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Pedagogical Remarks on Free Field Relativistic Wave Equations and their Geometrical Nature[1]

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Abstract

A technique of regarding the line elemement of space-time appropriate to each field variable as an operator acting on this variable allows a simple derivation of the free field relativistic wave equations, the geometrical nature of which is the line element. An illustrative example is the derivation of Einstein's linear equation for weak gravitational fields and the exact equation in the absence of matter.

Key-words: Fields; Wave equations; Line element geometrization of physics; Dirac's equation; Einstein's equations.

[1]Dedicated to Paulo Leal Ferreira, in the occasion of his 70^{th} birthday.

A tribute to Paulo Leal Ferreira on the occasion of his 70th birthday

Dear Paulo: You may remember that back in 1946, recently appointed to the chair of Theoretical Physics at the Faculdade Nacional de Filosofia in Rio, I met you and wanted to have you at my side to help me in research work and in the task of educating young physicists. But the administration of the University in Rio at that time did not favour the development of research in physics and there was no full time. I had to fight against this situation and had the occasion of meeting your belovedfather, Engineer J.H. Leal Ferreira, who was also against the lack of creativity in our university administrations and was gathering elements to create a Theoretical Physics Institute. At that time, I was in touch with Cesar Lattes, then in Bristol, to convince him to join me at the University in Rio and help change the situation there.

Finally, in 1949, we created the Centro Brasileiro de Pesquisas Físicas, and your father got enough help to create the Instituto de Física Teórica in São Paulo. You had the visit and the help of young and famous German and Japonese Physicists like Reinhard Oehme, Von Weiszöcher and M. Taketani, we had young and famous physicists like Richard Feynman, Daniel Amati, Alberto Sirlin and many others.

From those difficult but glovious times emerged a climate of research in physics in Brazil which is promosing to give good results for the country and for science.

I join my colleagues to congratulate you - and your brother Jorge, who left us so prematurely - on the brilliant work you accomplished during all these years, and wish you many years ahead of a productive and happy life.

1 Introduction

Sometime ago I tried to characterize Dirac's equation as a spinor geodesic in space-time. The idea was to postulate a variational principle of the form:

$$\delta \int_A^B d\Sigma = 0 \tag{1}$$

where

$$d\Sigma = \gamma_\alpha dx^\alpha \psi(x), \tag{2}$$

$\psi(x)$ is a Dirac spinor and the form

$$\gamma_\alpha dx^\alpha \tag{3}$$

is the well-known matrix which linearizes the line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{4}$$

where $\eta_{\mu\nu}$ is the flat space-time metric and

$$\frac{1}{2}\{\gamma_\alpha, \gamma_\beta\} = \eta_{\alpha\beta} \tag{5}$$

The idea is not correct since equation (1) leads to the condition

$$\partial_\alpha(\gamma_\beta\psi) - \partial_\beta(\gamma_\alpha\psi) = 0$$

which led me to write $\gamma_\alpha\psi$ as the gradient of ψ, namely:

$$\gamma_\alpha\psi = 4i\frac{\hbar}{m_0 c}\partial_\alpha\psi \tag{6}$$

from which one obtains Dirac's equation:

$$i\hbar\gamma^\alpha\partial_\alpha\psi - m_0 c\psi = 0 \tag{7}$$

The condition (6) is, however, too restrictive and is not valid and therefore the deduction of equation (7) is not correct.

2 Spin 1/2 Field Geometrical Equation

The idea of the geometric nature of the relativistic wave equations, however, pursued me and led me to a few pedagogical, trivial, remarks about these equations.

First of all the linearisation of the line-element (4) by means of the matrix (3), having (5) in mind, is valid only as a condition on a spinor ψ, as an equation which defines the spinor as a solution of the equation:

$$\gamma_\alpha dx^\alpha\psi = ds I\psi \tag{8}$$

where I is the unit matrix and ds is a number.

We shall develop the idea that the line element - linear or quadratic - is to be regarded as an operator acting on the field representative.

What follows is then trivial, since the momentum of a classical particle is

$$P_\alpha = m_0 c \frac{dx_\alpha}{ds} \tag{9}$$

Therefore we deduce from equation (8)

$$(\gamma_\alpha P^\alpha - m_0 c)\psi = 0$$

and thus the quantum - mechanical equation (7)

$$(i\hbar\gamma^\alpha \partial_\alpha - m_0 c)\psi = 0.$$

which is Dirac's equation.

In a similar way it follows from equations (4) the relationship

$$P_\alpha P^\alpha = (m_0 c)^2 \tag{10}$$

and hence the Klein-Gordon equation for all function $f(x)$ representative of the Poincaré group:

$$\left\{ \Box + \left(\frac{m_0 c}{\hbar}\right)^2 \right\} f(x) = 0 \tag{11}$$

Besides the consideration of the line-element ds^2 or the matrix (3), we must take into account the spin s of the field, the number of independent components of which is $2s+1$, for s integer and $2(2s+1)$ for s half-integer, and for massive particles.

3 Spin 1 Field Geometric Equation

Massive spin $s = 1$ particles are described by a four-vector ϕ_μ, of which only three components are independent.

We therefore write for the geometric equation for ϕ_μ:

$$\eta_{\alpha\beta} dx^\alpha dx^\beta \phi^\mu = ds^2 \phi^\mu \tag{12}$$

and the condition:

$$dx^\mu \eta_{\mu\nu} \phi^\nu = 0 \tag{13}$$

The equation (13) means that there is no scalar field built from ϕ_μ in the neighbourhood of every point x of the manifold where ϕ^μ is defined:

$$dx_\mu \phi^\mu(x) = 0.$$

From equations (12) and (13) one obtains trivially:

$$\eta_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\phi^\mu = \phi^\mu \tag{14}$$

and

$$\frac{dx^\mu}{ds}\phi_\mu = 0 \tag{15}$$

whence, in view of (9) and its transcription in quantum mechanics:

$$\left\{\Box + \left(\frac{m_0 c}{\hbar}\right)^2\right\}\phi^\mu(x) = 0, \tag{16}$$

$$\partial_\mu \phi^\mu(x) = 0 \tag{17}$$

4 Proca's Equation

Proca's equation incorporates equations (16) and (17) for massive spin 1 fields. It has the form:

$$\partial_\nu G^{\mu\nu} + \left(\frac{m_0 c}{\hbar}\right)^2 \phi^\mu = 0 \tag{18}$$

where

$$G_{\mu\nu} = \partial_\nu \phi_\mu - \partial_\mu \phi_\nu. \tag{19}$$

It is this equation which allows, as is well known, the construction of a lagrangean for spin 1 fields.

We take advantage of the fact that besides the line element (12) there exists a possible term constructed with the field, bilinear in dx^μ and which may enter $ds^2\phi^\mu$. It is $dx_\nu \phi^\nu dx^\mu$. So we postulate:

$$ds^2\phi^\mu = dx^\alpha \eta_{\alpha\nu}(dx^\nu \phi^\mu - dx^\mu \phi^\nu) \tag{20}$$

Equation (20) is the geometrical transcription of equations (18) and (19) in view of equation (9) and the replacement of P_μ by the differential operator $i\hbar\partial_\mu$.

Contraction of equation (20) with dx_μ gives:

$$dx_\mu \phi^\mu = 0 \quad if \quad ds^2 \neq 0$$

So equation (20) is equivalent to equations (14) and (15).

In general we write $ds^2\phi^\mu$ as a linear combination of the two possible terms

$$ds^2\phi^\mu = dx^\alpha \eta_{\alpha\nu}(a\ dx^\nu \phi^\mu + b\ dx^\mu \phi^\nu)$$

and require the vanishing of $dx_\mu \phi^\mu$:

$$ds^2 \; dx_\mu \phi^\mu = dx_\alpha dx^\alpha (a+b) dx_\nu \phi^\nu = 0$$

whence $a = -b = 1$ if $ds^2 \neq 0$, $dx_\mu \phi^\mu = 0$.

5 Photons

For photons it is well known that classically the line element vanishes:

$$ds^2 = 0 \tag{21}$$

We postulate a four-vector field A^α for which:

$$\eta_{\alpha\beta} dx^\alpha dx^\beta A^\mu = 0$$

and that there exists no scalar photon in the neighbourhard of any point x:

$$dx_\alpha A^\alpha = 0$$

These are the geometrical transcriptions of the dynamical equations (by means of a parameter in the place of s):

$$\Box A^\mu = 0, \qquad \partial_\mu A^\mu = 0$$

Equations (20) and (21) lead to the equation

$$\partial_\nu F^{\mu\nu} = 0, \qquad F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

but here the gauge $\partial_\mu A^\mu = 0$ has to be postulated.

6 Spin 3/2 Fields

Spin 3/2 fields are described by a spinor-vector $\psi_a^\mu(x)$, where $\mu = 0, 1, 2, 3$ is the vector index and $a = 1, 2, 3, 4$, the spinor index. We postulate the equation (8) applied to ψ_a^μ:

$$\gamma_\alpha dx^\alpha \psi^\mu = ds I \psi^\mu$$

ψ_a^μ has 16 components, 8 of which describe spin 3/2 particles and two sets of four other components each are spinors which describe two types of spin 1/2-particles. The latter are obtained from ψ_a^μ in the following way:

$$\begin{aligned}\chi_a &\equiv (\gamma_\mu \psi^\mu)_a \\ \phi_a &\equiv dx_\mu \psi_a^\mu\end{aligned}$$

They must vanish, which leaves eight independent components to describe a spin 3/2 field only. The geometrical equations are thus (since we do not want any spin $\frac{1}{2}$ field in any neighbourhood of every point):

$$\begin{aligned}(\gamma_\alpha)_{ab} dx^\alpha \psi_b^\mu &= ds \delta_{ab} \psi_b^\mu \\ (\gamma_\mu)_{ab} \psi_b^\mu &= 0 \\ dx_\mu \psi_a^\mu &= 0\end{aligned} \tag{22}$$

which correspond to the Rarita-Schwinger type of equations

$$\begin{aligned}(i\hbar \gamma_{ab}^\mu \partial_\mu - m_0 c\ \delta_{ab}) \psi_b^\alpha &= 0 \\ (\gamma_\mu)_{ab}\ \psi_b^\mu &= 0 \\ \partial_\mu \psi_a^\mu &= 0\end{aligned} \tag{23}$$

In the same way that for spin 1 fields we took into account the existence of a line element bilinear in the coordinate differentials constructed with the vector field and different from the usual $\eta_{\alpha\beta} dx^\alpha dx^\beta \phi^\mu$, we may appeal to new terms in dx and the gamma matrices and the vector-spinor, to add to

$$\gamma_\alpha dx^\alpha \eta^{\mu\nu} \psi_\nu$$

namely: $\gamma^\mu dx^\nu \psi_\nu, \gamma^\nu dx^\mu \psi_\nu$ and $\gamma^\mu (\gamma_\alpha dx^\alpha + ds) \gamma^\nu \psi_\nu$. We therefore postulate the following equation:

$$\begin{aligned}&\{(\gamma_\alpha)_{ab}\, dx^\alpha - ds\ \delta_{ab}) \eta_{\mu\nu} - (\gamma_\mu)_{ab} dx_\nu - (\gamma_\nu)_{ab} dx_\mu + \\ &+ (\gamma_\mu)_{ac} [(\gamma_\alpha)_{cd} dx^\alpha + ds\ \delta_{cd}] (\gamma_\nu)_{db}\}\ \psi_b^\nu = 0\end{aligned} \tag{24}$$

and this geometrical equation will lead to the following one:

$$\begin{aligned}&(i\hbar \gamma^\alpha \partial_\alpha - m_0 c) \eta_{\mu\nu} - i\hbar \gamma_\mu \partial_\nu - i\hbar \gamma_\nu \partial_\mu + \\ &\gamma_\mu [i\hbar \gamma^\alpha \partial_\alpha + m_0 c] \gamma_\nu\} \psi^\nu = 0\end{aligned} \tag{25}$$

where we have omitted the spinor indices.

Equation (24) is equivalent to the equations

$$\begin{aligned}&(\gamma_\alpha dx^\alpha - ds)_{ab}\psi_b^\mu = 0,\\ &(\gamma_\nu)_{ab}\psi_b^\nu = 0,\\ &dx_\mu \psi_a^\mu = 0.\end{aligned}$$

For this, we differentiate equations (25) with respect to x^μ and multiply by γ^μ on the left, respectively.

The geometrical equation which defines a free spin 3/2 field is therefore :

$$i\sigma^{\alpha\beta} ds\psi_\beta = \{\gamma_\mu dx^\mu \eta^{\alpha\beta} - \gamma^\alpha dx^\beta - \gamma^\beta dx^\alpha + \gamma^\alpha\gamma_\mu dx^\mu \gamma^\beta\}\psi_\beta \tag{26}$$

where

$$\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha]$$

This equation may be written in compact form:

$$\varepsilon^{\alpha\beta\mu\nu}\gamma^5\gamma_\beta(dx_\mu - \frac{1}{2}\gamma_\mu ds)\psi_\nu = 0 \tag{27.a}$$

which corresponds to the equation:

$$\begin{aligned}&\varepsilon^{\alpha\beta\mu\nu}\gamma^5\gamma_\beta(\partial_\mu + \frac{i}{2}\frac{m_0 c}{\hbar}\gamma_\mu)\psi_\nu = 0,\\ &\gamma^5 = \frac{i}{4!}\varepsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d.\end{aligned} \tag{27.b}$$

In general there is a family of equations of the form:

$$\{(\gamma_\mu dx^\mu - ds)\eta_{\alpha\beta} - A\gamma_a dx_\beta - B\gamma_\beta dx_\alpha + \gamma_\alpha[C\gamma_\mu dx^\mu + Dds]\gamma_\beta\}\,\psi_\beta = 0$$

The requirement that the equation be spinor-gauge-invariant[1], namely under the transformation: (ϕ is an arbitrary spinor):

$$\psi_\beta \to \psi_\beta + \partial_\beta\phi$$

in the limiting case $ds = 0$ (vanishing mass), gives:

$$A = B = C = D = 1$$

This requirement uniquely defines the form of the equation (27a) and (27b) for spin 3/2 free particles. It is the equation which describes the massless gravitino in supergravity.

[1] J. Leite Lopes, D. Spehler and N. Fleury, Lettere al Nuovo Cimento 35, N.2, 60 (1982).

7 Spin Two Fields and Einstein's Equation for the Gravitational Field

We follow our technique: consider the gravitational i.e. the Riemann space line-element:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

and apply it to Einstein's field which is the metric tensor g itself. But besides the obvious term

$$ds^2 g^{\alpha\beta} = g_{\mu\nu}dx^\mu dx^\nu g^{\alpha\beta}$$

we must take into account other possible terms as we did for spin 1 and for spin 3/2 fields. Namely, we have the following terms to consider (the non-linearity of the equation is assured by the fact that there occur products like $g_{\mu\nu}g^{\alpha\beta}$:

$$\begin{aligned}
&g_{\mu\nu}dx^\mu dx^\nu g^{\alpha\beta};\\
&g_{\mu\nu}dx^\mu(dx^\alpha g^{\nu\beta} + dx^\beta g^{\alpha\nu});\\
&g_{\mu\nu}dx^\alpha dx^\beta g_{\mu\nu};\\
&g^{\alpha\beta}dx^\mu dx^\nu g_{\mu\nu};\\
&g^{\alpha\beta}dx_\lambda dx^\lambda \eta_{\mu\nu}g^{\mu\nu};\\
&dx^\mu g_{\mu\nu}dx^\nu g^{\alpha\beta}\\
&dx^\mu g_{\mu\nu}(dx^\alpha g^{\nu\beta} + dx^\beta g^{\alpha\nu});\\
&dx^\alpha g_{\mu\nu}dx^\mu g^{\nu\beta};\\
&dx^\beta g_{\mu\nu}dx^\mu g^{\nu\alpha};\\
&dx^\alpha g_{\mu\nu}dx^\beta g^{\mu\nu};\\
&dx_\mu g^{\alpha\beta}dx_\nu g^{\mu\nu};\\
&dx^\lambda g^{\alpha\beta}dx_\lambda \eta_{\mu\nu}g^{\mu\nu}.
\end{aligned}$$

We must have in mind that $\frac{dx^\mu}{ds}$ being proportional classically to the momentum P^μ, $\frac{dx^\mu}{ds}g_{\mu\nu}(x)$ is not the same quantum mechanically as $g_{\mu\nu}(x)\frac{dx^\mu}{ds}$. That is why the terms above are not identical to some terms previously written. And this is consistent with our idea of taking our line-element as an operator defined on the field variable. We are thinking on the fact that if coordinates commmute: $[x^\alpha, x^\mu] = 0$ the same cannot be said of the commutator between a position coordinate and a displacement dx^μ:

$$[x^\alpha, dx^\mu] = x^\alpha dx^\mu - (dx^\mu)x^\alpha = \frac{ds}{m_0 c}(x^\alpha P^\mu - P^\mu x^\alpha) \neq 0.$$

In this sense we are distinguishing for instance $g_{\mu\nu}dx^\mu dx^\alpha g^{\nu\beta}$ from $(dx^\mu g_{\mu\nu}).(dx^\alpha g^{\nu\beta})$.

If we take this for granted I will have the following equation:

$$\begin{aligned}
& ds^2 g^{\alpha\beta} = A g_{\mu\nu} dx^\mu dx^\nu g^{\alpha\beta} + \\
& + B g_{\mu\nu} dx^\mu (dx^\alpha g^{\nu\beta} + dx^\beta g^{\alpha\nu}) + \\
& + C g_{\mu\nu} dx^\alpha dx^\beta g^{\mu\nu} + D g^{\alpha\beta} dx_\lambda dx^\lambda \eta_{\mu\nu} g^{\mu\nu} + \\
& + E g^{\alpha\beta} dx_\mu dx_\nu g^{\mu\nu} + A' dx^\mu g_{\mu\nu} dx^\nu g^{\alpha\beta} + \\
& B' dx^\mu g_{\mu\nu} (dx^\alpha g^{\nu\beta} + dx^\beta g^{\alpha\nu}) + \\
& + C' [dx^\alpha g_{\mu\nu} (dx^\mu g^{\nu\beta} + dx^\beta g^{\mu\nu}) + dx^\beta g_{\mu\nu} (dx^\mu g^{\nu\alpha} + dx^\alpha g^{\mu\nu})] + \\
& + D' dx^\lambda g^{\alpha\beta} dx_\lambda \eta_{\mu\nu} g^{\mu\nu}
\end{aligned} \tag{28}$$

We write:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$h_{\mu\nu}$ is the effective gravitational potential; and consider the transition $\frac{dx^\mu}{ds}$ to $\frac{1}{m_0 c} P^\mu$ to $\frac{1}{m_0 c} i\hbar \partial^\mu$ to get the equation:

$$\begin{aligned}
& \left(\frac{m_0 c}{\hbar}\right)^2 h^{\alpha\beta} + A(\Box h^{\alpha\beta} + h_{\mu\nu}\partial^\mu \partial^\nu h^{\alpha\beta}) + \\
& + B[\eta_{\mu\nu}\partial^\mu (\partial^\alpha h^{\nu\beta} + \partial^\beta h^{\alpha\nu}) + h_{\mu\nu}\partial^\mu (\partial^\alpha h^{\nu\beta} + \partial^\beta h^{\alpha\nu})] + \\
& + C(\partial^\alpha \partial^\beta h + h_{\mu\nu}\partial^\alpha \partial^\beta h^{\mu\nu}) + \\
& + D(\eta^{\alpha\beta}\Box h + h^{\alpha\beta}\Box h) + E(\eta^{\alpha\beta}\partial_\mu \partial_\nu h^{\mu\nu} + h^{\alpha\beta}\partial_\mu \partial_\nu h^{\mu\nu}) + \\
& + A' \partial^\mu h_{\mu\nu} \partial^\nu h^{\alpha\beta} + \\
& + B' \partial^\mu h_{\mu\nu} (\partial^\alpha h^{\nu\beta} + \partial^\beta h^{\alpha\nu}) + \\
& + C' [\partial^\alpha h_{\mu\nu} \partial^\mu h^{\nu\beta} + \partial^\beta h_{\mu\nu} (\partial^\mu h^{\nu\alpha} + \partial^\alpha h^{\mu\nu})] + \\
& + D' (\partial^\lambda h^{\alpha\beta})(\partial_\lambda h) = 0
\end{aligned} \tag{29}$$

where

$$h = \eta_{\mu\nu} h^{\mu\nu}$$

For a weak gravitational field we pose $m_0 = 0$ (or $ds = 0$) and retain only terms linear in h. I have:

$$\begin{aligned}
& A\Box h^{\alpha\beta} + B(\partial_\nu \partial^\alpha h^{\nu\beta} + \partial_\nu \partial^\beta h^{\alpha\nu}) + \\
& C\partial^\alpha \partial^\beta h + D\eta^{\alpha\beta}\Box h + \\
& E\eta^{\alpha\beta}\partial_\mu \partial_\nu h^{\mu\nu} = 0
\end{aligned} \tag{30}$$

Take the derivative with respect to x^α:

$$(A+B)\Box\partial_\alpha h^{\alpha\beta} + (B+E)\partial^\beta\partial_\mu\partial_\nu h^{\mu\nu} + (C+D)\Box\partial^\beta h = 0 \tag{31}$$

from which we deduce:

$$A = -B, B = -E, C = -D$$

Thus if we take a scale with $\frac{A}{C} = 1$ we shall have

$$\Box(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h) + (\partial^\alpha\partial^\beta - \frac{1}{2}\eta^{\alpha\beta}\Box)h + \eta^{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} - \partial_\nu(\partial^\alpha h^{\nu\beta} + \partial^\beta h^{\alpha\nu}) = 0$$

This is the Einstein equation in its linearized form for a weak gravitational field.

Much more complex is the expression, in terms of the field $g^{\alpha\beta}$, of the exact Einstein's equation:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -kT_{\mu\nu}.$$

We know that:

$$\begin{aligned}
&R_{\alpha\beta} = -\frac{1}{2}\partial_\beta(g_{\mu\nu}\partial_\alpha g^{\mu\nu}) - \frac{1}{2}\partial_\lambda\left[g^{\lambda\nu}g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})\right] + \\
&+\frac{1}{4}g^{\lambda\mu}(\partial_\alpha g_{\mu\eta} + \partial_\eta g_{\alpha\mu} - \partial_\mu g_{\alpha\eta})g^{\eta\nu}(\partial_\beta g_{\nu\lambda} + \partial_\lambda g_{\nu\beta} - \partial_\nu g_{\lambda\beta}) + \\
&+\frac{1}{4}g^{\lambda\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})g_{\zeta\eta}\partial_\lambda g^{\zeta\eta}
\end{aligned}$$

and the scalar curvature is then $R = g^{\mu\nu}R_{\mu\nu}$.

Thus the geometric form of Einstein's equation in the absence of matter would be:

$$\begin{aligned}
&ds^2 g^{\alpha\beta} = \frac{1}{4}[dx^\alpha(g_{\mu\nu}dx^\beta g^{\mu\nu}) + dx^\beta(g_{\mu\nu}dx^\alpha g^{\mu\nu})] + \\
&\frac{1}{2}dx^\mu\{g_{\mu\nu}[dx^\alpha g^{\nu\beta} + dx^\beta g^{\nu\alpha} - dx^\nu g^{\alpha\beta}]\} \\
&-\frac{1}{4}g_{\mu\nu}\{(dx^\alpha g^{\nu\lambda}) + dx^\lambda g^{\alpha\nu} - dx^\nu g^{\alpha\lambda}\}g_{\lambda\eta}\{dx^\beta g^{\eta\mu} + dx^\mu g^{\eta\beta} - dx^\eta g^{\mu\beta}\} \\
&-\frac{1}{4}g_{\lambda\nu}\{dx^\alpha g^{\nu\beta} + dx^\beta g^{\nu\alpha} - dx^\nu g^{\alpha\beta}\}(g_{\mu\eta}dx^\lambda g_{\mu\eta})
\end{aligned} \tag{32}$$

if we associated the coordinate differential to the derivative. But this correlation through the momentum is questionable and so we should consider only the case of the weak gravitational field.

8 Proof of the rule that establishes that spin 1/2 fermions with negative energy propagate backward in time

This rule gave rise to the Feynman-Wheeler interpretation of anti-particles, as is well-known.

Dirac's equation, we know

$$(i\gamma^\mu \hbar \partial_\mu - m_0 c)\psi(x) = 0 \tag{33}$$

leads to homogeneous algebraic equations of the form:

$$\begin{aligned}
(E - m_0c^2)u_1 - c(p_1 - ip_2)u_4 - cp_3u_3 = 0 \\
(E - m_0c^2)u_2 - c(p_1 + ip_2)u_3 + cp_3u_4 = 0 \\
(E + m_0c^2)u_3 - c(p_1 - ip_2)u_2 - cp_3u_1 = 0 \\
(E + m_0c^2)u_4 - c(p_1 + ip_2)u_1 + cp_3u_2 = 0
\end{aligned} \tag{34}$$

when one puts:

$$\psi(x) = u(\vec{p}, E)e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})} \tag{35}$$

and uses the representation of Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & v \end{pmatrix} \tag{36}$$

The condition for the existence of solutions of equations (33) is that the determinant of their coefficients vanish which gives.

$$\left((p^0)^2 - c^2(\vec{p})^2 - (m_0c^2)^2\right)^2 = 0$$

and so the roots:

$$p^0 = \pm\, c(\vec{p}^{\ 2} + m_0^2c^2)^{1/2} = \pm E \tag{37}$$

are double.There are thus two solutions with positive energy and two other solutions with negative energy. The Feynmann-Wheeler interpretation, namely negative energy electrons travel backward in time, follows immediately from the geometrical equation

$$\gamma_\alpha dx^\alpha \psi(x) = ds\psi(x)\ . \tag{38}$$

If one chooses the representation (33) for the gammas, the equation (8) will give the homogeneous equations similar to equations (33):

$$(dx^0 - ds)\psi_1 - (dx^1 - idx^2)\psi_4 - dx^3\psi_3 = 0$$
$$(dx^0 - ds)\psi_2 - (dx^1 + idx^2)\psi_3 + dx^3\psi_4 = 0$$
$$(dx^0 + ds)\psi_3 - (dx^1 - idx^2)\psi_2 - dx^3\psi_1 = 0$$
$$(dx^0 + ds)\psi_4 - (dx^1 + idx^2)\psi_1 + dx^3\psi_2 = 0$$

The determinant of the coefficients of these equations must vanish which gives:

$$\left((dx^0)^2 - ds^2 - (d\vec{x})^2\right)^2 = 0$$

and so the roots

$$dx^0 = \pm\left(ds^2 + (d\vec{x})^2\right)^{1/2}$$

are double. <u>The time intervals are either positive or negative. It is natural to associate positive energy solutions to time intervals which are always positive. Then the negative energy solutions will have negative time intervals</u> – they represent negative energy fermions running always backward in time (the usual interpretation was elaborated by analogy with the classical equation of motion of the electron).