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Constraints on the Detectability of Cosmic Topology from Observational Uncertainties

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Abstract

Recent observational results suggest that our universe is nearly flat and well modelled within a ΛCDM framework. The observed values of Ω_m and Ω_Λ inevitably involve uncertainties. Motivated by this, we make a systematic study of the necessary and sufficient conditions for undetectability as well as detectability (in principle) of cosmic topology (using pattern repetition) in presence of such uncertainties. We do this by developing two complementary methods to determine detectability for nearly flat universes. Using the first method we derive analytical conditions for undetectability for infinite redshift, the accuracy of which is then confirmed by the second method. Estimates based on WMAP data together with other measurements of the density parameters are used to illustrate both methods, which are shown to provide very similar results for high redshifts.

Key-words: Detectability of Cosmic Topology; Cosmic Topology; Observational Cosmology; FLRW Universes; Detection of Cosmic Topology.

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1 Introduction

An important feature of general relativity is that it is a local metrical theory, and therefore the corresponding Einstein field equations do not fix the global topology of space-time. This freedom has fueled a great deal of interest in the possibility that the universe may possess compact spatial sections with a non-trivial topology (see for example [1, 2]). Whatever the nature of cosmic topology may turn out to be, the issue of its detectability is of fundamental importance.

Motivated by recent observational results, a study was recently made of the question of detectability of the cosmic topology in nearly flat universes where the ratio of the current total density to the critical density of the universe, Ω_0, is very close to one. It was demonstrated that as $\Omega_0 \to 1$ increasing families of possible topologies become undetectable by methods based on image (or pattern) repetitions (see [3] – [8]).

Measurements of the density parameters unavoidably involve observational uncertainties, and therefore any study of the detectability of the cosmic topology should take such uncertainties into account. This is particularly crucial for nearly flat universes, which are favoured by the current observations.

In this paper we study the sensitivity of the detectability of cosmic topology to the uncertainties in the density parameters. We present two complementary methods for deciding the detectability of cosmic topology in terms of the density parameters within the uncertainty region, for any given survey depth. The first method provides sufficient (but not necessary) conditions for undetectability of cosmic topology. The second method provides necessary (but not sufficient) conditions for undetectabilty. The converses of the latter conditions also give sufficient (but not necessary) conditions for detectability in principle of cosmic topology. Using the former method in the limiting case $z \to \infty$ we derive an exact closed form, which expresses the sufficient conditions for undetectability of cosmic topology of nearly flat universes.

Both methods were devised to be suitable where the values of density parameters include uncertainties, which lie in a region in the neighbourhood of $\Omega_0 = 1$, and are shown to be accurate for high redshifts. Numerical criteria for both undetectability and detectability in principle (collectively denoted in what follows by (un)detectability to be succinct) are also presented.

The structure of the paper is as follows. In section 2 we give an account of the cosmological models employed, present a brief discussion of a topological indicator we will use, and make a brief analytical study of the question of detectability of cosmic topology of nearly flat universes. In section 3 we develop two complementary methods for deciding the detectability of cosmic topology taking into account the uncertainties in the density parameters, for any given survey depth. In section 4 we present a number of concrete examples, and discuss their connection with some results in the literature. We construct tables for specific topologies which provide support for the assertion that the closed form expression that ensures sufficient conditions for undetectability is very accurate for deciding the undetectability of cosmic topology of nearly flat universes. Finally, section 5 contains a discussion of our main results and conclusions.

2 Preliminaries

In the context of standard cosmology, the universe is modelled by a 4-manifold $\mathcal{M} = \mathcal{R} \times M$, with a locally isotropic and homogeneous Robertson-Walker (RW) metric

$$ds^2 = -c^2 dt^2 + R^2(t) \left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2) \right] \; , \tag{2.1}$$

where t is a cosmic time, $f(\chi) = \chi$, $\sin\chi$, or $\sinh\chi$ depending on the sign of the constant spatial curvature ($k = 0, \pm 1$), and $R(t)$ is the scale factor. For non-flat models ($k \neq 0$), the scale factor is identified with the curvature radius of the spatial section of the universe at time t. Usually the 3-space M is taken to be simply-connected; namely Euclidean E^3, spherical S^3, or hyperbolic H^3 spaces. In general, however, the 3-space may take the form of an infinite set of other possible quotient (multiply connected) manifolds $M = \widetilde{M}/\Gamma$, where Γ is a discrete group of freely acting isometries of the covering space $\widetilde{M}$.[1]

Recent observations have provided important information concerning the nature of the energy content of the universe. In particular, recent measurements by WMAP [9] of the position of the first acoustic peak in the angular power spectrum of cosmic microwave background radiation anisotropies (CMBR), which refine and to a great extent corroborate previous data [10] – [12]), seem to suggest that the universe is flat or nearly so ($\Omega_0 \sim 1$).

There is also ample evidence from observations, including the CMBR power spectrum, galaxy clustering statistics [13], peculiar velocities [14] and the baryon mass fraction in clusters of galaxies [15, 16] that the density of the clumped (including baryonic and dark) matter in the universe is substantially lower, being of the order of 0.3 of the critical value. Furthermore, observations of high redshift Type Ia Supernovae [17] seem to suggest that the universe is presently undergoing accelerated expansion.

One way of reconciling these diverse set of observations is to postulate that a substantial proportion of the energy density of the universe is in the form of a dark component which is smooth on cosmological scales and possesses a negative pressure. An important candidate for this is the cosmological constant.

In the present work we therefore assume that the current matter content of the universe is well approximated by dust (of density ρ_m) plus a cosmological constant Λ, with associated

[1] In this article, in line with the usage in the literature, by topology of the universe we mean the topology of the space-like section M of the space-time manifold $\mathcal{M}$.

fractional densities $\Omega_m = 8\pi G\rho_m \,/\, (3H^2)$ and $\Omega_\Lambda \equiv \Lambda\, c^2/\, (3H^2)$, with $\Omega_0 = \Omega_m + \Omega_\Lambda$. In this setting, for non-flat models, the redshift-(comoving)-distance relation in units of curvature radius takes the form

$$\chi(z) = \sqrt{|1-\Omega_0|} \int_0^z \left[(1+x)^3\Omega_{m0} + \Omega_{\Lambda 0} - (1+x)^2(\Omega_0 - 1)\right]^{-\frac{1}{2}} dx \,, \tag{2.2}$$

where the redshift z measures the depth of the survey, and the subscript '0' refers to present values of the density parameters. The horizon radius d_{hor} is then defined by (2.2) for $z = \infty$. For simplicity, on the right hand side of (2.2) and in many places in the remainder of this article, we have left implicit the dependence of the function χ on the density components.

An important feature of the expression (2.2) is that it is very sensitive to changes in the quantity $1 - \Omega_0$ near the flat line, falling rapidly to zero as $\Omega_0 \to 1$ (as will become quantitatively evident below). This limit plays a crucial role in the detectability of any non-trivial topology [3] – [5].

To proceed we recall some topological background that we shall use below. To begin with we note that the classes of topologies allowed for spherical, flat and hyperbolic 3-manifolds are qualitatively different, and any analysis must deal with each family separately. Furthermore, in three dimensions there are complete classifications for flat and spherical topologies, but not for the hyperbolic ones. Also, geometrical quantities of hyperbolic and spherical manifolds, expressed in terms of the curvature radius, are topological invariants. For flat manifolds, however, we have no such natural unit of length, and they are not rigid. They should therefore be dealt with separately (see [6] for a study of detectability in such cases). Here we shall confine ourselves to non-flat cases only.

Compact orientable hyperbolic 3-manifolds can be constructed by the so called Dehn surgery procedure, denoted by two coprime winding numbers, (n_1, n_2), applied to a cusped seed manifold. For example, in the case of the manifold $m003(-3,1)$ (the smallest known orientable hyperbolic manifold), $m003$ is the seed cusped manifold, and -3 and 1 are the winding numbers. Using the software SnapPea [18], Hodgson and Weeks [19] compiled a census with 11031 of such manifolds, ordered by volume.

In order to study the (possibly non-trivial) topology of the spatial sections M of the universe, we need a topological invariant length that could be put into correspondence with depth of surveys. We shall employ the so-called injectivity radius r_{inj}, the radius of the smallest sphere 'inscribable' in M, which is defined as half the length ℓ_M of the smallest closed geodesics (for details see [3]),

$$r_{inj} = \frac{\ell_M}{2} \,. \tag{2.3}$$

The indicator T_{inj}, i.e. the ratio of the injectivity radius to the depth $\chi(z_{obs})$ of a given astro-cosmological survey up to a maximum redshift $z = z_{obs}$, is then defined as

$$T_{inj} = \frac{r_{inj}}{\chi(z_{obs})} \,. \tag{2.4}$$

In a universe with $T_{inj} > 1$ there would be no observed multiple images of objects up to the maximum survey depth, and therefore the cosmic topology would not be detectable observationally by any observer looking for patterns repetition. Similarly for universes with $T_{inj} < 1$ the topology is in principle observationally detectable through pattern repetitions, at least for some observers.

An important issue (and the main goal for this work) is to determine the which manifolds (topologies) are undetectable, and which are detectable (in principle), for given values of the density parameters (Ω_{m0}, $\Omega_{\Lambda 0}$) subject to observational uncertainties. If Ω_{m0} and $\Omega_{\Lambda 0}$ were known precisely, it would be a simple matter to calculate $\chi(z_{obs})$ and then obtain T_{inj} for any non-flat universe (with the associated r_{inj}). The density parameters, however, inevitably involve observational uncertainties, with an associated uncertainty region in the parameter plane. The most recent estimates [9] specify this region to be $\Omega_0 \in [0.99, 1.05]$ and $\Omega_\Lambda \in [0.69, 0.79]$ with a 2σ confidence, straddling the flat line where $\chi(z) = 0$. So, for any 3-manifold there is a set of values for Ω_{m0} and $\Omega_{\Lambda 0}$ within the uncertainty region for which the corresponding topology is undetectable. The (un)detectability issue has been investigated, for some specific values of the density parameters in [3] – [5] and [7]. Here we systematically extend those results by developing methods that for each manifold M separates the uncertainty region into undetectable and detectable (in principle) sub-regions, for any given survey depth z_{obs}.

3 Conditions for undetectability and detectability in principle

To motivate our methods we begin by noting that a first estimate of the constraints on detectability of cosmic topology can be obtained from the horizon radius function $\chi_{hor}(\Omega_{m0}, \Omega_{\Lambda 0}, z)$ given by (2.2) for $z = \infty$, in the neighbourhood of the flat line $\Omega_0 = \Omega_{m0} + \Omega_{\Lambda 0} = 1$ favoured by recent diverse set of observations. Figure 1 shows the behaviour of χ_{hor} as a function of Ω_{m0} and $\Omega_{\Lambda 0}$. The curves in the parametric plane are contour curves defined by $\chi(\Omega_{m0}, \Omega_{\Lambda 0}, z_{obs}) = r^M_{inj}$ for a given manifold with $r_{inj} = r^M_{inj}$, and a fixed survey depth z_{obs}. Since in parametric plane the flat universes are characterized by the straight line $\Omega_0 = 1$, this figure makes clear that as $|\Omega_0 - 1| \to 0$, $\chi_{hor} \to 0$, hence showing that, for a given manifold M with injectivity radius r^M_{inj} there are values of $\Omega_{\Lambda 0}$ and Ω_{m0} for which the topology of the universe is either undetectable ($T_{inj} > 1$, i.e. $r^M_{inj} > \chi_{hor}$) or detectable in principle ($r^M_{inj} > \chi_{hor}$).

Such scheme has been employed for specific values of the density parameters (see, e.g., [3] – [5]). Here we extend this approach so as to include the whole observational uncertainty region $\mathcal{U}$, which is determined by a diverse set of observations (CMB, SNIa, lensing, large scale structure observations). Furthermore, in the limiting case $z \to \infty$ we shall derive an exact closed form, which expresses the sufficient conditions for undetectability of cosmic topology of nearly flat universes, no matter how complicated is the spatial topology of M.

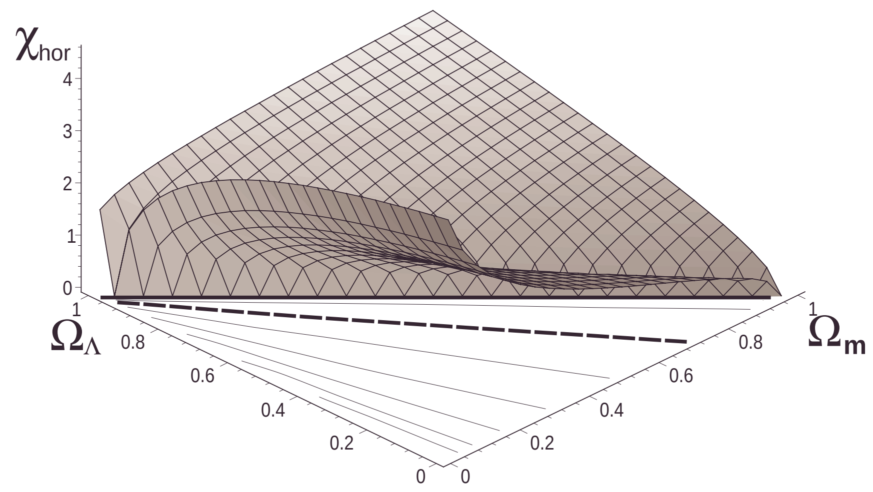

Figure 1: The behaviour of the horizon radius χ_{hor} in units of curvature radius as a function of the density parameters Ω_Λ and Ω_m . Clearly a similar behaviour holds for χ_{obs} for any fixed z_{obs} . The curves in the parametric plane are contour curves defined by $\chi(\Omega_{m0}, \Omega_{\Lambda 0}, z_{obs}) = r^M_{inj}$ for a given manifold with $r_{inj} = r^M_{inj}$, and a fixed survey depth z_{obs} .

From now on we shall focus our attention in the parametric plane $\Omega_{\Lambda 0} - \Omega_{m0}$. To better understand undetectability regarding this plane consider a manifold M with $r_{inj} = r^M_{inj}$. For a fixed survey depth $z = z_{obs}$ a contour curve is defined by $\chi(\Omega_{m0}, \Omega_{\Lambda 0}, z_{obs}) = r^M_{inj}$. This curve defines two sub-regions in the parametric plane: one between the flat line and the contour curve where the topology of a universe with space section M is undetectable for a survey with depth z_{obs}, and another beyond the contour line where it is detectable in principle for the same survey depth (see Fig. 2). If the contour curve does not intersect the current uncertainty region $\mathcal{U}$ the topology of M is undetectable for the values of the density parameters in $\mathcal{U}$. If it does, a criterion is needed to determine the set of values of the density parameters in $\mathcal{U}$ for which the topology is undetectable.

We shall use two different methods to obtain linear approximations of the contour curves. The first approximation method (called secant method) provides sufficient (but not necessary) conditions for undetectabilty of cosmic topology. As such, these conditions do not apply to the whole region where the topology of M is undetectable, since there is a sub-region of $\mathcal{U}$ above the secant line for which the topology of M is still undetectable (see Fig. 2). To ensure that the secant line approximation method is an efficient method to decide the detectability in practice, we need to show that this sub-region is small. We do this by employing a second linear approximation method (which we refer to as the tangent

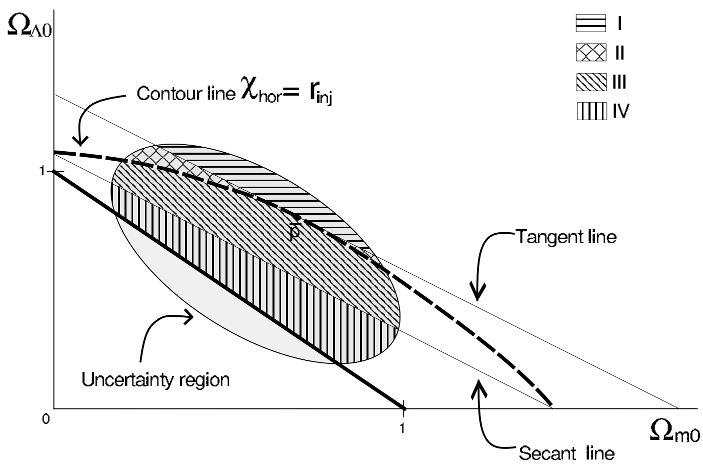

Figure 2: A schematic representation of the secant line (SL) and tangent line (TL) methods. The convexity of the contour curve for $\Omega_0 > 1$ can be proven analytically, as discussed below.

method) that provides necessary (but not sufficient) conditions for undetectabilty. Since both methods rely on linear approximation, the conditions provided by them will be given in terms of linear inequalities in the density parameters.

The secant line method simply states that a universe with space-like section M has an undetectable topology if

$$\alpha\,\Omega_{m0} + \beta\,\Omega_{\Lambda 0} \begin{cases} > 1\,, & \text{for} \quad \Omega_0 < 1\,, \\ < 1\,, & \text{for} \quad \Omega_0 > 1\,, \end{cases} \tag{3.5}$$

where the coefficients α, β are "normalized" constants fixed by the topology (via r^M_{inj}) and the depth of the survey z_{obs}.

Similarly, the tangent method states that the necessary conditions for M to have undetectable topology are

$$\mu\,\Omega_{m0} + \nu\,\Omega_{\Lambda 0} \begin{cases} < 1\,, & \text{for} \quad \Omega_0 > 1\,, \\ > 1\,, & \text{for} \quad \Omega_0 < 1\,, \end{cases} \tag{3.6}$$

where again the coefficients μ, ν are "normalized" constants fixed by the topology (via r^M_{inj}) and the depth of the survey z_{obs}. We note that the converses of conditions (3.6) provide sufficient (but not necessary) conditions for detectability in principle. So, for example $\mu\,\Omega_{m0} + \nu\,\Omega_{\Lambda 0} > 1$ is a sufficient condition for detectability in principle of the topology of spherical universes ($\Omega_0 > 1$).

Before we proceed further it is worth stressing that the conditions (3.5) and (3.6) are global (observer independent), since they depend on r^M_{inj}, which is half the length of the smallest geodesic in M. However, it is also possible to define $r_{inj}(x)$, half the length of the smallest geodesic that passes through the point $x \in M$. Then (3.5) and (3.6) would become location (observer)-dependent conditions for undetectability of cosmic topology.

3.1 Secant method

Consider the contour curve $\chi(z_{obs}, \Omega_{m0}, \Omega_{\Lambda 0}) = r_{inj}^{M}$. For a given survey depth z_{obs} we define the secant line as the line joining the points $(\widetilde{\Omega}_{m0}, 0)$ and $(0, \widetilde{\Omega}_{\Lambda 0})$ where the contour curve intersects the axes Ω_{m0} and $\Omega_{\Lambda 0}$, respectively. Clearly the equation of this line is given by

$$\frac{\Omega_{m0}}{\widetilde{\Omega}_{m0}} + \frac{\Omega_{\Lambda 0}}{\widetilde{\Omega}_{\Lambda 0}} = 1 \,. \tag{3.7}$$

For these intersecting points the redshift-distance relation (2.2) can also be more easily integrated. By writing explicitly $\chi\,(z_{obs}, \widetilde{\Omega}_{m0}, 0) = r_{inj}^{M}$ and $\chi\,(z_{obs}, 0, \widetilde{\Omega}_{\Lambda 0}) = r_{inj}^{M}$, and setting $\varepsilon = [\mathrm{sign}(\,1 - \widetilde{\Omega}_0)\,]^{\frac{1}{2}}$ we obtain, respectively[2],

$$\begin{aligned} r_{inj}^{M} &= 2\,\varepsilon \left[\text{arctanh }(1 - \widetilde{\Omega}_{m0})^{-1/2} - \text{ arctanh }\left(\frac{1 \,+\, z\,\widetilde{\Omega}_{m0}}{1 \,-\, \widetilde{\Omega}_{m0}}\right)^{1/2} \right] \,, \\ r_{inj}^{M} &= \varepsilon \log \left\{ \frac{(1+z)\,(1-\widetilde{\Omega}_{\Lambda 0})^{1/2} \,+\, [\,(1+z)^2 - z(2+z)\,\widetilde{\Omega}_{\Lambda 0}\,]^{1/2}}{1 + (1-\widetilde{\Omega}_{\Lambda 0})^{1/2}} \right\} \,. \end{aligned} \tag{3.8}$$

We now wish to solve the equations (3.8) for $\widetilde{\Omega}_{m0}$ and $\widetilde{\Omega}_{\Lambda 0}$. Although this can be done analytically, the (very long) resulting expressions are not in general very useful, except in the limiting case $z \to \infty$. It can, however, always be done numerically.

By treating $\Omega_{\Lambda 0}$ as a function of Ω_{m0}, it is possible to show that for any fixed $z = z_{obs}$ the contour curves $\chi(z_{obs}, \Omega_{\Lambda 0}, \Omega_{m0}) = r_{inj}^{M}$ are convex (concave) for $\Omega_0 < 1 \;\; (>1\,)$, i.e., $d^2\Omega_{\Lambda 0}/d\Omega_{m0}^2 > 0 \;\; (<0\,)$. This property can also be gleaned from the parametric plot of $\chi(\Omega_{\Lambda 0}, \Omega_{m0})$ (see for example [3]). It also ensures that the secant line crosses the contour line only at the Ω_{m0} and $\Omega_{\Lambda 0}$ axes, and that any sub-region of $\mathcal{U}$ lying between the secant and the flat lines (region IV in Figure 2) will also lie between the contour line and the flat line. Therefore the topology of M is undetectable for the values of the density parameters in such region (IV). To sum up, if Ω_{m0} and $\Omega_{\Lambda 0}$ are such that inequality (3.5) holds with α and β given by (3.7), then the topology of M is undetectable.

Now we shall obtain a closed form expression for (3.7) in the limiting case $z \to \infty$. Unless there is a horizon for a finite $z_{\max}$ (which is not the case for currently accepted density parameters), $\chi(z)$ increases monotonically with z. Therefore, by taking $z \to \infty$ we obtain an upper bound for $\chi(z_{obs})$. Using (3.8) we obtain explicitly $\widetilde{\Omega}_{m0}$ and $\widetilde{\Omega}_{\Lambda 0}$ in terms of r_{inj}^{M} in this limiting case. The undetectability conditions (3.5) [with (3.7)] then reduce to:

[2] Incidentally, note that in these equations the functions arctanh and log are defined as the analytical extensions to the complex plane of the ordinary real functions, and as such they apply to both spherical and hyperbolic cases.

A universe with space section M has undetectable topology if

$$\begin{aligned} \cosh^2 \Big(\frac{r_{inj}^{M}}{2} \Big) \, \Omega_{m0} + \Omega_{\Lambda 0} \;\; &> \;\; 1 \, , \quad \text{for} \quad \Omega_0 < 1 \, , \\ \cos^2 \Big(\frac{r_{inj}^{M}}{2} \Big) \, \Omega_{m0} + \Omega_{\Lambda 0} \;\; &< \;\; 1 \, , \quad \text{for} \quad \Omega_0 > 1 \, . \end{aligned} \tag{3.9}$$

Despite its simple form, this result is of considerable interest in that it gives a test for undetectability for *any* z. Note, however, that for nearly flat universes the conditions (3.9) and (3.5) for $z \sim 1100$ (CMB) provide very close results. Yet as these results are also close to those provided by the tangent method (discussed in the next section), it follows that for the cases of physical interest (3.9) is, in practice, an efficient criterion for undetectability of cosmic topology. Its closed form greatly enhances its usefulness, as we shall illustrate below.

A word of clarification is in order here. The expression used for $\chi(z_{obs})$ here assumes only matter and cosmological constant components. For $z \gg 1100$ the photon energy density becomes significant. But since its presence only decreases the actual value of $\chi(z_{obs})$ (increasing undetectability), the expressions (3.9) remain valid nonetheless.

It is worth stressing that the coefficients obtained by the secant line method do not depend on the values of density parameters or their associated uncertainties, and can be tabulated for any nearly flat universe to be checked against present or future observations.

To close this subsection we note that there is a region in the parameter plane (lying between the secant and contour curves, depicted as region III in Figure 2) that does not meet the condition (3.5), but for which the topology of M is still undetectable. In the next subsection we shall present the tangent line method, and use it in section 4 to show that this region is indeed very narrow, and can be disregarded in practical cases of physical interest (nearly flat universes).

3.2 Tangent method

The tangent line method discussed in more details in this section provides necessary conditions for undetectability of cosmic topology of nearly flat universes. As we have mentioned before their converses also furnish conditions for detectability in principle of cosmic topology of these universes.

We obtain a tangent to the contour line $\chi(z_{obs}, \Omega_{m0}, \Omega_{\Lambda 0}) = r_{inj}^{M}$ by taking a line passing through the point $\bar{P} = (\bar{\Omega}_{m0}, \bar{\Omega}_{\Lambda 0})$ that is perpendicular to the gradient of $\bigtriangledown\chi(z_{obs})$ at $\bar{P}$. The equation for this tangent line can be written as

$$\frac{\Omega_{\Lambda 0} - \overline{\Omega}_{\Lambda 0}}{\Omega_{m0} - \overline{\Omega}_{m0}} = - \frac{\left. \frac{\partial \chi}{\partial \Omega_{m0}} \right|_{P=\bar{P}}}{\left. \frac{\partial \chi}{\partial \Omega_{\Lambda 0}} \right|_{P=\bar{P}}} \; . \tag{3.10}$$

We note that the convexity (concavity) property mentioned above ensures that this line will never cross the contour curve. Rearranging the terms in (3.10) we obtain

$$\frac{\frac{\partial\chi}{\partial\Omega_{m0}}}{\frac{\partial\chi}{\partial\Omega_{m0}}\,\overline{\Omega}_{m0}+\frac{\partial\chi}{\partial\Omega_{\Lambda0}}\,\overline{\Omega}_{\Lambda0}}\,\Omega_{m0}+\frac{\frac{\partial\chi}{\partial\Omega_{\Lambda0}}}{\frac{\partial\chi}{\partial\Omega_{m0}}\,\overline{\Omega}_{m0}+\frac{\partial\chi}{\partial\Omega_{\Lambda0}}\,\overline{\Omega}_{\Lambda0}}\,\Omega_{\Lambda0}=1\;, \tag{3.11}$$

where (although we have not explicitly indicated for simplicity) all partial derivatives are evaluated at $P=\bar{P}$.

The choice of $\bar{P}$ is not crucial, but in practice we take it as the point in the contour curve that lies along the gradient $\bigtriangledown\chi(z_{obs})$ from the point of maximum $\chi(z_{obs})$ within the uncertainty region (see Fig. 2). The extreme points of uncertainty range given by WMAP data, together with other measurements [9], are $P_{hyp}=(0.23,0.69)$ for the hyperbolic, and $P_{sph}=(0.31,0.79)$ for the spherical case, respectively.

In this way for a fixed $z=z_{obs}$ and for any given manifold M, we obtain the tangent line at a point $\bar{P}$ of the contour line $\chi(z_{obs})=r_{inj}^{M}$. Thus the necessary (but not sufficient) conditions for the topology of nearly flat universes to be undetectable are provided by (3.6) with μ and ν given by (3.11). Clearly if Ω_{m0} and $\Omega_{\Lambda0}$ are such that the converses of the inequalities (3.6) hold with μ and ν given by (3.11), then the topology of the universe is detectable in principle.

In terms of Figure 2, this means that for points in region I the inequalities (3.6) hold, and therefore the topology will be detectable in principle. There are also points (region II) where the inequalities do not hold, and yet the topology is still detectable in principle. This is however a very small region, as we shall discuss below.

To close this section we note that for each pair $(r_{inj}^{;M},z_{obs})$ we can calculate numerically the coefficients μ and ν, and compare their values to α and β obtained by the secant method to make clear that the sub-region of $\mathcal{U}$ between the secant and the tangent lines (regions II and III in Figure 2) is indeed very small for manifolds whose contour curves intersect the uncertainty region. This amounts to saying that in practice, for currently accepted values of the density parameters, the secant line method as formulated in eq. (3.5) [or its stronger closed form (3.9)] gives an efficient criterion for undetectability of nearly flat spherical and hyperbolic universes. In the next section we shall discuss this point further.

4 Applications

In section 3 we described how to obtain the coefficients (α,β) and (μ,ν) required by the secant and tangent methods to ensure, respectively, the sufficient and necessary conditions for undetectability [see (3.5) and (3.6)], providing therefore a systematic criterion for undetectability of cosmic topology of nearly flat universes, given the unavoidable observational uncertainties in the values of the density parameters. We are mostly interested in undetectability conditions for large redshifts ($z = 1100$ and $z \to \infty$), so here we shall tabulate these coefficients for specific topologies to provide support for the assertion that the closed form expression (3.9) is an accurate criterion for undetectability of cosmic topology of nearly flat universes. For smaller z, however, the coefficients must be calculated numerically.

The conditions (3.5) [with α and β fixed by (3.7)], their closed forms (3.9), as well as (3.6) [with the μ and ν given by (3.11)] can be trivially rewritten in terms of Ω_0 and Ω_{X0}, where the latter variable can be either $\Omega_{\Lambda 0}$ or Ω_{m0}. These new variables make clearer for which exact values of the total density parameter the topology of M becomes undetectable. The expression (3.5) then becomes

$$\begin{aligned} \Omega_0 &> K + \gamma\,\Omega_{X0}\ , \quad \text{for} \quad \Omega_0 < 1\ , \\ \Omega_0 &< K + \gamma\,\Omega_{X0}\ , \quad \text{for} \quad \Omega_0 > 1\ . \end{aligned} \tag{4.12}$$

To make a direct comparison with previous results we choose Ω_{X0} as $\Omega_{\Lambda 0}$ or Ω_{m0}. In these cases one has $K = 1/\alpha$ and $\gamma = (\alpha - \beta)/\alpha$ for the pair $(\Omega_0, \Omega_\Lambda)$, while for the pair (Ω_0, Ω_{m0}) one obtains $K = 1/\beta$ and $\gamma = (\beta - \alpha)/\beta$. Now for the limit $z \to \infty$ the undetectability conditions (3.9) take the form:

A universe with space section M has undetectable topology if

$$\begin{aligned} \Omega_0 \ &> \ \operatorname{sech}^2(\,r_{inj}^M/\,2\,) \,+\, \tanh^2(\,r_{inj}^M/\,2\,)\ \Omega_{\Lambda 0}\ , \quad \text{for} \quad \Omega_0 < 1\ , \\ \Omega_0 \ &< \ \sec^2(\,r_{inj}^M/\,2\,) \ -\ \tan^2(\,r_{inj}^M/\,2\,)\ \Omega_{\Lambda 0}\ , \quad \text{for} \quad \Omega_0 > 1\ , \end{aligned} \tag{4.13}$$

or, in terms of Ω_{m0}, if

$$\begin{aligned} \Omega_0 \ &> \ 1 - \sinh^2(\,r_{inj}^M/\,2\,)\ \Omega_{m0}\ , \quad \text{for} \quad \Omega_0 < 1\ , \\ \Omega_0 \ &< \ 1 + \ \sin^2(\,r_{inj}^M/\,2\,)\ \Omega_{m0}\ , \quad \text{for} \quad \Omega_0 > 1\ . \end{aligned} \tag{4.14}$$

As an example, we follow [3] and tabulate K and γ for the first seven manifolds in the Hodgson-Weeks census of hyperbolic manifolds, ordered here by decreasing r_{inj} .

Table 1 shows that both methods provide very close numerical values for the coefficients K and γ when $z = 1100$ and $z \to \infty$ (for the secant method). This means that, on the one hand, the sufficient and necessary undetectability conditions provided, respectively, by the secant and tangent line methods are consistent (as expected), and give very close results. On the other hand, this numerical closeness also makes clear how narrow is the thin strip between the secant and the tangent lines.

As an example, in [3] (see also [5]) it was shown that in the range $\Omega_{\Lambda 0} \in [0.63, 0.73]$ the topology of both manifolds $m007(3,1)$ and $m009(4,1)$ are undetectable for $\Omega_0 \in [0.99, 1)$, and detectable in principle for $\Omega_0 \in [0.98, 0.99)$. Using the values in Table 1 from the

Manifold	r_{inj}	TL Method $z = 1100$		SL Method $z = 1100$		SL Method $z \to \infty$	
		K	γ	K	γ	K	γ
$m007(3,1)$	0.416	0.955	0.040	0.956	0.045	0.958	0.041
$m009(4,1)$	0.397	0.959	0.037	0.959	0.040	0.961	0.039
$m003(-3,1)$	0.292	0.977	0.020	0.978	0.022	0.978	0.022
$m003(-2,3)$	0.289	0.978	0.020	0.978	0.022	0.979	0.020
$m003(-4,3)$	0.288	0.978	0.020	0.978	0.022	0.980	0.020
$m004(6,1)$	0.240	0.985	0.014	0.985	0.015	0.985	0.014
$m004(1,2)$	0.183	0.991	0.008	0.991	0.009	0.991	0.008

Table 1: Coefficients $K = 1/\alpha$ and $\gamma = (\alpha - \beta)/\alpha$ for undetectability of hyperbolic manifolds provided by the necessary conditions from the tangent line (TL) method, and the sufficient conditions given by the secant (SL) method, for two distinct survey depths.

conditions (4.13) we can state more precisely that the topology of $m007(3,1)$ is undetectable for $\Omega_{\Lambda 0} = 0.63$ if $\Omega_0 > 0.984$ and also for $\Omega_{\Lambda 0} = 0.73$ if $\Omega_0 > 0.986$. Likewise, the topology of $m009(4,1)$ is undetectable for $\Omega_{\Lambda 0} = 0.63$ if $\Omega_0 > 0.986$ and for $\Omega_{\Lambda 0} = 0.73$ if $\Omega_0 > 0.988$.

On the other hand, using the coefficients obtained with the tangent method, it is clear that the topology of $m007(3,1)$ is detectable in principle for $\Omega_0 < 0.980$ if $\Omega_{\Lambda 0} = 0.63$ and the topology of $m009(4,1)$ is detectable in principle for $\Omega_0 < 0.982$ if $\Omega_{\Lambda 0} = 0.73$.

These results agree with those obtained in [7], where it was found that the probability of detection of the topology of these manifolds is zero, if $\Omega_0 = 0.99$. The above examples also demonstrate clearly that the conditions for (un)detectability of cosmic topology we have obtained in this article extend previous results [3] – [5], in which the undetectability of cosmic topology was investigated only for specific values of the density parameters, rather than values within the whole uncertainty region.

It is clear that χ_{obs} is not strongly dependent on $\Omega_{\Lambda 0}$, and it can be viewed as a function of Ω_0 for an appropriate value (fixed by observation, for example) of $\Omega_{\Lambda 0}$. Furthermore, as we have seen for manifolds of physical interest the secant line with $z \to \infty$ is a good approximation of the contour curve. Therefore (4.13) can be understood as an analytical relation between χ_{obs} and Ω_0 which can be used to answer specific questions about whole classes of manifolds, as we shall discuss in examples below.

We now discuss some important examples related to spherical manifolds. In Table 2 we tabulate all single action spherical manifolds with their respective r_{inj}, as well as the lower and upper bounds for the minimal value of Ω_0 for undetectability provided, respectively, by the secant line and the tangent line methods. Single action manifolds are globally homogeneous and therefore the (un)detectability conditions provided by both methods are location independent, for any fixed survey depth. As a consequence, the tangent line method also gives sufficient conditions for detectability (not only in principle). For direct comparison with a similar table in [8] we have taken $\Omega_{m0} = 0.35$. The minimal values of the total density in Table 2 given by the tangent and secant line methods are in good agreement with one another, as well as with those found in ref. [8].

Group	r_{inj}	Sup. Ω_0 TL Method $z = 1100$	Inf. Ω_0 SL Method $z \to \infty$
Binary icosahedral I^*	$\frac{\pi}{10}$	1.011	1.009
Binary octahedral O^*	$\frac{\pi}{8}$	1.018	1.013
Binary tetrahedral T^*	$\frac{\pi}{6}$	1.031	1.023
Binary dihedral D^*_m	$\frac{\pi}{2m}$	—	$1 + \sin^2\left(\frac{\pi}{4m}\right)\Omega_{m0}$
Cyclic Z_n	$\frac{\pi}{n}$	—	$1 + \sin^2\left(\frac{\pi}{2n}\right)\Omega_{m0}$

Table 2: Upper and lower bounds for the minimal value of Ω_0 for undetectability of topology, obtained, respectively, from the necessary conditions given by the tangent line (TL) method, and the sufficient conditions provided by the secant (SL) method, for $\Omega_{m0} = 0.35$.

It is useful to employ the analytical expression (4.14) to provide a lower bound for undetectability in the case of binary dihedral and cyclic spaces, since it contains an explicit dependence on Ω_{m0}, allowing a more systematic study of detectability. For these manifolds the undetectability conditions are given as functions of n and m, respectively. Up to second order they reduce to $1 + 0.86/(2m)^2$ and $1 + 0.86/n^2$, for $\Omega_{m0} = 0.35$, in agreement with table 2 in [8].

m	n	r_{inj}	Max. Ω_0 TL Method $z = 1100$
2	4	0.785	1.072
3	6	0.524	1.031
4	8	0.393	1.018
5	10	0.314	1.011
6	12	0.262	1.008
9	18	0.175	1.005

Table 3: Upper bound for the minimal value for Ω_0 for undetectability of cosmic topology obtained from the (necessary) conditions provided by the tangent line method for sample binary dihedral D^*_m and cyclic spaces Z_n, for $\Omega_{m0} = 0.35$.

The values in Tables 2 and 3 show that the results given by the tangent and secant line methods are in good agreement with each other, as well as with those in [8]. For manifolds with smaller r_{inj} the results of course become closer.

The values in Tables 2 and 3 show that the results given by the tangent and secant line methods are in good agreement with each other, as well as with those in [8]. If we compare the bounds in Table 2 with numerical estimates of the contour curves with $\Omega_{m0} = 0.35$, it becomes clear that the actual minimal value of Ω_0 for detectability is very close (agreeing to 3 decimal places) to the bound given by the tangent method. This is not surprising, given that the tangent line is the best linear fit of the contour curve. We note, however, that the use of the secant line instead does not introduce significant errors.

To complete the picture we also show in Table 3 the maximum values of Ω_0 for detectability provided by the tangent method for some members of the D^*_m and Z_n classes, again with $\Omega_{m0} = 0.35$. These two classes are particularly important because as $\Omega_0 \to 1$ from above, there is always a n_* (or m_*) such that the topology corresponding to Z_n (or D^*_m) is detectable for $n > n_*$ (or $m > m_*$). In particular, the single action cyclic spaces are globally homogeneous and constitute a subclass of the lens spaces $L(n, q)$, namely $L(n, 1)$. The general lens spaces $L(n, q)$ for $q \neq 1$ are not globally homogeneous, but have the same injectivity radius, $r_{inj} = \pi/n$, of the homogeneous family. Thus, although for the homogeneous cyclic spaces the condition $n > n_*$ is global and ensures that the topology is detectable, for the inhomogeneous cases it becomes a condition for detectability in principle only. For some specific density values n_* has been obtained [3]. Here we extend these results by solving (4.14) for n_* and m_* to obtain

$$\begin{aligned} n_* &= \mathrm{Int}\left\{ \frac{\pi}{2} \left[\arcsin \sqrt{\frac{\Omega_0 - 1}{\Omega_{m0}}} \right]^{-1} \right\} , \\ m_* &= \mathrm{Int}\left\{ \frac{\pi}{4} \left[\arcsin \sqrt{\frac{\Omega_0 - 1}{\Omega_{m0}}} \right]^{-1} \right\} , \end{aligned} \tag{4.15}$$

where $\mathrm{Int}[x]$ denotes the integer part of x.

These expressions can be used to calculate n_* and m_* for any set of density parameters. For $z \to \infty$ expressions (4.15) are in agreement with results obtained [3], which in turn are very close to the results for $z = 3000$ tabulated in [3]. For smaller z the values of n_* and m_* must be obtained numerically, by using equation (3.8). We stress that these expressions also provide an example of how our results can be used to study the detectability of cosmic topology for classes of manifolds systematically without resorting to numerical calculations of the contour lines.

5 Conclusions and final remarks

Current estimates of the cosmological density parameters suggest that the universe is nearly flat, with the associated 2σ confidence level allowing for the possibility of a spherical, flat or hyperbolic universe. Motivated by this we have reexamined the question of detectability of the cosmic topology taking into account the uncertainty region in the $\Omega_\Lambda - \Omega_m$ parametric plane. Since the detectability or undetectability of the cosmic topology crucially depends on the values of the density parameters, the true determination of the topology requires taking into account a detailed analysis of the effects of the related uncertainties.

We present two complementary methods (secant and tangent methods) that give, respectively, sufficient (but not necessary) conditions for undetectability, and necessary (but not sufficient) conditions for undetectability of cosmic topology. The converses of the latter also constitute a set of sufficient (but not necessary) conditions for detectability in principle. These methods are systematic in the sense that they determine (un)detectability for any values of the density parameters in the uncertainty region, except for a negligible thin strip.

When applied to specific manifolds, both methods provide conditions that are shown to be in good agreement, for any fixed survey depth, accurately separating the parameter plane into undetectable and detectable regions. Since they are both systematic and accurate, these criteria extend previous results [3] – [5], (see also [7]) in which the undetectability of cosmic topology was investigated only for specific values of the density parameters, rather than within an uncertainty region that include such values.

Clearly the tangent line obtained from (3.11), being the best linear fit for the contour curve, is a better approximation of the contour curve than the secant line (3.7). However, as was discussed in Sections 3 and 4, in practice the difference between the coefficients given by the tangent and secant line methods is small, and become even smaller as $\Omega_0 \to 1$, and which one is used is a matter of convenience. The closed form of the latter in the limit $z \to \infty$ makes it more useful to study classes of manifolds rather than specific examples.

An important result of this article is the closed form conditions (3.9) for undetectability of cosmic topology or nearly flat universes obtained from the secant method in the $z \to \infty$ limit. These inequations can be seen, to a very good approximation, as establishing conditions for detectability in principle as well, as can be shown by comparison with numerical values obtained from both methods for $z = 1100$. For high redshifts we can therefore use (3.9) to separate the parameter plane into undetectable and detectable sub-regions with great accuracy. The closed form of these conditions makes its application quite straightforward and potentially more useful. Equation (4.15) is a good example of such application, because it extends previous results to a more general, and yet simpler, form.

There are other advantages in the use of (3.9) and its counterparts (4.13) and (4.14). If, as expected, new observations further constrain the uncertainty region nearer to the flat line, the conditions provided by the secant and tangent methods will become closer to each other, and either (3.9), (4.13) or (4.14) will become even more accurate. In that case, our expressions can be used to address a number of questions regarding the detectability of cosmic topology for classes of manifolds, as for example in the detectability conditions $n > n^*$ and $m > m^*$ with n^* and m^* given by (4.15). Note also that, for any given survey depth, the coefficients in the conditions (3.9) or its counterparts for each (fixed) manifold do not depend on the size and shape of the uncertainty region.

In a recent paper Weeks [7] introduced the so called injectivity profile, in which for each manifold the probability of detecting the cosmic topology is plotted as a function of the horizon distance $\chi(z_{hor})$. The probability is defined as the fraction of the manifold where $\chi(z_{obs}) \leq r_{inj}(x) \leq \chi(z_{obs}) + \Delta\chi$. We note that even though the (un)detectability conditions presented here were obtained for a global r_{inj}^M, expressions such as (3.9) still hold if the injectivity radius function $r_{inj}(x)$ is used instead. Such location dependent conditions could be used to calculate injectivity profiles in terms of Ω_0 instead of $\chi(z_{hor})$. This would allow us to determine, for instance, how the probability of detectability changes as the total density approaches its critical value.

Finally even though we have used a ΛCDM framework, similar methods could be developed for other cosmological models with different redshift-distance relations in order to obtain conditions for undetectabilty of cosmic topology.

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