# 4 Concluding remarks

The solution that we obtained (29-31), in spite of the transformation (40) leading to the Kerr solution, is different from it for the following reasons. For a given source $(M, a)$, the solution (29-31) can produce a toroidal topology for its exterior ergosphere even in the case $M > a$, which is never the case for the Kerr solution with the same $(M, a)$: to be produced, it is sufficient that its third parameter satifies $m_1 > M/a$. Likewise, if $M < a$, the solution (29-31) can produce a spherical topology for its exterior ergosphere, - it suffices that $m_1 < M/a$ -, which is never the case for the Kerr solution with that source $(M, a)$. If, far from the source, we can associate the potentials $f$ and $\omega$ to observable quantities, then up to the post-Newtonian approximation, $O(r^{-2})$, we can determine the parameters $M$ and $a$. The metric up to this order is indescernible from that of Kerr. But if we make observations up to $O(r^{-3})$, then, in principle, we can determine $m_1$, and if $m_1 \neq 1$ then we can conclude that the metric is different from Kerr with a topology depending upon its value.

Furthermore, if we compare the solution (29-31) to the one found in [1] we observe that the topology of the ergospheres are different. The angle aperture in [1] appeared in the equatorial plane, while here the equatorial plane includes, on both sides the axis $z$, the ergosphere for any value of $m_1$.

For (29-31) the horizons do not exist when $m_1 > M/a$, while the singularity is naked for increasing angle apertures, centered at the axis $z$, while $m_1$ increases. While in the solution found in [1] the singularity is always dressed because the event horizon is always defined.

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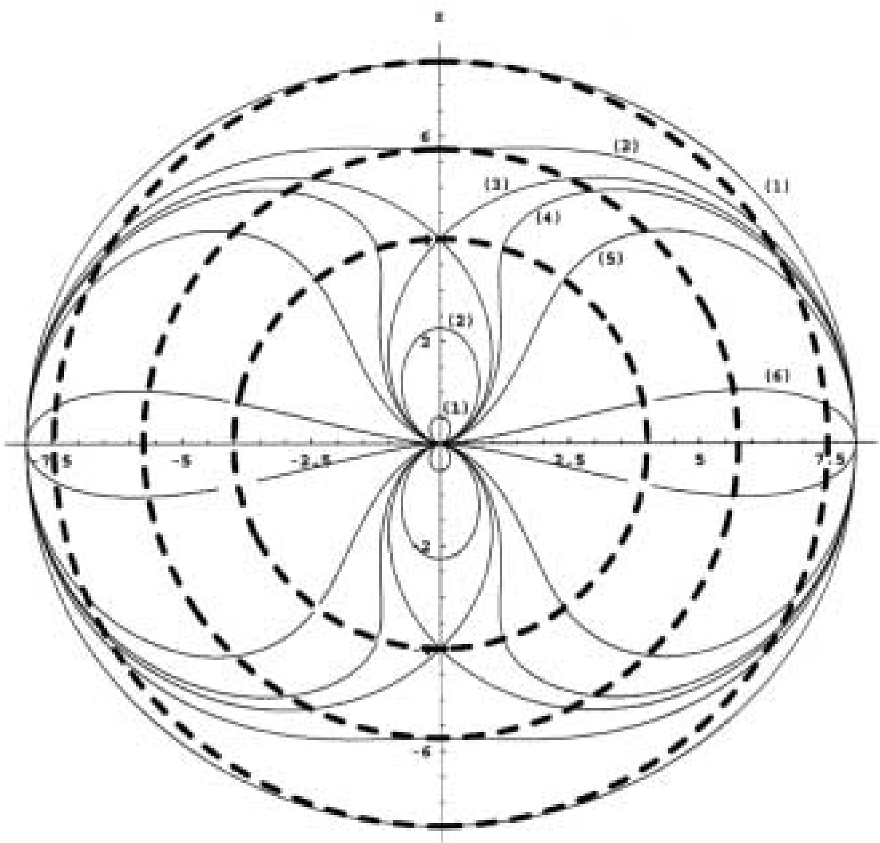


Figure 1: Plots of the exterior and interior ergospheres ($a$ to $f$), and the corresponding event horizons (dashed lines) for $M = 4$, $a = 2$ and different values of the parameter $m_1$: i) $m_1 < M/a$, $m_1 = 1$ (Kerr) curves $a$, $m_1 = 1.8$ curves $b$; ii) $m_1 = M/a = 2$ (extreme black hole where the two horizons are the same), curve $c$ ; iii) $m_1 > M/a$ (the horizons are no more defined), $m_1 = 2.1$ curve $d$, $m_1 = 2.5$ curve $e$, $m_1 = 10$ curve $f$.