GRAVITATIONAL PECULIARITIES OF A SCALAR FIELD

A. Kleber

Observatório Nacional - CNPq
20921 Rio de Janeiro, Brasil

Carlos A.P. Galvão* and A.F. da F. Teixeira
Centro Brasileiro de Pesquisas Físicas
22290 Rio de Janeiro, Brazil

*On leave of absence from Departamento de Física - UFRN, Natal-RN, Brazil.

GRAVITATIONAL PECULIARITIES OF A SCALAR FIELD

ABSTRACT

The zero-adjoint of a time-static Ricci-flat solution to Einstein's field equations is investigated. It represents a spacetime curved solely by a massless scalar field. The cylindrical symmetry is assumed, to permit both planar and non-planar geodetic motions. Unusual, velocity-dependent gravitational features are encountered from these geodesics.

1. INTRODUCTION

The study of exact solutions of gravity coupled to other fields is important to clearly understand the physical and mathematical structure of spacetime (Duncan 1977). For many reasons, the coupling of scalar fields to gravitation has been object of special attention in recent years (Bronnikov 1978, Kodama et al. 1978, Buchdahl 1978, Chung et al. 1977, Bekenstein 1974, 1975). In most cases, systems have been studied in which the scalar field coexists with other constituents, such as diffused matter or electromagnetic fields (Banerjee and Dutta Choudhury 1977, Teixeira et al. 1974, 1975, 1976). In such complex systems, however, the nonlinearity of the field equations generally makes it difficult to see the gravitational peculiarities of each constituent, separately.

In this paper, we study the gravitation associated to a massless, real scalar field, in the absence of any material source or other field. Differently from the electromagnetic fields, the scalar field under static condition can be described in terms of cosmic time. We consider a system with cylindrical symmetry, to permit both planar and nonplanar geodesics. From the investigation of these geodesics, an interesting, velocity dependent acceleration field is found, acting differently upon each component of the velocity vector.

2. GRAVITATIONAL AND SCALAR POTENTIALS

We concern with the line element

$$ds^2 = dt^2 - \left[(dr^2 + dz^2)r^{2b} + r^2 d\phi^2\right] , \quad b = \text{const} \geq 0 . \tag{1}$$

It satisfies the Einstein-scalar field equations

$$R_{\mu\nu} = -2\partial_\mu S \partial_\nu S \quad , \quad S = \pm\sqrt{b}\ \ell n\ r \quad , \tag{2}$$

where the dimensionless constant $\pm\sqrt{b}$ represents the strenght of the long range, attractive scalar field S. The important feature of (1) is that it represents the gravitation from the pure scalar field, as is explained in some detail at the end of Section 4.

The line element (1) can be obtained, without solving the field equations, in a variety of ways starting from the static, Ricci-flat solution with cylindrical symmetry (Weyl 1917)

$$ds^2_{Weyl} = (r/a)^{4\lambda} dt^2 - (r/a)^{-4\lambda}\left[(dr^2 + dz^2)(r/a)^{8\lambda^2} + r^2 d\phi^2\right] . \tag{3}$$

We are using c = G = 1; the constant a has dimension of length and is set equal to one, for simplicity. Following Buchdahl (1978), we should simply write the zero-adjoint of (3) and set $\lambda^2 = b/4$ to obtain (1). Alternatively, we can use the prescriptions of Teixeira et al. (1976) and set the constant $c^2 = 1$ in their attractive scalar field. Also, we could follow the method of Janis et al. (1969) and set their constant $A^2 \to \infty$.

3. GEODESICS

To investigate the gravitational features of (1), we consider the geodetic differential equations

$$\ddot{t} = 0 \quad , \quad \ddot{z} = -2b\dot{z}\dot{r}/r \quad , \quad \ddot{\phi} = -2\dot{\phi}\dot{r}/r \quad , \tag{4}$$

$$r\ddot{r} = -b\dot{r}^2 + b\dot{z}^2 + r^{2(1-b)}\dot{\phi}^2 \quad , \tag{5}$$

where a dot means d/ds. With the restriction $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, valid for timelike geodesics, we find the first integrals

$$s' = (1 - v^2)^{1/2} \quad , \tag{6}$$

$$z' = vh/r^{2b} \quad , \quad \phi' = v\ell/r^2 \quad , \tag{7}$$

$$r' = \pm\, vr^{-b}\left(1 - h^2r^{-2b} - \ell^2r^{-2}\right)^{1/2} \quad , \tag{8}$$

where a prime means d/dt, and where the three parameters $0 \leq v < 1$, h and ℓ are constants of integration.

A trivial solution of (6) to (8) is obtained when v = 0, and corresponds to a particle at rest in the presence of the anisotropic fields. This interesting result is discussed in Sec. 4. Other trivial solutions are obtained when b = 0, and correspond to the rectilinear, uniform motions in the flat spacetime.

The nontrivial solutions of (6) to (8) correspond to three types of motion:

3.1 Motion on Planes Normal to the z-Axis

Setting h = 0 in (7) and (8) we obtain

$$dr/d\phi = \pm\left[(r/\ell)^2 - 1\right]^{1/2} r^{1-b} \quad . \tag{9}$$

$|\ell|$ is then the minimal radial location of the particle in its motion. Table 1 presents exact solutions of (9), obtained for some values of the parameter b. In Figure 1 are drawn some solutions corresponding to $\ell = 1.5$; as in all cases where $\ell^2 \geq 1$, these solutions represent spiral motions around the z-axis. In the cases where $\ell^2 < 1$, however, a different behavior of the test particle is found near the z-axis; in Fig. 2, corresponding to motions with $\ell = 0.5$, we remark that all trajectories bend outwards for small values of r.

It can be shown that the shape of orbits given by (9) can also be obtained from a non-relativistic, static, cylindically symmetric potential

$$V(r) = -\frac{1}{2} v^2 \left[(1 - r^{-2b})(\ell/r)^2 + r^{-2b}\right] \quad . \tag{10}$$

However, the relativistic velocity of motion in the orbit differs from its non-relativistic analogue.

3.2 Motion on Planes Containing the z-Axis

Setting $\ell = 0$ in (7) and (8) we obtain

$$dr/dz = \pm \left[(r/m)^{2b} - 1\right]^{1/2} \quad , \tag{11}$$

where m, given by $m^b = |h|$, is the minimal value of the radial coordinate along the motion. Table 2 presents solutions of (11) for several values of b, while Figure 3 shows the corresponding orbits. We find that the trajectory of the particle always bends outwards, what indicates repulsion from the axis of symmetry.

As before, a non-relativistic potential can be obtained, producing the same orbits as (11):

$$V(r) = -\frac{1}{2} (vr^b/h^2)^2 \quad . \tag{12}$$

However, the relativistic and non-relativistic velocity of motion again differ.

3.3 Non-planar Motions

When h and ℓ are non-zero, we obtain the following exact solution of (6) to (8), for b = 1:

$$z = \sin\beta \cosh^{-1} R \quad , \quad R = r/m \quad , \tag{13}$$

$$\phi = \cos\beta \cosh^{-1} R \quad , \tag{14}$$

$$vt = \frac{1}{2} m^2 \left[R(R^2 - 1)^{1/2} + |\cosh^{-1} R| \right] \quad , \tag{15}$$

$$s = t\ (1 - v^2)^{1/2} \quad , \tag{16}$$

where $m = (h^2 + \ell^2)^{1/2}$ is the minimal distance from the axis of symmetry, and $\beta = \tan^{-1} h/\ell$ is the angle of incidence on the plane z = 0. For t < 0 the particle is approaching the z-axis in a helical motion, and reaches the minimal radial distance r = m when t = 0. For t > 0 a helical motion is found with increasing radius.

The solutions belonging to other values of b present similar basic features, but the corresponding mathematical expressions are rather involved. It can be shown that the non--planar orbits do not derive from any non-relativistic potential which is static and cylindrically symmetric.

4. DISCUSSIONS

The source of gravitation of the system is concentrated around the axis of symmetry, as is seen from the scalar curvature, the square of the Ricci tensor, and the Kretschmann scalar:

$$R = 2b/r^{2(b+1)} \quad , \quad R^{\mu\nu}R_{\mu\nu} = R^2 \quad , \quad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 3R^2 \ . \tag{17}$$

Since $b \geq 0$, all these quantities tend to zero at radial infinity. The same happens to the energy momentum tensor, which is diagonal with components

$$T^0_0 = -T^1_1 = T^2_2 = T^3_3 = R/(16\pi) \geq 0 \ . \tag{18}$$

We found, in Sec. 3, that a particle once at rest remains at rest. This is a consequence of the staticity of the metric with $g_{00} = 1$. Such type of metric seems not possible when electromagnetic fields are present.

The anisotropic state of stresses (18) is responsible for the peculiar, velocity-dependent gravitation originating from the scalar field. A radial acceleration field is found from (5), acting attractively upon the radial component of velocity of test particles, and repulsively upon the longitudinal component. We next compare the radial acceleration associated to the azimuthal velocity, $\ddot{r}_1 = r(r/a)^{1-2b}\,\dot{\phi}^2$, with its analogue in the case of rectilinear motion, $\ddot{r}_2 = r(r/a)\,\dot{\phi}^2$: Since $\ddot{r}_2 \gtrless \ddot{r}_1$ implies respectively $r \gtrless a$, we find that the azimuthal component of the velocity is acted upon attractively when $r > a$, and repulsively when $r < a$. This explains the shapes of orbits in Fig. 2, drawn for $a = 1$.

One finds, from (6), that v represents the modulus of velocity of the test particle. All the results obtained for time-like geodesics are then also valid for lightlike geodesics, provided one sets v = 1.

A final comment concerns the physical interpretation of the metric (1). This metric is now explicitly obtained following the prescriptions given in Teixeira et al. (1976). We start from the line element (3), where λ is linear density of matter in weak field approximation, and get the intermediate solution

$$ds^2 = r^{4\mu}\, dt^2 - r^{-4\mu} \left[(dr^2 + dz^2)\, r^{8\lambda^2} + r^2\, d\phi^2 \right] , \quad (19)$$

$$R_{\alpha\beta} = -2\partial_\alpha S\, \partial_\beta S \quad , \quad S = 2c\lambda \ln r \quad , \quad \mu \equiv \lambda(1 - c^2)^{1/2} \quad , \quad (20)$$

where c = const. For weak fields, this intermediate solution corresponds to a linear density of matter μ , together with a linear source of scalar field $c\lambda$. The original vacuum solution (3) corresponds to the special value c = 0, when the source of scalar field vanishes and $\mu = \lambda$. If we now start from the vacuum solution, fix λ and let c^2 increase, then the source of scalar field $|c\lambda|$ gradually increases, while the matter parameter μ gradually decreases. We interprete this process as gradual substitution of the original matter by attractive scalar source. The substitution is completed when $c^2 = 1$, in which case $\mu = 0$ and the line element becomes (1), with $b = 4\lambda^2$. This fact strongly suggests to interprete (1) as the gravitation from the scalar field alone, at least in the weak field approximation.

REFERENCES

Banerjee A and Dutta Choudhury S B 1977 Phys. Rev. D 10 3062-4

Bekenstein J D 1974 Ann. Phys. NY 82 535-47

——————— 1975 Ann. Phys. NY 91 75-82

Bronnikov K A 1978 Gen. Rel. Grav. 9 271-5

Buchdahl H A 1978 Gen. Rel. Grav. 9 59-70

Chung K C, Kodama T, and Teixeira AFF 1977 Phys. Rev. D 16 2412-6

Duncan C 1977 Phys. Rev. D 16 1688-90

Dwight H B 1961 "Tables of Integrals and other Mathematical Data", 4th. ed., MacMillan, New York

Kodama T, Chung K C, and Teixeira AFF 1978 Nuovo Cim. 46 B 206-15

Teixeira A F F, Wolk I, and Som M M 1974 J. Math. Phys. 15 1756-9

——————— 1975 Phys. Rev. D 12 319-22

——————— 1976 J. Phys. A 9 53-8

Weyl H 1917 Ann. Phys., Lpz 54 117-45

Janis A I, Robinson DC and Winicour J 1969 Phys. Rev. 186 1729-31

TABLE 1

b	$\lvert \ell \rvert^{-b}\ \phi(r)$
0	$\sec^{-1} R \qquad (R \equiv r/\lvert \ell \rvert)$
1/2	$\sqrt{2}\ F\ (\sec^{-1} \sqrt{R}\ ,\ 1/\sqrt{2}\ )\ \left[\equiv G(r)\right]$
1	$\cosh^{-1} R$
3/2	$2(R-R^{-1})^{1/2} + G(R) - 2\sqrt{2}\ E(\sec^{-1} \sqrt{R}\ ,\ 1/\sqrt{2})$
2	$(R^2 - 1)^{1/2}$
5/2	$\frac{1}{3}\left\{2\left[R(R^2 - 1)\right]^{1/2} + G(R)\right\}$
3	$\frac{1}{2}\left[R(R^2 - 1)^{1/2} + \cosh^{-1} R\right]$

TABLE 2

b	$m^{-1}z(r)$
1/2	$2(R - 1)^{1/2} \qquad (R \equiv r/m)$
1	$\cosh^{-1} R$
3/2	$3^{-1/4}\, F\left(\cos^{-1} \frac{\sqrt{3} + 1 - R}{\sqrt{3} - 1 + R} ,\ \sin \pi/12\right)$
2	$2^{-1/2}\, F(\sec^{-1} R ,\ 1/\sqrt{2})$

CAPTIONS FOR THE TABLES AND FIGURES

Table 1 — Orbits $\phi(r)$ in planes z = const, for several values of b. The functions E and F are elliptic integrals (Dwight 1961).

Table 2 — Orbits $z(r)$ in planes containing the z-axis, for several values of b. The functions $F(\phi,k)$ are elliptic integrals (Dwight 1961).

Figure 1 — Orbits in planes z = const, for $\ell = 1.5$ and several values of b. Particles are attracted to the axis of symmetry, nevertheless these always escape to the radial infinity.

Figure 2 — Orbits in planes z = const, for $\ell = 0.5$ and several values of b. Particles are repelled from the z-axis for short radial distances, and attracted for larger values of r.

Figure 3 — Orbits in planes containing the z-axis for several values of b and for arbitrary m. Particles are repelled from the z-axis.

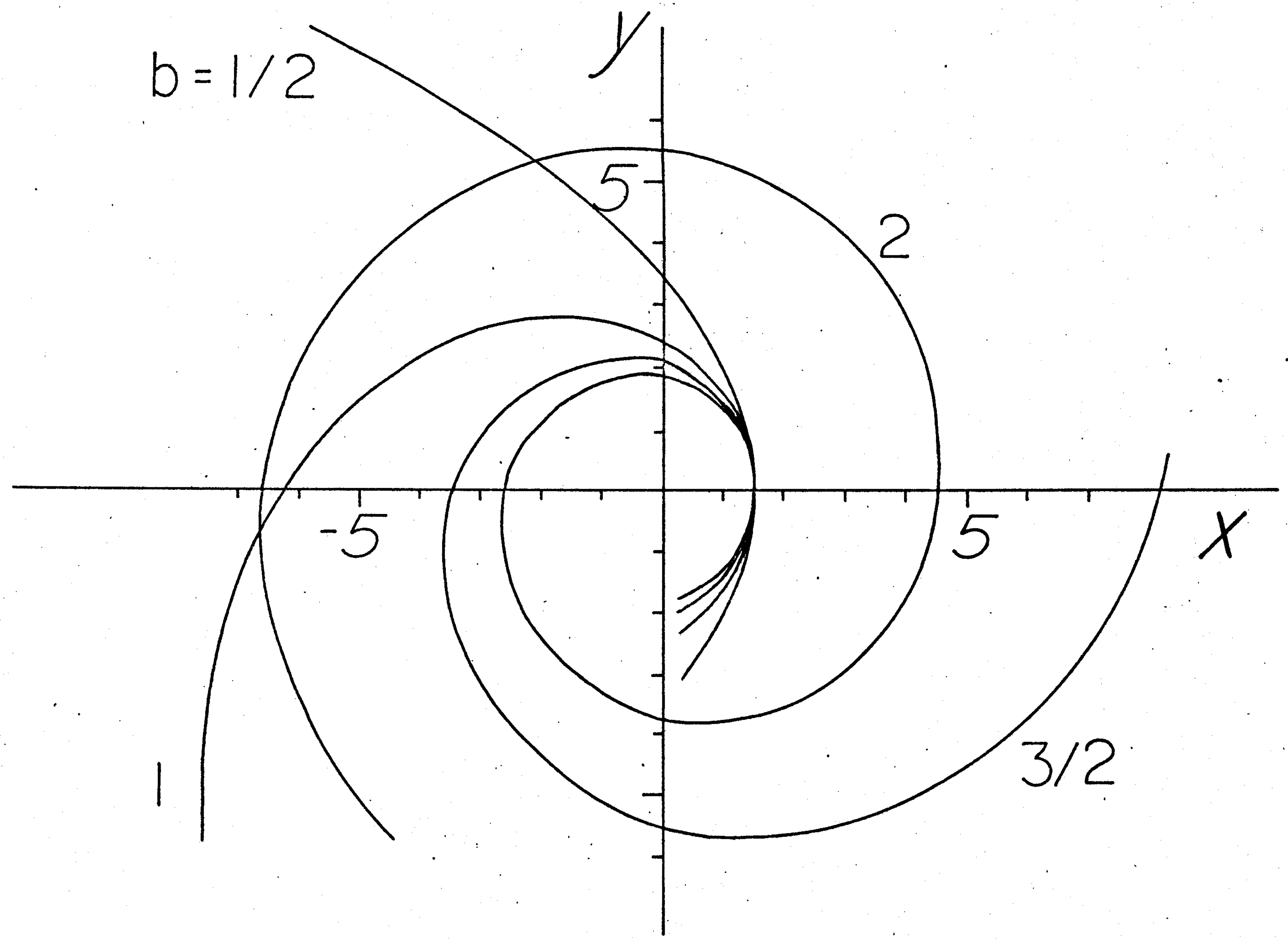

Fig. 1

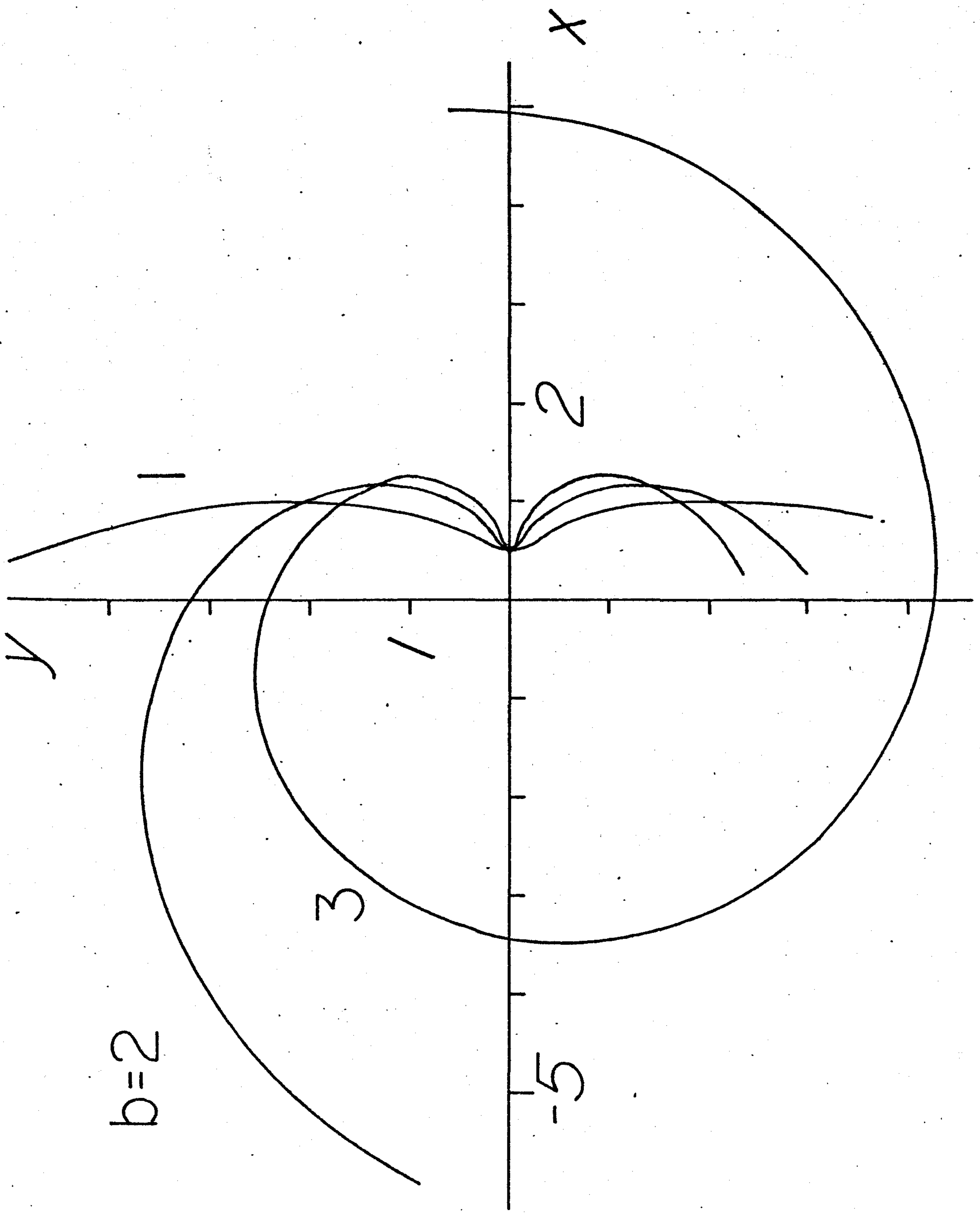

Fig. 2

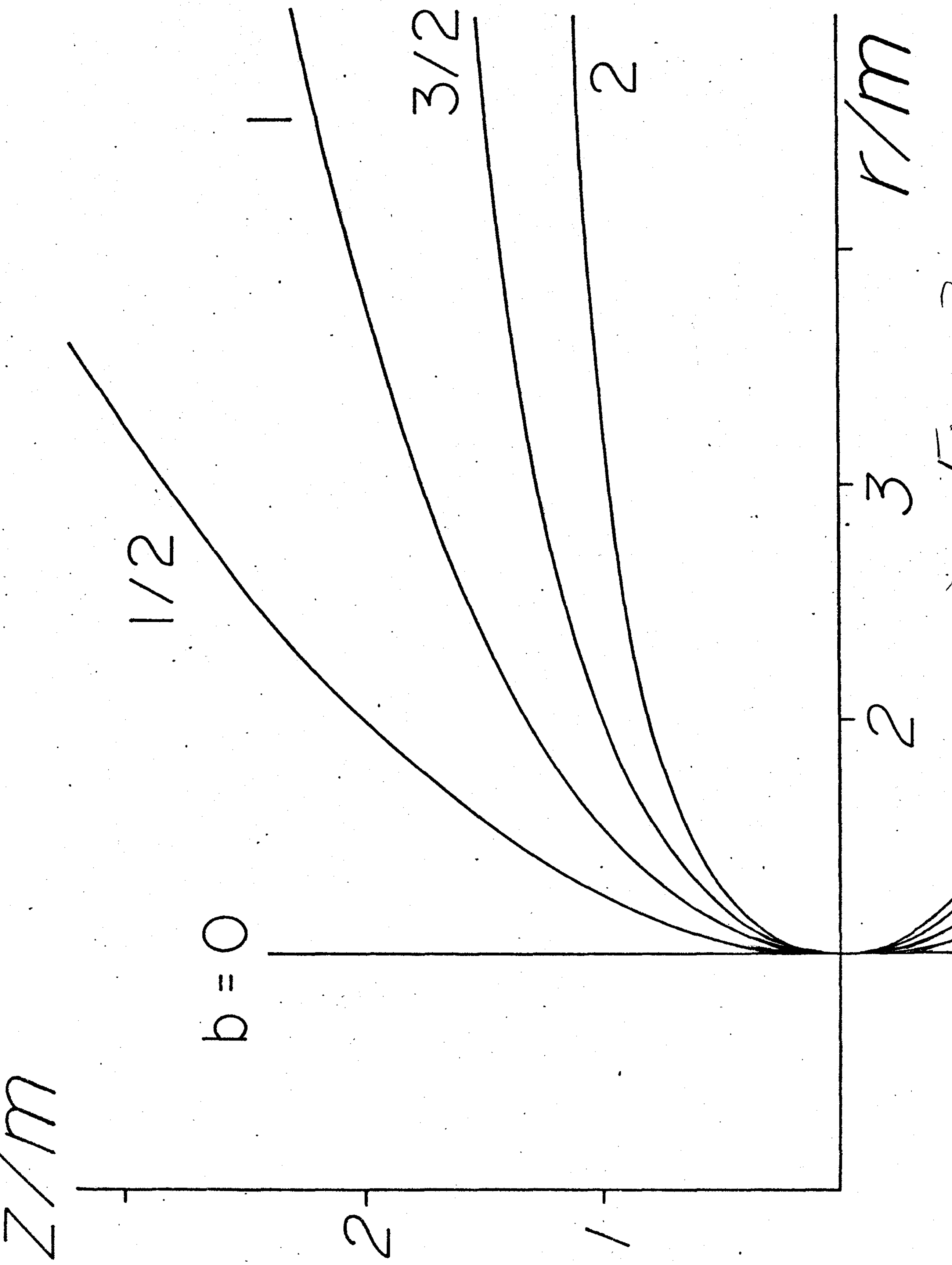

Fig. 3

O CBPF PUBLICA PERIODICAMENTE AS DUAS SÉRIES DE RELATÓRIO A SEGUIR.

SÉRIE A: PRÉ—PUBLICAÇÃO DE RELATÓRIOS DE TRABALHOS QUE NECESSITAM DE RÁPIDA CIRCULAÇÃO.

SÉRIE B TRABALHOS ORIGINAIS, MONOGRAFIAS, TABELAS E OUTROS DADOS NÃO DESTINADOS A PUBLICAÇÃO.

CBPF PUBLISHES PERIODICALLY THE FOLLOWING TWO SERIES OF REPORTS:

SERIES A: PRE-PUBLICATION OF REPORTS OF PAPERS NEEDING QUICK CIRCULATION.

SERIES B: ORIGINAL PAPERS, MONOGRAPHIES, TABLES AND OTHER DATA WHICH ARE NOT INTENDED TO BE PUBLISHED ELSEWHERE.

PEDIDOS DE CÓPIA DESTE RELATÓRIO FAVOR DIRIGIR-SE A:

REQUEST FOR COPY OF THIS REPORT PLEASE ADDRESS TO:

BIBLIOTECÁRIA – CHEFE
CBPF
AV. WENCESLAU BRAZ, 71 FUNDOS, ZC-82
RIO DE JANEIRO, RJ 20.000
BRASIL