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HOMOCLINIC PHENOMENA IN THE GRAVITATIONAL COLLAPSE

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A class of Bianchi IX cosmological models is shown to have chaotic gravitational collapse, due to Poincaré's homoclinic phenomena. We can program such models so that for *any* given positive integer N(N=∞ included) the universe undergoes N non-periodic oscillations (each oscillation requiring a long time) before collapsing. For N=∞ the universe undergoes periodic oscillations.

Key-words: Cosmological models; Gravitational collapse; Chaotic behavior; Homoclinic phenomena.

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Stochastic properties in the gravitational collapse of a cosmological model were first discussed in the papers of Belinskii, Khalatnikov and Lifshitz.[1] They showed that the approach to the singularity ($t \to 0$) of a general Bianchi IX cosmological solution is an oscillatory mode, consisting of an infinite sequence of periods (called eras) during which two of the scale functions oscillate and the third one decreases monotonically; on passing from one era to another the monotonic behaviour is transferred to another of the three scale functions. The "length" of each era is determined by a sequence of numbers $\{X_s \mid 0<X_s<1,\ s = \text{integer}\}$, each of which arises from the preceding one by the transformation X_{s+1} = Fractional Part of $\frac{1}{X_s}$. From the properties of this map it is obtained that the behaviour of the model becomes stochastic on approaching to the singularity ($t=0$) for arbitrary initial condition given at a $t_0>0$. For the vacuum case (Mixmaster universe[2]) and other variables, Chernoff and Barrow[3] derived maps which also exhibit strong stochastic properties. The stochasticity is an intrinsic feature of the maps which approximate the dynamics of the model (described by Einstein equations) close to the cosmological singularity.

In the present letter we produce an exact example in which the stochastic behaviour in the gravitational collapse of a Bianchi IX cosmological model is due to Poincaré's homoclinic phenomena.[4] This strongly suggests that these phenomena are the basic feature of chaotic behaviour in a general case.[1,3] We use some new modes of doing *exact* perturbations of the Einstein universe. The necessary mathematical prerequisites are given without proof but we provide a number of references.

The topology of the Einstein universe[5] is RxS^3. Here S^3 is Hopf's fiber bundle with base space S^2 and fiber homeomorphic to S^1.[6] Let (X_1, X_2, X_3) be left invariant vector fields over S^3, with X_1 tangent to the fiber S_1 and σ^1 its corresponding dual 1-form. The tangent space of S^3 at any point can then be split into the X_1 component with line element $g_V = (\sigma^1)^2$ and into the orthogonal complement whose line element g_H is pulled back from the geometry of the base space S^2. The temporal coordinate is defined on R, and the geometry of the Einstein universe is then split into

$$ds^2 = dt^2 - \lambda^2 (g_V + g_H) \tag{1}$$

according to the local fibering RxS^1xU where $U \subset S^2$, and λ^2 is a constant parameter. Starting from the geometry (1) we construct the following family of models: the radius of the 2-sphere S^2 and the radius of S^1 are made time-dependent, with respective time dependence $B(t)$ and $A(t)$. We then obtain in RxS^3 the time-dependent geometry

$$ds^2 = dt^2 - \{A^2(t) g_V + B^2(t) g_H\} \tag{2}$$

The dynamics of the models is described by Einstein equations with the cosmological constant term. We take for the matter content of the model a perfect fluid with matter-energy density ρ, pressure p and four-velocity field $\partial/\partial t$. Einstein equations for (2) reduce to three independent differential equations. Two of them define ρ and p, whereas the third one yields the differential equation

$$\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 - \frac{\ddot{A}}{A} - \frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} - \frac{A^2}{B^4} = 0 \qquad (3)$$

The physically admissible solution of (3) are restricted by the energy conditions[7] which ρ and p must satisfy. For all cases considered in this letter the energy conditions are always satisfied.

We distinguish from (3) the following cases: (I) *The Einstein universe*: $A^2 = B^2 = \lambda^2$ and the field equations imply $k\rho = -2\Lambda = = \frac{1}{2\lambda^2}$, $p=0$. (II) *Exact perturbation of the Sector* (base space) S^2 *of the geometry*: $A^2 = \lambda^2$, and from (3) the dynamics can be described by the Hamiltonian $H = \frac{1}{2}(\dot{q})^2 + V(q) = C$ with $V(q) = 2q - 2\lambda^2 \ell n q$ and $C = \text{const}$, where we have denoted $q = B^2(t)$. The graph of $V(q)$ is depicted in Fig. 1. The minimum of the potential occurs for $q = \lambda^2 = B_E^2$, that is, the configuration of the Einstein universe is a point of stability of the class of models (II).

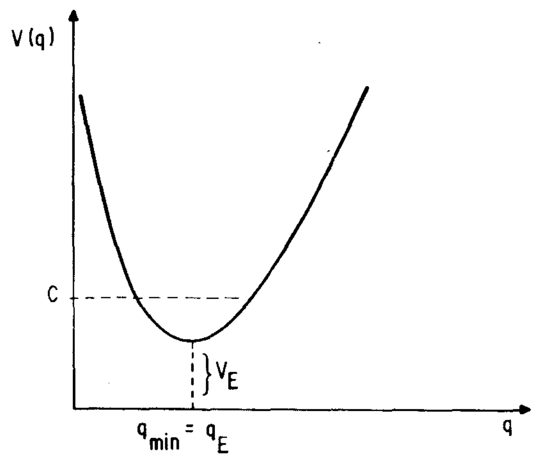

Fig. 1

The trajectories of the system in the phase plane $(q,\dot{q})$ are closed curves about $(q_E,0)$. Each periodic trajectory corresponds to an exact stable cosmological solution depending on the energy parameter C and which can be confined to any neighborhood of the stability point $(q_E,0)$ by a suitable variation of C. In this sense these are denoted *exact* stable perturbations of the Einstein universe. (III) *Exact Perturbation of the Sector (fiber)* S^1: $B^2=\lambda^2$ and the dynamics of A(t) is described by the time independent Hamiltonian $H=\frac{1}{2}(\dot{A})^2+V(A)=D$, where $V(A)=\frac{1}{4\lambda^4}(A^4-2\lambda^2A^2)$. The graph of V(A) is depicted in Fig. 2. The minimum of the potential also corresponds to the configuration of the Einstein universe.

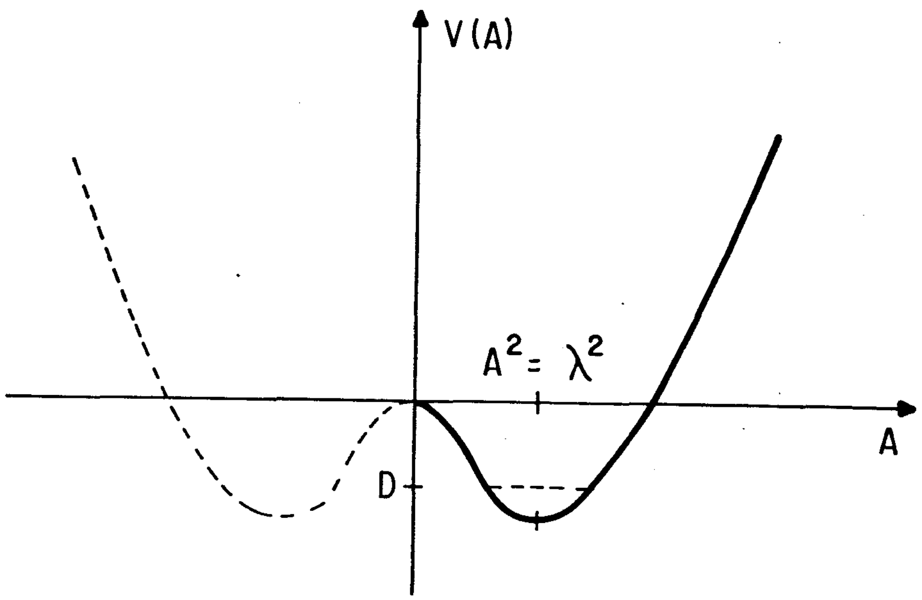

Fig. 2

We restrict ourselves to the non-negative range of A since A=0 corresponds to the physical singularity of the model. From the point of view of mathematical dynamical systems, both regions are admissible. The trajectories of the system in the phase plane $(A,\dot{A})$ are depicted in Fig. 3. It is most important to re mark that the unstable equilibrium point (0,0) is homoclinic[8,9] and its homoclinic trajectories H^+ and H^- correspond to the value of the energy parameter D=0,

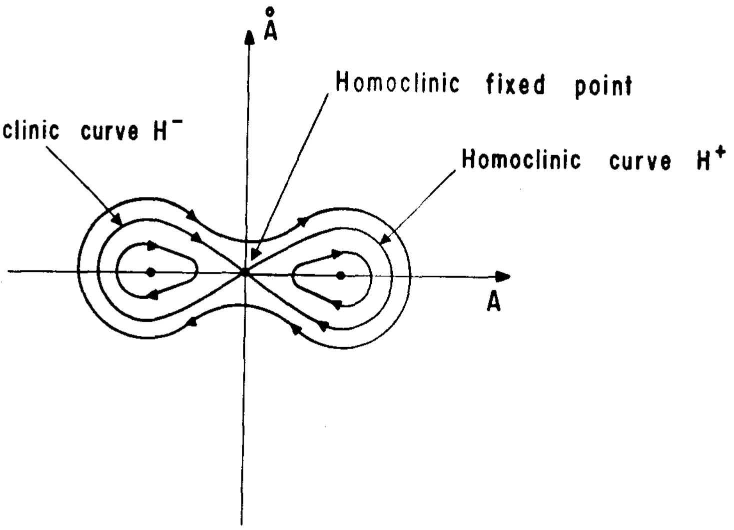

Fig. 3

(IV) This is the case of interest, the interaction (via gravitation) of the degrees of freedom of the two sectors S^1 and S^2. We consider the special mode in which the oscillations in the sector S^2 excite the degree of freedom of the sector S^1, via gravitational interaction. Namely, taking for B(t,C) a periodic exact solution of (II) and substituting in (3) we obtain

$$A'' + \frac{1}{B^2}(A^3-\lambda^2 A) = 0 \qquad (4)$$

Here a prime denotes $d/d\eta$ where the variable η is defined by $d\eta = B^{-1}dt$. For $B^2 = \text{const} = \lambda^2$ the curves in the phase plane are given as in Fig. 3. The Hamiltonian system obtained from (4) is *ergodic* and *even mixing*[10] for a large set of initial conditions, which depend on the value of C determining the amplitude and period of $B^2(t,C)$ in (II). We shall not discuss these properties here, but for our purposes in the present letter it is sufficient to consider $C = V_E + \varepsilon^2$, ε^2 infinitesimal. We obtain from (II), with $\lambda^2 = 1$,

$$B^2 = 1 + \varepsilon \cos\sqrt{2}\eta \tag{5}$$

This infinitesimal oscillation is enough to generate chaotic evolution of the universe, as we proceed to show.

The phase-plane trajectories of the dynamical system (4) are perturbations of the curves of Fig. 3 for the conservative case (III). We concentrate now on the homoclinic curves of Fig. 3 and we use a method due to Melnikov[11] to detect Poincaré's homoclinic phenomena.[8,12] In our case the homoclinic fixed point (0,0) is unchanged under the perturbation. By a general result of dynamical systems, the Poincaré map[13] to period $T=\sqrt{2}\pi$ possesses invariant curves Γ^s and Γ^u (respectively called the stable and unstable 1-dim manifolds), close (to order ε, for future and past times, respectively) to the unperturbed homoclinic curve, as in Fig. 4).

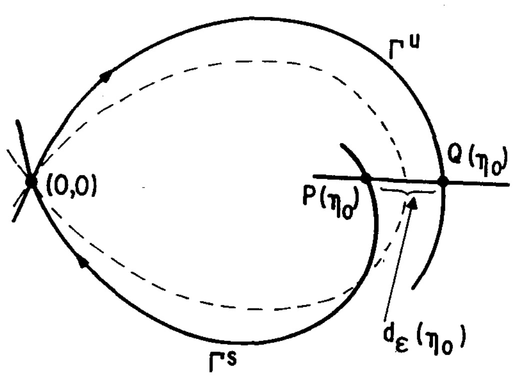

Fig. 4

The *Melnikov's distance function* $d_\varepsilon(\eta_0)$ can be written as

$$d_\varepsilon(\eta_0) = \varepsilon M(\eta_0) + O(\varepsilon^2)$$

where η_0 is the variable η corresponding to the initial unperturbed homoclinic trajectory. If the function M has simple zeros, then for $\varepsilon > 0$ sufficiently small Γ^s and Γ^u intersect transversally. This is our case since we obtain

$$M(\eta_0) = K \sin\sqrt{2}\eta_0$$

where K is a non-zero constant, given by a residue integral.[14] In fact $M(\eta_0)$ has an infinity of simple zeros which implies that Γ^s and Γ^u intersect transversally infinite times. Due to unicity of solutions none of the two curves intersect itself, but the area-preserving property of the Poincaré map of a Hamiltonian dynamical system[10,13] implies that each one bends in a complex way that it intersects all the loops of the other in infinitely many times.[12,15] The above properties imply that these invariant curves are "area filling", namely any arbitrary point eventually maps arbitrarily close to any other point in the region considered. These are the so-called homoclinic phenomena of Poincaré,[4,12,15] and the complexity of behaviour of the curves introduce a stochastic element in the dynamics. In fact, let us call the first intersection point J (see Fig. 5) and let N(J) be a neighborhood of J. It can be shown that N(J) has a certain subset I with the following properties: it is uncountable, closed, of measure zero[16] and contains points which are arbitrarily close to J.[15,17] This set I is invariant under iterates of the Poincaré map and has chaotic dynamics in the sense that a suitable power of the Poincaré map is equivalent to a Bernoulli shift on two symbols.[12,15] Introducing the notation of symbolic dynamics where +1 (respectively -1) corresponds to the part of a trajectory which remains near (in a neighborhood of order ε) the unperturbed homoclinic orbit H^+ (respectively the homoclinic orbit H^-), cf. Fig. 3 - until it reaches the neighborhood N again - we can always find points of I such that orbits of these points visit neighborhoods of H^+ and H^- in any specified order.

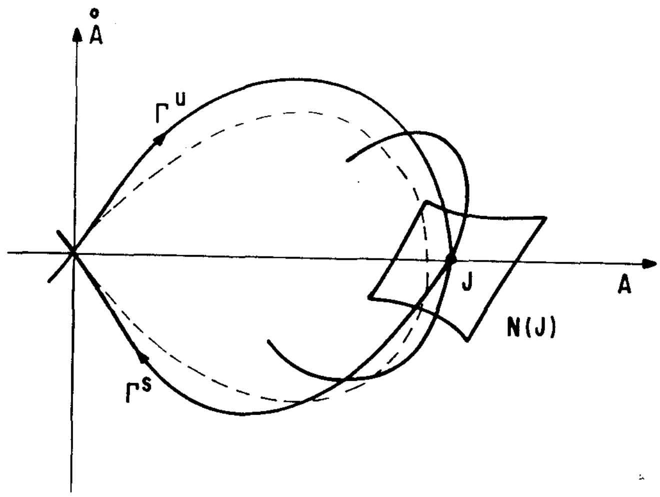

Fig. 5

In other words, for any given symbolic sequence of 1's and -1's, say (...1111-11-111-1....) there will be a point in I such that its orbit will be about H^+ and H^- in the specified order. Notice that the appearance of two adjacent symbols 1 -1 in the specified symbolic sequence means that the orbit *crosses* the axis A=0 on passing from the neighborhood of H^+ to the neighborhood of H^-.

However we know that A=0 corresponds to the physical singularity of the cosmological model (divergence of energy density, curvature invariants, etc): thus the appearance of the symbol -1 for the first time in the symbolic sequence corresponds to the gravitational collapse of the system. Using the above properties we now make the following *program* for the dynamics of the universe: for *any* given positive integer N we construct the symbolic sequence

$$\underbrace{111\ldots\ldots11}_{N\text{ times}}-1\ldots\ldots$$

and find a point $Q_0 \in I$ corresponding to the above sequence. We then have - for Q_0 as the initial value of our system - that the model oscillates (more precisely oscillates non-periodically about the homoclinic curve H^+, each oscillation requiring a long time)[18] N times before undergoing gravitational collapse. The case $N = \infty$ deserves a special comment. It is a remarkable mathematical fact[17] that the doubly infinite sequence of 1's corresponds to a periodic trajectory for which $A > 0$. Therefore this case corresponds to an universe undergoing periodic oscillations, without collapsing.

The probabilistic programming of the gravitational collapse of the present model was shown for exact perturbations of the sectors S^1 enhanced via gravitational interaction by oscillations in the sector S^2 of the geometry of the Einstein universe. This result could in principle be extended to more general Bianchi IX models. A more complicated interaction of the oscillations of the sector S^1 and the sector S^2 via Einstein equations could give analogous properties (at last to first order approximation) although a complete analysis shall require a more elaborate exam of the general system (3) restricted by the energy conditions.

The important feature of our example is the existence, in the case (III), of the homoclinic curves H^+ and H^- linking the fixed point (0,0) to itself. The introduction of a small perturbation (via gravitational interaction) by the oscillations of the sector S^2 is sufficient to break this smooth link and produce homoclinic phenomena of Poincaré in a small neighborhood of $H^+ \cup H^- \cup \{(0,0)\}$. The homoclinic phenomena are the basis of the chaotic behaviour of the system and of its chaotic approach to the cosmological singularity.

Although we have shown this for a particular example, it strongly suggests that, for the general case, homoclinic phenomena are the basic ingredient of the stochastic behaviour in the gravitational collapse of closed cosmological models.

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