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AN OPTIMIZATION ALGORITHM FOR THE PILE DRIVER PROBLEM

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ABSTRACT

In this paper we present the analysis of an algorithm of Uzawa type to compute solutions of the quasi variational inequality

$$\text{(QVI)} \quad \left(\frac{\partial^2 u}{\partial t^2}, v - \frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial^2 u}{\partial x \partial t}\right) + \left(\frac{\partial^2 u}{\partial x \partial t}, \frac{\partial v}{\partial x} - \frac{\partial^2 u}{\partial x \partial t}\right) +$$

$$+ \left[u(1,t) + \frac{\partial u}{\partial t}(1,t)\right]\left[v(1) - \frac{\partial u}{\partial t}(1,t)\right] + J(u;v) - J(u;\frac{\partial u}{\partial t}) \geq$$

$$(f, v - \frac{\partial u}{\partial t}) + F(t)\left[v(0) - \frac{\partial u}{\partial t}(0,t)\right], \quad t > 0, \ \forall\, v \in H^1(0,1),$$

which is a model for the dynamics of a pile driven into the ground under the action of a pile hammer. In (QVI) $(\cdot,\cdot)$ is the scalar product in $L^2(0,1)$ and $J(u;\cdot)$ is a convex functional on $H^1(0,1)$, for each u, describing the soil-pile friction effect.

1. INTRODUCTION

Our purpose is to approximate the solution of an initial value problem for a quasi variational inequality of evolution type, introduced in [4] as a model of a problem appearing in Foundation Engineering. The general idea is to discretize the time variable and solve at each time level a mathematical programming problem for the approximate solution at that level.

With the usual notation

$$\Omega = (0,1) \quad , \quad H^1(\Omega) = \{v \varepsilon L^2(\Omega) | \frac{\partial v}{\partial x} \varepsilon L^2(\Omega)\} \quad ,$$

$$(u,v) = \int_\Omega u(x)\ v(x)\ dx \quad ,$$

$$(u,v)_1 = (u,v) + (\frac{\partial u}{\partial x} , \frac{\partial v}{\partial x}) \quad ,$$

$$a(u;v) = a\ (\frac{\partial u}{\partial x} , \frac{\partial v}{\partial x}) + k_1\ u(1)v(1) \quad ,$$

$$b(u;v) = b\ (\frac{\partial u}{\partial x} , \frac{\partial v}{\partial x}) + k_2\ u(1)v(1) \quad ,$$

$$J(u;v) = c\int_\Omega H(x+u-1)(x+u-1)|v(x)|dx \quad ,$$

$$H(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda \leq 0 \ , \\ 1 & \text{if} \quad \lambda > 0 \ ; \end{cases}$$

$$L^p(0,T;V) = \{v:[0,T] \to V \Big| |v|_{L^p(0,T;V)} = \left[\int_0^T |v(t)|_V^p\ dt\right]^{1/p} < \infty \},$$

$$0 < T \leq \infty \quad , \quad 1 \leq p < \infty \quad ,$$

$$L^\infty(0,T;V) = \{v:[0,T] \to V \Big| |v|_{L^\infty(0,T;V)} = \operatorname*{ess\,sup}_{t\varepsilon(0,T)} |v(t)|_V < \infty \} \ ,$$

where a, k_1, b, k_2, c, T are positive physical parameters and V can be either $L^2(\Omega)$ or $H^1(\Omega)$, the continuous problem is to find $u \varepsilon L^\infty(0,T;H^1(\Omega))$ such that

$$\frac{\partial u}{\partial t} \ \varepsilon\ L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^1(\Omega)) \quad , \qquad (1.1)$$

$$\frac{\partial^2 u}{\partial t^2} \ \varepsilon\ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \quad , \qquad (1.2)$$

$$u(0) = 0 \quad , \tag{1.3}$$

$$\frac{\partial u}{\partial t}(0) = 0 \quad , \tag{1.4}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\, ,\; v - \frac{\partial u}{\partial t}\right) + a\left(u; v - \frac{\partial u}{\partial t}\right) +$$

$$b\left(\frac{\partial u}{\partial t}\, ;\; v - \frac{\partial u}{\partial t}\right) + J(u;v) - J\left(u; \frac{\partial u}{\partial t}\right) \geq \tag{1.5}$$

$$\left(f, v - \frac{\partial u}{\partial t}\right) + F(t)\left[v(0) - \frac{\partial u}{\partial t}(0,t)\right] \quad ,$$

$$\forall\, v \in H^1(\Omega) \quad , \quad \text{a.e. } t \in (0,T) \quad ,$$

with f and F given.

The justification of (1.3)-(1.5) as a description of the motion of a one-dimensional visco-elastic pile penetrating into the soil under the action of the characteristic force of the pile driver, the resistance force of the soil and the friction on the contact surface, is made in [4] . The unknown function $u(x,t)$ is the displacement field, which maps a "particle" x from the initial configuration of the pile to its position $x + u(x,t)$ at time t.

From the mathematical side, the basis for that justification is the following theorem, quoted here for future reference:

<u>Theorem 1.1</u> - Given $f \in L^2(0,\infty;L^2(\Omega))$ and $F \in L^2(0,\infty;R)$ such that $\frac{\partial f}{\partial t} \in L^2(0,\infty;L^2(\Omega))$, $\frac{\partial F}{\partial t} \in L^2(0,\infty;R)$ and $F(0) = 0$, then, for any given $0 < T \leq \infty$, there exists a unique $u \in L^\infty(0,T;H^1(\Omega))$ satisfying (1.1)-(1.5). Furthermore, the motion is stable and tends to rest as $t \to \infty$, in the sense that

$$|u|_1(t) \text{ is bounded on } [0,+\infty) \quad ,$$

$$\left|\frac{\partial u}{\partial t}\right|_1 (t) \to 0 \quad \text{as} \quad t \to \infty ,$$

where $|v|_1 = \sqrt{(v,v)_1}$.

Proof. See [4] .

We shall be concerned with approximations of the function $u(x,t)$ and its related fields with physical significance. To define the approximation scheme we first introduce the discretizations

$$x_j = jh \quad , \quad h = \frac{1}{M} \quad , \quad j = 0,1,\ldots, M \quad ,$$
$$t_n = nk \quad , \quad k = \frac{T}{N} \quad , \quad n = 0,1,\ldots, N \quad ,$$

where T is a fixed time level and M, N are given positive integers. After that, take the basic functions

$$\lambda_h^j(x) = \begin{cases} 1 , & jh \leq x < (j+1)\,h \quad , \\ 0 & \text{otherwise}, \; j = 0,1,\ldots, M-1 \quad , \end{cases}$$

$$\theta_k^n(t) = \begin{cases} 1 & nk \leq t < (n+1)\,k \quad , \\ 0 & \text{otherwise}, \; n = 0,1,\ldots, N-1 \quad , \end{cases}$$

that allow us to introduce the spaces of approximants

$$V_h^1 = \{ v \in H^1(\Omega) \,|\, v(x) = \sum_{j=0}^{M-1} \left[v^j + (v^{j+1} - v^j)\left(\frac{x}{h} - j\right)\right] \lambda_h^j(x) \quad , \quad v^j \in R \},$$

$$V_k^0 = \{ v \in L^2(0,T) \,|\, v(t) = \sum_{n=0}^{N-1} v^n \theta_k^n(t) \quad , \quad v^n \in R \} .$$

If we have a function S defined at the time levels t_n, we denote by S^n the value of S at t_n, and define

$$S^{n+1/2} = \frac{1}{2}\,(S^{n+1} + S^n) \quad ,$$
$$\partial_t S^n = \frac{S^{n+1} - S^n}{k} \quad ,$$

$$\delta_t S^n = \frac{S^{n+1} - S^{n-1}}{2k} = \frac{\partial_t S^n + \partial_t S^{n-1}}{2} ,$$

$$\partial_t^2 S^n = \frac{S^{n+1} - 2S^n + S^{n-1}}{k^2} = \frac{\partial_t S^n - \partial_t S^{n-1}}{k} .$$

We shall need also the notation

$$S_{h,k}(x,t) = \sum_{n=0}^{N} S^n(x)\theta_k^n(t) ,$$

$$\partial_t S_{h,k}(x,t) = \sum_{n=0}^{N-1} \partial_t S^n(x)\theta_k^n(t) ,$$

$$\delta_t S_{h,k}(x,t) = \sum_{n=1}^{N-1} \delta_t S^n(x)\theta_k^n(t) ,$$

$$\partial_t^2 S_{h,k}(x,t) = \sum_{n=1}^{N-1} \partial_t^2 S^n(x)\theta_k^n(t) ,$$

where $S^n \varepsilon V_h^1$, $n = 0,1,\dots,N$.

The approximation we propose for the solution u of (1.1)-(1.5) is the function $U_{h,k}(x,t) = \{U^n(x)\}_{n=0}^{N}$, characterized by the following conditions:

$$U^n \ \varepsilon\ V_h^1 , \quad n = 0,1,\dots,N \tag{1.6}$$

$$U^0 = 0 , \tag{1.7}$$

$$U^1 = 0 , \tag{1.8}$$

$$(\partial_t^2 U^n, V - \delta_t U^n) + a\,(U^n; V-\delta_t U^n) + b(\delta_t U^n; V-\delta_t U^n) + J(U^n; V) - \tag{1.9}$$

$$-J(U^n;\delta_t U^n) \geq (f_n, V-\delta_t U^n) + F_n\left[V(0)-\delta_t U^n(0)\right] ,$$

$$\forall\ V\ \varepsilon\ V_h^1 , \quad n \geq 1 ,$$

where $f_n = f(x,t_n)$, $F_n = F(t_n)$. Such scheme is consistent with (1.1)-(1.5) with local error of order k^2.

We observe that in view of the relation

$$\partial_t^2 U^n = \frac{2}{k} (\delta_t U^n - \partial_t U^{n-1}) \quad ,$$

inequality (1.9) can be written in the form

$$(\delta_t U^n, V-\delta_t U^n) + \frac{k}{2} b(\delta_t U^n; V-\delta_t U^n) +$$

$$+ \frac{k}{2} J(U^n;V) - \frac{k}{2} J(U^n;\delta_t U^n) \geq$$

$$\frac{k}{2} (f_n, V-\delta_t U^n) - \frac{k}{2} a(U^n;V-\delta_t U^n)$$

$$+ \frac{k}{2} F_n \left[V(0) - \delta_t U^n(0)\right] + (\partial_t U^{n-1}, V-\delta_t U^n) \quad ,$$

$$\forall V \in V_h^1 .$$

Hence, if we define

$$A: \quad H^1(\Omega) \times H^1(\Omega) \to R \tag{1.10}$$

$$\{u,v\} \to (u,v) + \frac{k}{2} b(u;v) ,$$

bilinear, symmetric, continuous and coercive form on $H^1(\Omega)$,

$$L_n: \quad H^1(\Omega) \to R \quad , \tag{1.11}$$

$$v \to \frac{k}{2} (f_n, v) + \frac{k}{2} F_n v(0) +$$

$$+ (\partial_t U^{n-1}, v) - \frac{k}{2} a(U^n;v) ,$$

linear and continuous forms on $H^1(\Omega)$,

$$j_n : H^1(\Omega) \rightarrow R \quad , \tag{1.12}$$

$$v \rightarrow \frac{k}{2} J(U^n; v) \quad ,$$

convex and continuous forms on $H^1(\Omega)$, the defining conditions (1.6)-(1.9) are equivalent to

$$U^n \in V_h^1 \quad , \quad n = 0,1,\ldots, N \quad , \tag{1.13}$$

$$U^0 = U^1 = 0 \quad , \tag{1.14}$$

$$A(\delta_t U^n; V-\delta_t U^n) + j_n(V) - j_n(\delta_t U^n) \geq \tag{1.15}$$

$$L_n(V - \delta_t U^n) \quad , \quad \forall\, V \in V_h^1 \quad ,$$

$$U^{n+1} = U^{n-1} + 2k\delta_t U^n \quad , \quad n = 1,2,\ldots,N-1 \; . \tag{1.16}$$

Two facts must be observed at this point:

(i) if we assume $f \in L^2(0,\infty;L^2(\Omega))$ and $F \in L^2(0,\infty;R)$, the families of functions $\{L_n\}$ and $\{j_n\}$ are equicontinuous on $H^1(\Omega)$ by virtue of the stability results we shall prove for U^n and $\partial_t U^{n-1}$ in section 3.

(ii) At each time level we can uniquely solve the "stationary problem" (1.15) for $\delta_t U^n$, since $A(\cdot;\cdot)$, $j_n(\cdot)$ and $L_n(\cdot)$ satisfy the hypothesis of a theorem of Lions-Stampacchia's. Furthermore, $\delta_t U^n$ can be characterized as the solution of the optimization problem

$$\inf_{v_h \in V_h^1} \{ \frac{1}{2} A(v_h; v_h) - L_n(v_h) + j_n(V_h) \} \; . \tag{1.17}$$

Hence equations (1.13)-(1.16) can give us an explicit algorithm to compute step by step the approximations at the various time levels if we are able to produce a numerical solution of the optimization problem (1.17) at each step.

The objective of this paper is to analize a lower level scheme of computation coupling (1.13), (1.14) and (1.16) with an Uzawa type algorithm to generate the solution of problem (1.17). We shall prove a convergence result in section 4, but first we present and analize the optimization algorithm in section 2, in a more general framework, and discuss two stability lemmas in section 3.

2. THE OPTIMIZATION ALGORITHM

In this section we drop the lower indices from j_n and L_n. We consider the "stationary" problem, equivalent to (1.17),

$$\begin{cases} \text{(i)} \qquad u \in V_h^1 \ , \\ \text{(ii)} \qquad A(u;v-u) + j(v)-j(u) \geq L(v-u) \ , \quad v \in V_h^1 \ , \end{cases} \tag{2.1}$$

where A, j and L retain the properties assumed in section 1 and, in particular,

$$j(v) = \int_\Omega g(x)|v(x)|dx \quad ,$$

where g is assumed given, positive and bounded on Ω.

Let us start proving the existence of multipliers for problem (2.1). Defining

$$\Lambda = \{q \in L^2(\Omega) \,|\, |q(x)| \leq g(x) \text{ a.e. } \} \ ,$$

we have

<u>Theorem 2.1</u> The solution u of (2.1) is characterized by the existence of a multiplier p such that

$$\begin{cases} u \in V_h^1 , \\ A(u;v)+\int_\Omega p(x)v(x)dx = L(v), \forall\ v \in V_h^1 , \end{cases} \tag{2.2}$$

$$\begin{cases} p \in \Lambda , \\ pu = g|u| \quad \text{a.e.} . \end{cases} \tag{2.3}$$

<u>Proof</u>. Assume (2.1) and take v = 0 to get

$$A(u;u) + j(u) \leq L(u) ,$$

and then v = 2u to get

$$A(u;u) + j(u) \geq L(u) ,$$

that is,

$$A(u;u) + j(u) = L(u) . \tag{2.4}$$

With $\varepsilon > 0$ we regularize j by j_ε defined by

$$j_\varepsilon(v) = \int_\Omega g(x) \sqrt{\varepsilon^2+v^2(x)}\ dx .$$

Since j_ε is convex and continuous on V_h^1, the regularized problem

$$\begin{cases} \text{(i)} \quad u_\varepsilon \in V_h^1 , \\ \text{(ii)} \quad A(u_\varepsilon;v-u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq L(v-u_\varepsilon) , \end{cases} \tag{2.5}$$

$$\forall\ v \in V_h^1 ,$$

has a unique solution.

We claim that u_ε converges strongly to u in V_h^1 when $\varepsilon \to 0$. Taking $v = u_\varepsilon$ in (2.1), v = u in (2.5) and adding we get

$$A(u_\varepsilon - u; u_\varepsilon - u) + j_\varepsilon(u_\varepsilon) - j(u_\varepsilon) \le j_\varepsilon(u) - j(u). \tag{2.6}$$

Now, from the inequality

$$0 < \sqrt{x^2+\varepsilon^2} - |x| = \frac{\varepsilon^2}{\sqrt{x^2+\varepsilon^2} + |x|} \le \varepsilon ,$$

valid for any $x \varepsilon R$, we deduce

$$0 < j_\varepsilon(v) - j(v) \le C\varepsilon \quad , \quad \forall\, v \;\varepsilon\; V_h^1 \quad ,$$

which, together with (2.6), implies that

$$A(u_\varepsilon - u; u_\varepsilon - u) \le C\varepsilon \quad , \tag{2.7}$$

and thus the claim is proved.

The functional j_ε being differentiable on V_h^1, problem (2.5) is equivalent to the variational equation

$$\begin{cases} \text{(i)} \; u_\varepsilon \;\varepsilon\; V_h^1 , \\ \text{(ii)} \; A(u_\varepsilon; v) + (j'_\varepsilon(u_\varepsilon), v) = L(v), \quad \forall\, v \;\varepsilon\; V_h^1 \quad , \end{cases} \tag{2.8}$$

where

$$(j'_\varepsilon(w), v) = \int_\Omega \frac{g(x)w(x)v(x)}{\sqrt{\varepsilon^2 + |w|^2}}\, dx, \; v \;\varepsilon\; V_h^1, \; w \;\varepsilon\; V_h^1. \tag{2.9}$$

If we define

$$p_\varepsilon = \frac{g\; u_\varepsilon}{\sqrt{\varepsilon^2 + |u_\varepsilon|^2}} \quad , \tag{2.10}$$

we have that $p_\varepsilon \;\varepsilon\; \Lambda$ and from (2.8):

$$\begin{cases} \text{(i)} \; u_\varepsilon \;\varepsilon\; V_h^1 \quad , \\ \text{(ii)} \; A(u_\varepsilon; v) + (p_\varepsilon, v) = L(v), \; v \;\varepsilon\; V_h^1 \; . \end{cases} \tag{2.11}$$

The set Λ is bounded and closed in $L^2(\Omega)$, hence it is weakly compact and we can extract from $\{p_\varepsilon\}_{\varepsilon>0}$ a sub-sequence, still denoted by $\{p_\varepsilon\}_{\varepsilon>0}$, such that

$$\begin{cases} \text{(i)} & p_\varepsilon \to p \quad \text{weakly in } L^2(\Omega) \quad , \\ \text{(ii)} & p \in \Lambda \quad . \end{cases} \tag{2.12}$$

The function p is the candidate for multiplier. Indeed, passing (2.11) to the limit, in view of (2.12) and (2.7) we obtain (2.2). Now, if we take $v = u$ in this equation, we get

$$A(u;u) + (p,u) = L(u) \quad ,$$

which compared with (2.4) gives

$$(p,u) - j(u) = \int_\Omega (pu - g|u|)dx = 0 \quad .$$

But $p \in \Lambda$, hence $pu \leq g|u|$ and so

$$pu = g|u| \quad \text{a.e. in } \Omega \ ,$$

that is, (2.3) is satisfied.

The converse implication results from a direct calculation: to obtain (2.1) we just take $v = w-u$, any $w \in V_h^1$, in (2.2),

$$A(u;w-u) + \int_\Omega pw dx - \int_\Omega pu dx = L(w-u) \quad ,$$

and then use the information contained in (2.3) to get

$$A(u;w-u) + \int_\Omega g|w|dx - \int_\Omega g|u|dx \geq L(w-u) \quad ,$$

which is (2.1)(ii).

The proof of the theorem is ended.

At this point we recall that problem (2.1) is equivalent to (1.17), that is,

$$\begin{cases} \text{(i)} \ u \in V_h^1 , \\ \text{(ii)} \ F(u) = \inf\limits_{v \in V_h^1} F(v) , \end{cases} \tag{2.13}$$

where

$$F(v) = \frac{1}{2} A(v;v) + j(v) - L(v) . \tag{2.14}$$

Now, since a continuous and convex function is the upper envelope of all affine functions lying bellow it, we can represent j as

$$j(v) = \sup_{q \in \Lambda} (q,v) , \quad v \in V_h^1 , \tag{2.15}$$

in view of

$$\int_\Omega qv dx \leq \int_\Omega |q| |v| dx \leq \int_\Omega g|v| dx .$$

Formulas (2.13)-(2.15) suggest us to introduce a Lagrangian function

$$\mathcal{L}(v,q) = \frac{1}{2} A(v;v) + (q,v) - L(v) , \tag{2.16}$$

defined on $V_h^1 \times \Lambda$ (Λ = closed, bounded and convex in $L^2(\Omega)$). In this situation problem (2.13) is equivalent to

$$\begin{cases} \text{(i)} \ u \in V_h^1 , \\ \text{(ii)} \ F(u) = \inf\limits_{v \in V_h^1} \sup\limits_{q \in \Lambda} \mathcal{L}(v,q) . \end{cases} \tag{2.17}$$

We can say more about the characterization of the solution u in terms of $\mathcal{L}(v,q)$: (u,p), where p is the multiplier of theorem 2.1 , is a saddle point of $\mathcal{L}(v,q)$. In fact we shall prove now the following

Theorem 2.2 The solution u of (2.1) is characterized by

$$F(u) = \inf_{v \in V_h^1} \sup_{q \in \Lambda} \mathcal{L}(v,q) = \sup_{q \in \Lambda} \inf_{v \in V_h^1} \mathcal{L}(v,q) . \tag{2.18}$$

Proof. The first relation is already clear, and since we have in general sup inf $\leq$ inf sup ,

$$\sup_{q \in \Lambda} \inf_{v \in V_h^1} \mathcal{L}(v,q) \leq F(u) . \tag{2.19}$$

The claim is: equality holds. For this we calculate explicitly

$$\inf_{v \in V_h^1} \mathcal{L}(v,q) ,$$

which is reached at v(q), the solution of

$$\begin{cases} \text{(i)} \quad v(q) \in V_h^1 , \\ \text{(ii)} \quad A(v(q);w) - L(w) + (q,w) = 0 , \quad \forall\, w \in V_h^1 . \end{cases} \tag{2.20}$$

The last equation implies

$$\mathcal{L}(v(q),q) = -\frac{1}{2} A(v(q);v(q)) ,$$

and then

$$\sup_{q \in \Lambda} \inf_{v \in V_h^1} \mathcal{L}(v,q) = \sup_{q \in \Lambda} \{-\frac{1}{2} A(v(q);v(q))\} .$$

Now if we take $q = p \varepsilon \Lambda$, a multiplier from theorem 2.1, and $v(p) = u$, we have

$$F(u) = -\frac{1}{2} A(u;u) \leq \sup_{q \varepsilon \Lambda} \inf_{v \varepsilon V_h^1} \mathcal{L}(v,q) \quad ,$$

which together with (2.19) proves the theorem.

Those two theorems put us in position to define the algorithm for the computation of the solution of problem (1.17) or (2.1). We are in the general framework considered by Glowinski, Lions and Trémolières in [1] (Chapter 2, section 4) for the searching of saddle points of functions of the type $\mathcal{L}(v,q)$, satisfying

$$\mathcal{L}(u,q) \leq \mathcal{L}(u,p) \leq \mathcal{L}(v,p) \quad , \quad \forall\, v \in V_h^1 \quad , \quad \forall\, q \in \Lambda \, . \tag{2.21}$$

In this situation we define the following procedure:

$$\left\{ \begin{array}{l} \text{(i) initialize with } p^0 \in \Lambda \,; \\ \quad p^{\ell} \in \Lambda \,, \ \ell \geq 0 \,, \text{ known;} \\ \text{(ii) minimize } \mathcal{L}(v,p^{\ell}) \text{ in } V_h^1 \text{ to obtain } u^{\ell} \in V_h^1 \,; \\ \text{(iii) } p^{\ell+1} = P_{\Lambda}\,(p^{\ell} + \rho_{\ell} u^{\ell}) \,. \end{array} \right. \tag{2.22}$$

Here P_{Λ} is the projector operator from $L^2(\Omega)$ onto Λ and ρ_{ℓ} is a parameter to be conveniently chosen for convergence.

We must remark that:

1 - Problem (2.22) (ii) is equivalent to solve the Neumann problem

$$\left\{ \begin{array}{l} \text{(i)} \quad u^{\ell} \in V_h^1 \,; \\ \text{(ii)} \quad A(u^{\ell};v) = L(v) - (p^{\ell},v), \ \forall\, v \in V_h^1 \,. \end{array} \right. \tag{2.22)(ii)'$$

2 - The projection operator P_Λ is given by the explicit formula

$$P_\Lambda v(x) = \frac{g(x)v(x)}{\sup\{g(x);|v(x)|\}} \quad , \quad v \in L^2(\Omega) \; . \qquad (2.23)$$

3 - After theorem 4.1 of [1] , procedure (2.22) is convergent, in the sense that $u^\ell \to u$ strongly in V_h^1 as $\ell \to \infty$. For this we have to take ρ_ℓ in between two bounds defined in the proof of this result.

4 - The sequence $\{p^\ell\}$ may have more than one limit point, and each of them, together with u, is a saddle point of $\mathcal{L}(v,q)$ on $V_h^1 \times \Lambda$.

We end this section with the final version of the complete algorithm:

$$\left\{\begin{array}{ll}
(A) & U^n \in V_h^1 \quad , \quad n = 0,1,\ldots, N \quad ; \\
(B) & U^0 = U^1 = 0 \quad ; \\
(C) & \text{for } n = 1,2,\ldots,N-1, \; \delta_t U^n(x) = u_h^\ell(n;x) \in V_h^1 \\
 & (\ell \text{ sufficiently large) is obtained by the convergent it-} \\
 & \text{erative procedure} \\
 & (C\text{-}1) \quad p_n^0 \in \Lambda \text{ given,} \\
 & \qquad\quad p_n^\ell \text{ known, } \ell = 0,1,\ldots, \\
 & (C\text{-}2) \quad A(u_h^\ell(n;\cdot);v_h) = L_n(v_h) - (p_n^\ell, v_n), \quad \forall\, v_h \in V_h^1 \quad , \\
 & (C\text{-}3) \quad p_n^{\ell+1}(x) = g_n(x) \dfrac{(p_n^\ell(x) + \rho_\ell u_h^\ell(n;x))}{\sup\{g_n(x);|p_n^\ell(x) + \rho_\ell u_h^\ell(n;x)|\}} \quad , \\
 & \qquad\quad \ell = 0,1,2,\ldots, \\
 & \qquad\quad \text{where } g_n(x) = cH(x+U^n(x)-1)(x+U^n(x)-1) \; ;
\end{array}\right. \qquad (2.24)$$

$$(D) \quad U^{n+1}(x) = U^{n-1}(x) + 2k\delta_t U^n(x) \quad , \quad n = 1,2,\ldots,N-1 \; .$$

3. STABILITY OF THE EVOLUTIONARY SCHEME

In this section we establish some stability properties of scheme (1.13)-(1.16). We shall have to impose a stability condition for k in terms of the stability function S(h) associated with the spaces $H^1(\Omega)$ and $L^2(\Omega)$ (see [3]).

Theorem 3.1 If k and h satisfy

$$k^2 S(h)^2 \leq (a + k_1)^{-1} \quad , \tag{3.1}$$

then

$$|\partial_t U_{h,k}|_{L^\infty(0,T;L^2(\Omega))} \leq \text{constant} \; , \tag{3.2}$$

$$|U_{h,k}|_{L^\infty(0,T;H^1(\Omega))} \leq \text{constant} \; , \tag{3.3}$$

$$|\delta_t U_{h,k}|_{L^2(0,T;H^1(\Omega))} \leq \text{constant} \; , \tag{3.4}$$

where the constants depend only on the data, but not on T, which can be even $T = +\infty$.

Proof. We take V = 0 in (1.9) and multiply by (-1). Since $J(U^n;\delta_t U^n) \geq 0$, we have

$$\frac{1}{2k}\left\{\left|\partial_t U^n\right|_0^2 - \left|\partial_t U^{n-1}\right|_0^2\right\} + a\,(U^n;\delta_t U^n) + \tag{3.5}$$

$$+ b(\delta_t U^n;\delta_t U^n) \leq (f_n,\delta_t U^n) + F_n \delta_t U^n(0) \; ,$$

where $|v|_0 = \sqrt{(v,v)}$.

By the Cauchy-Schwarz inequality and Sobolev's embedding theorem we can bound the right hand side of (3.5) by

$$(|f_n|_0 + |F_n|)|\delta_t U^n|_1 .$$

Now,

$$a(U^n;U^{n+1} - U^{n-1}) = \frac{1}{2}\left[a(U^{n+1};U^{n+1}) - a(U^{n-1};U^{n-1})\right]$$

$$- \frac{1}{2}\left[a(U^{n+1} - U^n;U^{n+1}-U^n) - a(U^n-U^{n-1};U^n-U^{n-1})\right] ,$$

so that if we multiply (3.5) by 2k and sum from 1 to j, we get

$$\left|\partial_t U^j\right|_0^2 + \frac{1}{2} a(U^{j+1};U^{j+1}) + \frac{1}{2} a(U^j;U^j) -$$

$$- \frac{1}{2} a(U^{j+1}-U^j;U^{j+1}-U^j) + 2k \sum_{n=1}^{j} b(\delta_t U^n;\delta_t U^n)$$

$$\leq \sum_{n=1}^{j} k(|f_n|_0 + |F_n|)|\delta_t U^n|_1 ,$$

or

$$\left|\partial_t U^j\right|_0^2 + a(U^j;U^j) + 2 \sum_{n=1}^{j} k\, b(\delta_t U^n;\delta_t U^n)$$

$$\leq -a(U^j;U^{j+1}-U^j) + \varepsilon \sum_{n=1}^{j} k\left|\delta_t U^n\right|_1^2 + \qquad (3.6)$$

$$+ C(\varepsilon) \sum_{n=1}^{j} k(\left|f_n\right|_0^2 + \left|F_n\right|^2) ,$$

where ε is any positive number to be chosen conveniently.

We have the following norm equivalences:

$$\frac{1}{a+k_1} a(u;u) \leq |u|_1^2 \leq 2 \max(a;k_1) a(u;u) \quad ,$$

$$\frac{1}{b+k_2} b(u;u) \leq |u|_1^2 \leq 2 \max(b;k_2) b(u;u) \quad .$$

Hence, the first term in the right of inequality (3.6) is estimated as

$$-a(U^j;U^{j+1}-U^j) \leq \frac{1}{2} a(U^j;U^j) + \frac{1}{2} k^2 a(\partial_t U^j;\partial_t U^j)$$

$$\leq \frac{1}{2} a(U^j;U^j) + \frac{1}{2} k^2(a+k_1)\left|\partial_t U^j\right|_1^2$$

$$\leq \frac{1}{2} a(U^j;U^j) + \frac{1}{2}(a+k_1) k^2 S(h)^2 \left|\partial_t U^j\right|_0^2 \quad .$$

If we choose ε carefully, we obtain from (3.6):

$$\left[1 - \frac{1}{2}(a+k_1)k^2 S(h)^2\right] \left|\partial_t U^j\right|_0^2 + \frac{1}{2} a(U^j;U^j) + \frac{1}{2}\sum_{n=1}^{j} k\left|\delta_t U^n\right|_1^2$$

$$\leq \text{constant} \sum_{n=1}^{j} k\left(\left|f_n\right|_0^2 + \left|F_n\right|^2\right) \quad .$$

This inequality, together with (3.1), implies

$$\left|\partial_t U^j\right|_0^2 + a(U^j;U^j) + \sum_{n=1}^{j} k\left|\delta_t U^n\right|_1^2 \leq \text{constant} \quad ,$$

that is (3.2), (3.3) and (3.4).

<u>Theorem 3.2</u> If k and h satisfy (3.1) and $F(k) = 0$, then

$$|\partial_t^2 U_{h,k}|_{L^\infty(0,T;L^2(\Omega))} \leq \text{constant} \quad , \tag{3.7}$$

$$|\partial_t U_{h,k}|_{L^\infty(0,T;H^1(\Omega))} \leq \text{constant} \quad , \tag{3.8}$$

$$|\partial_t^2 U_{h,k}|_{L^2(0,T;H^1(\Omega))} \leq \text{constant} \quad , \tag{3.9}$$

where the constants depend on the data, including T.

<u>Proof</u>. We write (1.9) at the levels n and n+1:

$$\begin{aligned}
&(\partial_t^2 U^n, V-\delta_t U^n) + a(U^n; V-\delta_t U^n) + \\
&+ b(\delta_t U^n; V-\delta_t U^n) + J(U^n;V) - \\
&- J(U^n;\delta_t U^n) \geq (f_n, V-\delta_t U^n) \\
&+ F_n\left[V(0) - \delta_t U^n(0)\right] \quad ,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&(\partial_t^2 U^{n+1}, V-\delta_t U^{n+1}) + a(U^{n+1}; V-\delta_t U^{n+1}) + \\
&+ b(\delta_t U^{n+1}; V-\delta_t U^{n+1}) + J(U^{n+1};V) - \\
&- J(U^{n+1};\delta_t U^{n+1}) \geq (f_{n+1}, V-\delta_t U^{n+1}) \\
&+ F_{n+1}\left[V(0) - \delta_t U^{n+1}(0)\right] \quad .
\end{aligned} \tag{3.11}$$

Now we take $V = \delta_t U^{n+1}$ in (3.10), $V = \delta_t U^n$ in (3.11) and add to get

$$(\partial_t^2 U^{n+1} - \partial_t^2 U^n , \delta_t U^{n+1} - \delta_t U^n) +$$

$$+ a(U^{n+1} - U^n; \delta_t U^{n+1} - \delta_t U^n) +$$

$$+ b(\delta_t U^{n+1} - \delta_t U^n; \delta_t U^{n+1} - \delta_t U^n)$$

$$\leq (f_{n+1} - f_n , \delta_t U^{n+1} - \delta_t U^n) + \tag{3.12}$$

$$+ (F_{n+1} - F_n)(\delta_t U^{n+1}(0) - \delta_t U^n(0))$$

$$+ J(U^n; \delta_t U^{n+1}) - J(U^n; \delta_t U^n) +$$

$$+ J(U^{n+1}; \delta_t U^n) - J(U^{n+1}; \delta_t U^{n+1}) .$$

But

$$\delta_t U^{n+1} - \delta_t U^n = \frac{\partial_t U^{n+1} - \partial_t U^{n-1}}{2} = \frac{k}{2} (\partial_t^2 U^{n+1} + \partial_t^2 U^n) ,$$

so that if we multiply (3.12) by $\frac{2}{k}$ we obtain:

$$\left|\partial_t^2 U^{n+1}\right|_0^2 - \left|\partial_t^2 U^n\right|_0^2 + a(\partial_t U^n; \partial_t U^{n+1} - \partial_t U^{n-1}) +$$

$$+ 2k\, b\left(\frac{\partial_t^2 U^{n+1} + \partial_t^2 U^n}{2} ; \frac{\partial_t^2 U^{n+1} + \partial_t^2 U^n}{2}\right)$$

$$\leq 2k \left|\partial_t f^n\right|_0 \left|\frac{\partial_t^2 U^{n+1} + \partial_t^2 U^n}{2}\right|_0 + \tag{3.13}$$

$$+ 2k \left|\partial_t F^n\right| \left|\frac{\partial_t^2 U^{n+1}(0) + \partial_t^2 U^n(0)}{2}\right| +$$

$$+ \frac{2}{k} \int_\Omega \{ g(x; U^n) \left[\left|\delta_t U^{n+1}\right| - \left|\delta_t U^n\right| \right] +$$

$$g(x; U^{n+1}) \left[\left|\delta_t U^n\right| - \left|\delta_t U^{n+1}\right| \right] \} dx ,$$

where g is defined after the notation introduced in section 1 as

$$g(x;\lambda) = c\, H(x + \lambda - 1)(x + \lambda - 1)\,.$$

The last term in the right of (3.13) can be estimated as follows:

$$\frac{2}{k} \int_\Omega \{g(x;U^n)\left[\left|\delta_t U^{n+1}\right| - \left|\delta_t U^n\right|\right] + g(x;U^{n+1})\left[\left|\delta_t U^n\right| - \right.$$

$$\left. - \left|\delta_t U^{n+1}\right|\right]\}\, dx = \frac{2}{k} \int_\Omega \left[g(x;U^n) - g(x;U^{n+1})\right] \cdot$$

$$\cdot \left[\left|\delta_t U^{n+1}\right| - \left|\delta_t U^n\right|\right] dx$$

$$\leq \frac{2c}{k} \cdot \left|U^{n+1} - U^n\right|_0 \left|\delta_t U^{n+1} - \delta_t U^n\right|_0 =$$

$$= 2ck\left|\partial_t U^n\right|_0 \left|\frac{\partial_t^2 U^{n+1} + \partial_t^2 U^n}{2}\right|_0 .$$

Hence if we put this result into (3.13) and add from 1 to j-1, taking into consideration Sobolev's embedding theorem,

$$\left|\partial_t^2 U^j\right|_0^2 - \left|\partial_t^2 U^1\right|_0^2 + \sum_{n=1}^{j-1} a(\partial_t U^n;\partial_t U^{n+1} - \partial_t U^{n-1})$$

$$+ 2\sum_{n=1}^{j-1} k\, b(\partial_t^2 U^{n+1/2};\partial_t^2 U^{n+1/2})$$

$$\leq c(\varepsilon) \sum_{n=1}^{j-1} k\left[\left|\partial_t f^n\right|_0^2 + \left|\partial_t F^n\right|^2\right] + \tag{3.14}$$

$$+ (4+2c)\, \varepsilon \max(b;k_2) \sum_{n=1}^{j-1} k\, b(\partial_t^2 U^{n+1/2};\partial_t^2 U^{n+1/2})$$

$$+ c(\varepsilon) \sum_{n=1}^{j-1} k\left|\partial_t U^n\right|_0^2 ,$$

where $\partial_t^2 U^{n+1/2} = 1/2(\partial_t^2 U^{n+1} + \partial_t^2 U^n)$ and $\varepsilon > 0$ is any number.

By (3.2) and a proper choice of ε we have

$$|\partial_t^2 U^j|_0^2 + \sum_{n=1}^{j-1} a(\partial_t U^n ; \partial_t U^{n+1} - \partial_t U^{n-1})$$

$$+ \sum_{n=1}^{j-1} k\, b(\partial_t^2 U^{n+1/2} ; \partial_t^2 U^{n+1/2}) \tag{3.15}$$

$$\leq \text{const}\left(\frac{\partial f}{\partial t}, \frac{\partial F}{\partial t}, T, b, k_2, C\right) +$$

$$+ |\partial_t^2 U^1|_0^2 \quad .$$

Now we focus our attention on the second term in the left hand side of (3.15). We have the splitting

$$a(\partial_t U^n ; \partial_t U^{n+1} - \partial_t U^{n-1}) = \frac{1}{2}\Big[a(\partial_t U^{n+1} ; \partial_t U^{n+1}) -$$

$$- a(\partial_t U^{n-1} ; \partial_t U^{n-1})\Big] - \frac{1}{2}\Big[a(\partial_t U^{n+1} - \partial_t U^n ; \partial_t U^{n+1} - \partial_t U^n)$$

$$- a(\partial_t U^n - \partial_t U^{n-1} ; \partial_t U^n - \partial_t U^{n-1})\Big] \quad .$$

Hence if we sum and consider (1.7) and (1.8):

$$\sum_{n=1}^{j-1} a(\partial_t U^n ; \partial_t U^{n+1} - \partial_t U^{n-1}) = \frac{1}{2}\Big[a(\partial_t U^j ; \partial_t U^j) + a(\partial_t U^{j-1} ; \partial_t U^{j-1})$$

$$- a(\partial_t U^j - \partial_t U^{j-1} ; \partial_t U^j - \partial_t U^{j-1})\Big] =$$

$$= a(\partial_t U^{j-1} ; \partial_t U^{j-1}) + a(\partial_t U^{j-1} ; \partial_t U^j - \partial_t U^{j-1}) \quad .$$

Moreover, the second term can be bounded as

$$|a(\partial_t U^{j-1};\partial_t U^j-\partial_t U^{j-1})| \ = |k\ a(\partial_t U^{j-1}\ ;\ \partial_t^2 U^j)|$$

$$\leq \frac{1}{2}\ a(\partial_t U^{j-1};\partial_t U^{j-1}) + \frac{1}{2}\ k^2\ a(\partial_t^2 U^j;\partial_t^2 U^j)$$

$$\leq \frac{1}{2}\ a(\partial_t U^{j-1};\partial_t U^{j-1}) + \frac{1}{2}\ |\partial_t^2 U^j|_0^2\ ,$$

where we made use of the stability condition (3.1).

Carrying this information into (3.15) we reach the following inequality:

$$\frac{1}{2}|\partial_t^2 U^j|_0^2 + \frac{1}{2}\ a(\partial_t U^{j-1};\partial_t U^{j-1}) \ + \qquad (3.16)$$

$$\sum_{n=1}^{j-1} k\ b(\partial_t^2 U^{n+1/2};\partial_t^2 U^{n+1/2}) \leq |\partial_t^2 U^1|_0^2 + \text{const}\ ,$$

so that to complete the proof of the present theorem we must bound $|\partial_t^2 U^1|_0$ in terms of the data of the problem.

For this we write inequality (1.9) for n = 1, considering the initial conditions:

$$(\partial_t^2 U^1, V-\delta_t U^1) + b(\delta_t U^1;V-\delta_t U^1) \ \geq \ (f_1,V-\delta_t U^1)\ .$$

Then we take $V = \partial_t U^0 = 0$ and observe that

$$V-\delta_t U^1 = \partial_t U^0 - \delta_t U^1 = -\frac{k}{2}\ \partial_t^2 U^1\ .$$

We shall get, multiplying the inequality by $-\frac{2}{k}$:

$$|\partial_t^2 U^1|_0^2 + \frac{2}{k}\ b(\delta_t U^1;\delta_t U^1) \ \leq \ \frac{1}{2}\ |f_1|_0^2 + \frac{1}{2}\ |\partial_t^2 U^1|_0^2\ ,$$

and this implies

$$|\partial_t^2 U^1|_0 \ \leq \ |f_1|_0\ ,$$

ending the demonstration of the theorem.

4. CONVERGENCE ANALYSIS

In this section we shall go through the convergence analysis of the algorithm, establishing the adequate setting for the limit process. The following theorem summarizes the question.

Theorem 4.1 Let u be the exact solution of the pile driver problem with properties specified in theorem 1.1. Let also $U_{h,k}$ be its approximation calculated through algorithm (2.24). Then, if h and k go to zero satisfying the stability condition (3.1) and $F(k) = 0$, we have that

$$U_{h,k} \to u \quad \text{strongly in } L^2(Q_T) \quad , \tag{4.1}$$

$$\partial_t U_{h,k} \to \frac{\partial u}{\partial t} \quad \text{strongly in } L^2(Q_T) \quad , \tag{4.2}$$

$$\partial_t^2 U_{h,k} \to \frac{\partial^2 u}{\partial t^2} \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad , \tag{4.3}$$

where $Q_T = \Omega \times (0,T)$.

Proof. Taking a finite $T > 0$, theorems 3.1 and 3.2 say that the sequences $\{U_{h,k}\}$, $\{\partial_t U_{h,k}\}$ and $\{\partial_t^2 U_{h,k}\}$ remain bounded in $L^2(0,T;H^1(\Omega))$. Hence we can take sub-sequences such that

$$\begin{cases} \text{(i)} & U_{h,k} \to U \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad , \\ \text{(ii)} & \partial_t U_{h,k} \to \dfrac{\partial U}{\partial t} \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad , \\ \text{(iii)} & \partial_t^2 U_{h,k} \to \dfrac{\partial^2 U}{\partial t^2} \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad . \end{cases} \tag{4.4}$$

The convergences $\partial_t \to \frac{\partial}{\partial t}$ and $\partial_t^2 \to \frac{\partial^2}{\partial t^2}$ result from the relations

$$\langle \partial_t U_{h,k}, \phi \rangle = - \langle U_{h,k}, \partial_t \phi \rangle = - \int_0^T U_{h,k}\, \partial_t \phi \, dt \quad ,$$

$$\langle \partial_t^2 U_{h,k}, \phi \rangle = \langle U_{h,k}, \partial_t^2 \phi \rangle = \int_0^T U_{h,k}\, \partial_t^2 \phi \, dt \quad ,$$

valid for any $\phi \in \mathcal{D}(0,T)$, k sufficiently small, and the fact that

$$\partial_t \phi \to \frac{\partial \phi}{\partial t} \quad \text{strongly in } L^2(0,T;R) \quad ,$$

$$\partial_t^2 \phi \to \frac{\partial^2 \phi}{\partial t^2} \quad \text{strongly in } L^2(0,T;R) \quad ,$$

together with (4.4)(i).

Here we understand $\mathcal{D}(0,T)$ as the space of test-functions for vector-valued distributions on (0,T).

Now, since the injection of $H^1(Q_T)$ into $L^2(Q_T)$ is compact (see [2], theorem 3.6), we have

$$\begin{cases} \text{(i)} \quad U_{h,k} \to U \quad \text{strongly in } L^2(Q_T) \quad , \\ \text{(ii)} \quad \partial_t U_{h,k} \to \frac{\partial U}{\partial t} \quad \text{strongly in } L^2(Q_T) \quad , \end{cases} \tag{4.5}$$

where we are again taking sub-sequences.

Our objective is to show that $U \in L^2(Q_T)$ is the solution of the continuous problem, that is, $U = u$. This would imply our theorem by (4.5)(i), (4.5)(ii) and (4.4)(iii).

Let us take a test function $v \in L^2(0,T;H^1(\Omega))$ and approximations $V^n \in V_h^1$ such that $V^n \to v(\cdot,t_n)$ strongly in $H^1(\Omega)$. We have, in the notation of the Introduction,

$$V_{h,k} \to v \quad \text{strongly in } L^2(0,T;H^1(\Omega)) \quad . \tag{4.6}$$

Well, if we take $V = V^n$ in (1.9), multiply by k and sum from 1 to N-1, we get

$$\sum_{n=1}^{N-1} k(\partial_t^2 U^n, V^n-\delta_t U^n) + \sum_{n=1}^{N-1} k\ a(U^n;V^n-\delta_t U^n)$$

$$+ \sum_{n=1}^{N-1} k\ b(\delta_t U^n;V^n-\delta_t U^n)+\sum_{n=1}^{N-1} k\ J(U^n;V^n) -$$

$$- \sum_{n=1}^{N-1} k\ J(U^n;\delta_t U^n) \geq \sum_{n=1}^{N-1} k(f_n,V^n-\delta_t U^n) \tag{4.7}$$

$$+ \sum_{n=1}^{N-1} k\ F_n \left[V^n(0) - \delta_t U^n(0)\right] ,$$

$$\forall\ v \in L^2(0,T;H^1(\Omega)).$$

Now,

$$\int_0^T (\partial_t^2 U_{h,k}, V_{h,k}-\delta_t U_{h,k})dt = \sum_{n=1}^{N-1}\sum_{m=1}^{N-1} (\partial_t^2 U^n, V^m-\delta_t U^m) \int_0^T \theta_k^n \theta_k^m\ dt$$

$$= \sum_{n=1}^{N-1} k(\partial_t^2 U^n, V^n-\delta_t U^n) ,$$

$$\int_0^T a(U_{h,k};V_{h,k}-\delta_t U_{h,k})dt = \sum_{n=1}^{N-1} k\ a(U^n;V^n-\delta_t U^n) ,$$

$$\int_0^T b(\delta_t U_{h,k};V_{h,k}-\delta_t U_{h,k})dt = \sum_{n=1}^{N-1} k\ b(\delta_t U^n;V^n-\delta_t U^n) ,$$

$$\int_0^T J(U_{h,k};V_{h,k})dt = \sum_{n=1}^{N-1} k\ J(U^n;V^n) + \mathcal{O}(k) ,$$

$$\int_0^T (f_k, V_{h,k} - \delta_t U_{h,k})dt = \sum_{n=1}^{N-1} k(f_n, V^n - \delta_t U^n) \quad ,$$

$$\int_0^T F_k\left[V_{h,k}(0) - \delta_t U_{h,k}(0)\right]dt = \sum_{n=1}^{N-1} k\ F_n\left[V^n(0)-\delta_t U^n(0)\right] \quad ,$$

where

$$f_k(x,t) = \sum_{n=1}^{N-1} f_n(x)\ \theta_k^n(t) \quad ,$$

$$F_k(t) = \sum_{n=1}^{N-1} F_n\ \theta_k^n(t) \quad ,$$

and, as $k \to 0$,

$$\begin{cases} f_k \to f \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \quad , \\ F_k \to F \quad \text{strongly in } L^2(0,T;R) \quad . \end{cases} \tag{4.8}$$

Hence inequality (4.7) can be written as

$$\begin{aligned} &\int_0^T(\partial_t^2 U_{h,k}, V_{h,k} - \delta_t U_{h,k})dt + \int_0^T a(U_{h,k}; V_{h,k}-\delta_t U_{h,k})dt \\ &\quad + \int_0^T b(\delta_t U_{h,k}; V_{h,k} - \delta_t U_{h,k})dt + \int_0^T J(U_{h,k}; V_{h,k})dt \\ &\quad - \int_0^T J(U_{h,k}; \delta_t U_{h,k})dt + \mathcal{O}(k) \ \geq \\ &\qquad \int_0^T (f_k, V_{h,k}-\delta_t U_{h,k})dt \\ &\qquad\quad + \int_0^T F_k\left[V_{h,k}(0)-\delta_t U_{h,k}(0)\right] dt \quad , \\ &\qquad\qquad \forall\ v \in L^2(0,T;H^1(\Omega)) \quad . \end{aligned} \tag{4.9}$$

We have now that

$$\int_0^T (\partial_t^2 U_{h,k}, V_{h,k})dt \to \int_0^T (\frac{\partial^2 U}{\partial t^2}, v)dt \quad , \tag{4.10}$$

by (4.4)(iii) and (4.6) ;

$$\int_0^T a(U_{h,k}; V_{h,k})dt \to \int_0^T a(U;v)dt \quad , \tag{4.11}$$

by (4.4)(i) and (4.6);

$$\int_0^T b(\delta_t U_{h,k}; V_{h,k})dt \to \int_0^T b(\frac{\partial U}{\partial t}; v)dt \quad , \tag{4.12}$$

by (4.4)(ii) and (4.6);

$$\int_0^T J(U_{h,k}; V_{h,k})dt \to \int_0^T J(U;v)dt \quad , \tag{4.13}$$

by (4.5)(i) and (4.6);

$$\int_0^T J(U_{h,k}; \delta_t U_{h,k})dt \to \int_0^T J(U; \frac{\partial U}{\partial t})dt \quad , \tag{4.14}$$

by (4.5);

$$\int_0^T (f_k, V_{h,k} - \delta_t U_{h,k})dt \to \int_0^T (f, v - \frac{\partial U}{\partial t})dt \quad , \tag{4.15}$$

by (4.5)(ii), (4.6) and (4.8) ;

$$\int_0^T F_k\left[V_{h,k}(0) - \delta_t U_{h,k}(0)\right]dt \to \int_0^T F(t)\left[v(0,t) - \frac{\partial U}{\partial t}(0,t)\right]dt \quad , \tag{4.16}$$

by (4.4)(ii), (4.6), (4.8) and the trace theorem in $H^1(\Omega)$.

On the other hand,

$$\int_0^T (\partial_t^2 U_{h,k}, \delta_t U_{h,k})dt + \int_0^T a(U_{h,k};\delta_t U_{h,k})\, dt +$$

$$+ \int_0^T b(\delta_t U_{h,k};\delta_t U_{h,k})dt = \frac{1}{2}\sum_{n=1}^{N-1}\left[|\partial_t U^n|_0^2 - |\partial_t U^{n-1}|_0^2\right] +$$

$$+ \frac{1}{2}\sum_{n=1}^{N-1} a(U^n;U^{n+1}-U^{n-1}) + \int_0^T b(\delta_t U_{h,k};\delta_t U_{h,k})dt$$

$$= \frac{1}{2}|\partial_t U^{N-1}|_0^2 + \frac{1}{2}a(U^N;U^N) + \frac{1}{2}k\, a(U^N;\partial_t U^N)$$

$$+ \int_0^T b(\delta_t U_{h,k};\delta_t U_{h,k})dt \quad ,$$

so that if we pass to the limit $k \to 0$, we get

$$\liminf_{k\to 0} \{ \int_0^T (\partial_t^2 U_{h,k}, \delta_t U_{h,k})dt + \int_0^T a(U_{h,k};\delta_t U_{h,k})dt$$

$$+ \int_0^T b(\delta_t U_{h,k};\delta_t U_{h,k})dt\} \geq \frac{1}{2}\left|\frac{\partial U}{\partial t}(T)\right|_0^2 + \tag{4.17}$$

$$+ \frac{1}{2}a(U(T);U(T)) + \int_0^T b\left(\frac{\partial U}{\partial t};\frac{\partial U}{\partial t}\right)dt \quad ,$$

where we used the weak lower semi-continuity property of the norm in a Banach space and the fact that $a(U^N;\partial_t U^N)$ is bounded independently of U^n or k.

Hence, writing (4.9) in the form

$$\int_0^T (\partial_t^2 U_{h,k}, V_{h,k})dt + \int_0^T a(U_{h,k}; V_{h,k})dt +$$

$$+ \int_0^T b(\delta_t U_{h,k}; V_{h,k})dt + \int_0^T J(U_{h,k}; V_{h,k})dt -$$

$$- \int_0^T J(U_{h,k}; \delta_t U_{h,k})dt + \mathcal{O}(k) \geq$$

$$\int_0^T (f_k, V_{h,k} - \delta_t U_{h,k})dt + \int_0^T F_k\left[V_{h,k}(0) - \delta_t U_{h,k}(0)\right]dt$$

$$+ \int_0^T (\partial_t^2 U_{h,k}, \delta_t U_{h,k})dt + \int_0^T a(U_{h,k}; \delta_t U_{h,k})dt$$

$$+ \int_0^T b(\delta_t U_{h,k}; \delta_t U_{h,k})dt \quad ,$$

$$\forall\, v \in L^2(0,T;H^1(\Omega)) \quad ,$$

passing to the limit $k \to 0$ and considering the results (4.10)-(4.17), we get

$$\int_0^T (\frac{\partial^2 U}{\partial t^2}, v)dt + \int_0^T a(U;v)dt + \int_0^T b(\frac{\partial U}{\partial t}; v)\, dt$$

$$+ \int_0^T J(U;v)dt - \int_0^T J(U; \frac{\partial U}{\partial t})\, dt \geq$$

$$\int_0^T (f, v - \frac{\partial U}{\partial t})dt + \int_0^T F(t)\left[v(0,t) - \frac{\partial U}{\partial t}(0,t)\right] dt$$

$$+ \frac{1}{2} \left|\frac{\partial U}{\partial t}(T)\right|_0^2 + \frac{1}{2} a(U(T);U(T)) + \int_0^T b(\frac{\partial U}{\partial t} ; \frac{\partial U}{\partial t}) \, dt \quad ,$$

$$\forall \; v \in L^2(0,T;H^1(\Omega)) \quad ,$$

that is

$$\int_0^T \{(\frac{\partial^2 U}{\partial t^2}, v - \frac{\partial U}{\partial t}) + a(U;v - \frac{\partial U}{\partial t}) + b(\frac{\partial U}{\partial t} ; v - \frac{\partial U}{\partial t})$$

$$+ J(U;v) - J(U; \frac{\partial U}{\partial t})\} \, dt \geq \tag{4.18}$$

$$\int_0^T \{(f, v - \frac{\partial U}{\partial t}) + F(t)\left[v(0,t) - \frac{\partial U}{\partial t}(0,t)\right]\} dt \quad ,$$

$$\forall \; v \in L^2(0,T;H^1(\Omega)) \quad .$$

In the proof of theorem 1.1 in [4] , it is shown that inequality (4.18) is equivalent to (1.5), and so the conclusion is U = u, by the uniqueness property of the problem.

To conclude, a final remark: (4.4)(i) and (ii) give us also information on the convergence of the computed strains and stresses of the physical problem.

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