PROPERTIES OF GENERAL RELATIVISTIC KINK SOLUTION

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ABSTRACT

Properties of the general relativistic kink solution of a nonlinear scalar field recently obtained (Phys.Rev. D15 , 1978) are discussed. It has been shown that the kink solution is stable against radial perturbations. Possible applications to hadron physics from the geometrodynamic point of view are suggested.

1 - INTRODUCTION

In last few years, considerable interest in searching classical solutions of nonlinear field equations has been found in order to construct a pure field theoretical model of elementary particles(1-3). Such attempts are of course not new at all(4-8), but recent developments of nonlinear field theories brought some interesting aspects of these models. One of the most prominent features of modern theories is to introduce a topological concept to distinguish a particle from the vacuum. Another one is the concept of degenerate vacuum.

A kink is an example of model based on the idea of degenerate vacuum which garantees its stability. The simplest form of kink solutions is found in the so-called $\lambda\Phi^4$ theory. For two dimensional space-time case the model is analytically soluble(9,10). However, it is proved(7,11,12) to be impossible to construct a kink solution for the realistic four dimensional space-time. Although there exist several attempts to construct solitary waves, like instantons(13) or monopoles(14,15), they are still far from giving a definite model of elementary particles. It is thus worthwhile to investigate further possibilities to construct classical solutions of nonlinear field models.

Very recently one of the authors(16) (T.K.) has shown that, even for the scalar $\lambda\Phi^4$ model, there exists a static singularity free, finite energy solution if one extends the concept of spacetime to that of the general theory of relativity. This general relativistic kink (GRK) is a topological soliton and is a natural generalization of the usual one-dimensional kink solution.

The spacetime geometry implied there resembles that of the Rosen-Einstein bridge[4,17] of the Schwarzschild geometry, i.e. two asymptotically flat spaces connected by a bridge of radius r_o. However there is an essential difference. The surface $r = r_o$ is a true boundary of the universe in our case so that the geometry naturally excludes the $r = 0$ singularity. On the other hand, in the Schwarzschild case, $r = r_o$ is a null surface so that the surface is an event horizon but not a real boundary. In this aspect, our bridge is a kind of wormhole, as called by Wheeler[18], of the spacetime.

In this paper, we discuss further some properties of GRK. We first show that there exists a more convenient system of coordinates to describe our model. This coordinate system enables us to study radial perturbations on GRK.

2 - GRK

In the previous paper[16] (refered to as I hereafter), we adopted the Schwarzschild coordinate system to describe the geometry. In this coordinate system the line element is given by

$$ds^2 = e^{2\eta} (dx^o)^2 - e^{2\alpha} dr^2 - r^2 d\Omega^2 \qquad (1)$$

where $x^o = ct$, and η and α are functions of x^o and r.

In this form, the GRK exhibits a singularity in $e^{2\alpha}$ at $r = r_o$. However the geometry itself is nonsingular everywhere so that there exists a more convenient coordinate system to describe the geometry. After a suitable coordinate transformation, we can rewrite the line element as

$$ds^2 = e^{2h} (dx^o)^2 - dR^2 - r^2 d\Omega^2 \qquad (2)$$

where now $h = h(x^o,R)$ and $r = r(x^o,R)$. For a static metric the new radial coordinate R is related to r by

$$(dr/dR)^2 = e^{-2\alpha} \qquad (3)$$

The Lagrangean density is the same as in I:

$$\kappa \mathcal{L} = (-g)^{1/2} \left\{ \frac{1}{2}\mathcal{R} - \left[S_{,\mu} S^{,\mu} - V(S^2) \right] \right\} \qquad (4)$$

where κ is the Einstein constant (= $8\pi G/c^4$), $\mathcal{R}$ the scalar curvature, g the determinant of the metric tensor, S a scalar field, and V the potential describing the self interaction of the scalar field. We take $V(S^2) = \frac{1}{2} (\frac{\mu}{f})^2 . (1-f^2S^2)^2$, where μ and f are constant[16]

As in I, we introduce dimensionless quantities and variables as

$$u = \mu R \qquad (5)$$

$$x = \mu r \qquad (6)$$

$$y = fS \qquad (7)$$

$$\tau = \mu ct \qquad (8)$$

$$v = (f/\mu)^2 V = \frac{1}{2} (1-y^2)^2 \qquad (9)$$

In these variables, Einstein's equations are written as

$$\frac{1}{x^2} (1 - x'^2 + \dot{x}^2 e^{-2h}) - \frac{2}{x} x'' = -f^{-2}(\dot{y}^2 e^{-2h} + y'^2 + v) \qquad (10)$$

$$\frac{1}{x}(\dot{x}' - \dot{x}h') = f^{-2}\dot{y}y' \tag{11}$$

$$\frac{2}{x}\left\{e^{-2h}(\ddot{x} - \dot{x}\dot{h}) - xh'\right\} + \frac{1}{x^2}(1-x'^2+\dot{x}^2e^{-2h}) = f^{-2}(\dot{y}^2e^{-2h}+y'^2-v) \tag{12}$$

$$\frac{1}{x}\left\{e^{-2h}(\ddot{x} - \dot{x}\dot{h}) - x'h'\right\} - (h''+h'^2) - \frac{x''}{x} = f^{-2}(\dot{y}^2e^{-2h}-y'^2 - v) \tag{13}$$

where the dot and prime signify the derivative with respect to τ and u, respectively.

The Bianchi identity is written as

$$e^{-2h}\left\{\ddot{y} + \dot{y}\left(\frac{2\dot{x}}{x} - \dot{h}\right)\right\} - \left\{y'' + y'\left(h' + \frac{2x'}{x}\right)\right\} = - y\frac{dv}{dy^2} \tag{14}$$

Of course Eq. (14) is not independent from Eqs.(10) — (13).

The static GRK then satisfies the following equations:

$$\frac{1}{\bar{x}^2}(1 - \bar{x}'^2) - \frac{2}{\bar{x}}\bar{x}'' = -f^{-2}(\bar{y}'^2 + \bar{v}) \tag{15}$$

$$-\frac{2}{\bar{x}}\bar{x}'\bar{h}' + \frac{1}{\bar{x}^2}(1 - \bar{x}'^2) = f^{-2}(\bar{y}'^2 - \bar{v}) \tag{16}$$

$$\frac{1}{\bar{x}}\bar{x}'\bar{h}' + (\bar{h}'' + \bar{h}'^2) + \frac{\bar{x}''}{\bar{x}} = f^{-2}(\bar{y}'^2 + \bar{v}) \tag{17}$$

$$\bar{y}'' + \bar{y}'\left(\bar{h}' + \frac{2\bar{x}'}{\bar{x}}\right) = \bar{y}\frac{d\bar{v}}{d\bar{y}^2} \tag{18}$$

where we attached bars to the static quantities, to distinguish them from time-dependent quantities.

In this form, the advantage of the new coordinate system becomes evident. All the quantities $\bar{x}, \bar{y}$ and $\bar{h}$ are analytic functions of u and contain no singular behaviour. This fact enables us to solve numerically the equations with high accuracy.

The result of I shows that $\overline{x} = \overline{x}(u)$ is a positive definite function which has the asymptotic behaviour,

$$\overline{x}(u) \to |u| \qquad \text{for} \qquad |u| \to \infty$$

and has a minimum at $u = 0$, $\overline{x}(0) = x_o$.

Thus we get at $u = 0$

$$\overline{y}'(0) = (2f^2 + x_o^2)^{1/2}/x_o \tag{19}$$

$$\overline{x}''(0) = f^2(2f^2 + x_o^2)/2x_o \tag{20}$$

where we used the fact that $y(0) = 0$ and $h'(0) = 0$. Again we see that the new coordinate system simplifies the initial conditions to solve Eqs. (15) – (18).

The static solution with these initial conditions has a definite parity with respect to the transformation $u \to -u$. Functions $\overline{x}(u)$ and $\overline{h}(u)$ are even functions and $\overline{y}(u)$ is an odd function of u. (See Fig. 1).

It should be noted that, in the construction of the static solution, there remains a degree of freedom to fix the origin of u coordinate. Eqs. (15) – (18) are invariant under the transformation $u \to u' = u + \text{const}$. Under this transformation the static solutions are displaced in u-coordinate but remain invariant in r-coordinate. Such a transformation is a kind of gauge transformation of metric tensor, and will be discussed later.

3 - RADIAL STABILITY

The main purpose of this paper is to study the stability of the GRK. Here we discuss only the stability of the GRK against radial perturbations. First we put

$$x(\tau,u) = \bar{x}(u) + \delta x(\tau,u) \tag{21}$$

$$y(\tau,u) = \bar{y}(u) + \delta y(\tau,u) \tag{22}$$

$$h(\tau,u) = \bar{h}(u) + \delta h(\tau,u) \tag{23}$$

where $\bar{x}$, $\bar{y}$ and $\bar{h}$ are static solutions obtained before, and δx, δy and δh are infinitesimal perturbations. Substituting these expressions into Eqs. (10) – (14) and neglecting higher order terms in δ, we get

$$\frac{\bar{x}'}{\bar{x}^2}\,\delta x' + \frac{1}{\bar{x}^3}(1-\bar{x}'^2)\delta x - \frac{\bar{x}''}{\bar{x}^2}\,\delta x + \frac{1}{\bar{x}}\,\delta x'' = f^{-2}(\bar{y}'\delta y' + \bar{y}\,\frac{dv}{d\bar{y}^2}\,\delta y) \tag{24}$$

$$\delta\dot{x}' - \bar{h}'\delta\dot{x} = f^{-2}\,\bar{x}\,\bar{y}'\delta\dot{y} \tag{25}$$

$$\delta h'' + 2\bar{h}'\delta h' + \frac{2}{\bar{x}}\,\delta x'' - \frac{2\bar{x}''}{\bar{x}^2}\,\delta x = 2f^{-2}(2\bar{y}'\delta y' + \bar{y}\,\frac{dv}{d\bar{y}^2}\,\delta y) \tag{26}$$

$$\frac{1}{\bar{x}}\left\{e^{-2\bar{h}}\delta\ddot{x} - \bar{h}'\delta x' - \bar{x}'\delta h'\right\} + \frac{1}{\bar{x}^2}\,\bar{x}'\bar{h}'\delta x - \frac{1}{\bar{x}^3}(1-\bar{x}'^2)\delta x - \frac{\bar{x}'}{\bar{x}^2}\,\delta x' = f^2(\bar{y}'\delta y' - \bar{y}\,\frac{dv}{d\bar{y}^2}\,\delta y) \tag{27}$$

$$e^{-2\bar{h}}\delta\ddot{y} - \left\{\delta y'' + (\bar{h}' + \frac{2\bar{x}'}{\bar{x}})\delta y' + \bar{y}'(\delta h' - \frac{2\bar{x}'}{\bar{x}^2}\,\delta x' + \frac{2}{\bar{x}}\,\delta x')\right\} = - \frac{d}{d\bar{y}}(\bar{y}\,\frac{dv}{d\bar{y}^2})\,\delta y \tag{28}$$

Again Eq. (28) is the consequence of the Bianchi identity and not independent of Eqs. (24) – (27). Furthermore, it can be verified that Eqs. (24) and (25) are not independent provided $\bar{x}$, $\bar{y}$ and $\bar{h}$ satisfy Eqs. (15) – (19). Thus only 3 equations are independent among Eqs. (24) – (28).

Now to investigate the stability of these radial perturbations, we look for a normal mode solution of the type

$$\delta x(\tau,u) = e^{i\omega\tau}X(u) \tag{29}$$

$$\delta y(\tau,u) = e^{i\omega\tau}Y(u) \tag{30}$$

$$\delta h(\tau,u) = e^{i\omega\tau}H(u) \tag{31}$$

Then we get the following equations:

$$X' - \bar{h}'X = f^{-2}\,\bar{x}\,\bar{y}'Y \tag{32}$$

$$H'' + 2\bar{h}'H' + \frac{2}{\bar{x}}X'' - \frac{2\bar{x}''}{\bar{x}^2}X = 2f^2\left\{2\bar{y}'Y' + \bar{y}\,\frac{dv}{d\bar{y}^2}\,Y\right\} \tag{33}$$

$$Y'' + (\bar{h} + \frac{2\bar{x}'}{\bar{x}})Y' + \bar{y}'\left\{H' - \frac{2\bar{x}'}{\bar{x}^2}X + \frac{2}{\bar{x}}X'\right\} + \\ + \left\{\omega^2 e^{-2\bar{h}} - \frac{d}{d\bar{y}}(\bar{y}\,\frac{dv}{d\bar{y}^2})\right\}Y = 0 \tag{34}$$

These equations can be resolved numerically from $u = 0$ to ∞ when initial conditions at $u = 0$ are given.

The boundary condition is that X, Y and H are finite everywhere including $u \to \infty$. Under this boundary condition, Eqs. (32) – (34) form an eigenvalue problem for ω^2. If all the eigen value of ω^2 are positive, then the system is stable against radial

$$\text{case (b)}\begin{cases} Y(0) = 1 \\ Y'(0) = 0 \\ H'(0) = \alpha \\ X(0) = 0 \end{cases} \qquad (36)$$

where we have used the fact that the equations are linear so that we may fix the scale factor, and α is a constant which should be determined from the asymptotic values of X, Y and H' in acoordance with the boundary conditions.

On the other hand, since $\bar{y} \to 1$, $\bar{y}' \to 0$, $\bar{x} \to u$, $\bar{x}' \to (1\text{-const}/u)$, $e^{-2\bar{h}} \to (1\text{-const}/u)$, the asymptotic properties of X, Y and H are as follows:

$$Y \to \begin{array}{ll} e^{\pm \text{const}.u} & \text{for} \quad \omega^2 \leq 2 \\ \sin(\text{const}.u + \delta) & \text{for} \quad \omega^2 \geq 2 \end{array} \qquad (37)$$

$$X \to \text{const}.\ e^{+\bar{h}} \qquad (38)$$

$$H' \to \text{const}.e^{-2\bar{h}} \qquad (39)$$

where only the leading terms are maintained.

4 - EIGENVALUES

With the aid of Eq. (35) or (36) we may solve (numerically) Eqs. (32) – (34) from $u = 0$ to ∞ . First we analyse the case (a). In this case ω^2 is the only parameter to determine the asymptotic behaviour of X, Y and H. It is seen from Eq. (38) that the boundary condition for X is always satisfied. It is found that the boundary condition for Y is satisfied for only one particular value of ω^2 which lies between 0 and 2, or for any value of $\omega^2 \geq 2$. However, any finite value of ω^2 gives always a positive asymptotic value of H'. This means the perturbed metric element e^{2H} increases exponentially in u so that there is no finite value of ω^2 for which all the boundary conditions are satisfied. Thus we conclude that the type (a) solution is forbidden.

A different situation is found in the case (b). In this case there are two variables ω^2 and α to determine the asymptotic behaviour of X, Y and H. Again the boundary condition for X is always satisfied. The function Y satisfies the boundary condition for one and only one value of α, say $\alpha_1(\omega^2)$, if $\omega^2 < 2$. For $\omega^2 > 2$, Y is oscillatory in u, satisfying the boundary condition independent of α. On the other hand the asymptotic value of H'(u) should be zero, and this happens for a unique value of α, say $\alpha_2(\omega^2)$. The eigenvalue ω^2 is then obtained from the following condition:

$$\alpha_1(\omega^2) = \alpha_2(\omega^2) \tag{40}$$

In fig. 2 we illustrated $\alpha_1(\omega^2)$ and $\alpha_2(\omega^2)$ for f = 1.25. Thus eigenvalues are obtained to be 0 and any $\omega^2 \geq 2$ (continuum).

The null eigenvalue is the consequence of translational invariance of the static solution mentioned in §2. In fact for $\omega^2 = 0$, functions X, Y and H satisfy the same equations and boundary conditions for $\bar{x}'$, $\bar{y}'$ and $\bar{h}'$, respectively. We conclude that

$$X = \varepsilon \bar{x}'$$
$$Y = \varepsilon \bar{y}' \qquad (41)$$
$$H = \varepsilon \bar{h}'$$

where ε is an infinitesimal constant. The perturbed solutions are

$$x(\tau,u) = \bar{x}(u) + \varepsilon \bar{x}'(u) = \bar{x}(u+\varepsilon)$$
$$y(\tau,u) = \bar{y}(u) + \varepsilon \bar{y}'(u) = \bar{y}(u+\varepsilon) \qquad (42)$$
$$h(\tau,u) = \bar{h}(u) + \varepsilon \bar{h}'(u) = \bar{h}(u+\varepsilon)$$

showing that the null eigenvalue of ω^2 corresponds to the translations of static solutions in u.

5 - DISCUSSION

As shown in the previous section, there is no negative eigenvalue of ω^2. The GRK is stable at least against radial perturbations. The null eigenvalue is related to the translation in u. The corresponding state is physically not observable.

The geometrical structure of GRK is essentially viewed as a bridge which connects two sheets of asymptotically flat spaces. In asymptotic regions of one of these sheets, the GRK behaves as a concentration of mass. Since geodesic lines connect analytically the two sheets of space a light signal can pass through the bridge. In general, any kind of field flux can pass through the bridge from one sheet of the universe to the other. Thus even if the field current is defined to be locally conserved, the existence of such a bridge, to an observer in one of the two sheets, appears as a source or sink of field flux. Furthermore, the field current defined on the one sheet of universe should have an opposite direction in the other sheet. In this aspect, a bridge with a field current passing through it is viewed in one of the two sheets as a particle with the charge corresponding to the current and with the mass of the GRK, whereas it is viewed as its antiparticle in the other sheet.

The above scheme is first suggested by Wheeler in his geometrical interpretation of electric charge[18]. The existence of a stable GRK suggests us an interesting possibility of generalization of Wheeler's idea. Suppose that our universe is composed of two sheets of space connected by many bridges. These bridges are characterized as particular particles or antiparticles depending on the field current and its direction passing through them. The field currents might be that of electromagnetic fields or gluons, or even strings. They circulate from one bridge to the other satisfying the local conservation of currents, thus the Lagrangian contains no source terms for these fields. Such an approach, at least at the classical level, would provide a pure field-theoretical, singularity free model of elementary particles including the concept of antiparticles. A further study in this direction is now in progress.

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FIGURE CAPTIONS

Fig. 1 - Static solution of GRK. a) Radial coordinate $\bar{x}$ as a function of u. b) Scalar field $\bar{y}$ as a function of u. From this figure the kink property of the solution is evident. c) Metric exponent $\bar{h}$ as a function of u. The value of f is taken as 1.25.

Fig. 2 - Plot of functions $\omega_1(\alpha)$ and $\omega_2(\alpha)$ in α-ω plane for f = 1.25. The discrete eigenvalue is obtained from the intersection of two curves, i.e. $\omega = 0$. For $\omega \geq 2$, eigenvalues are continuum.

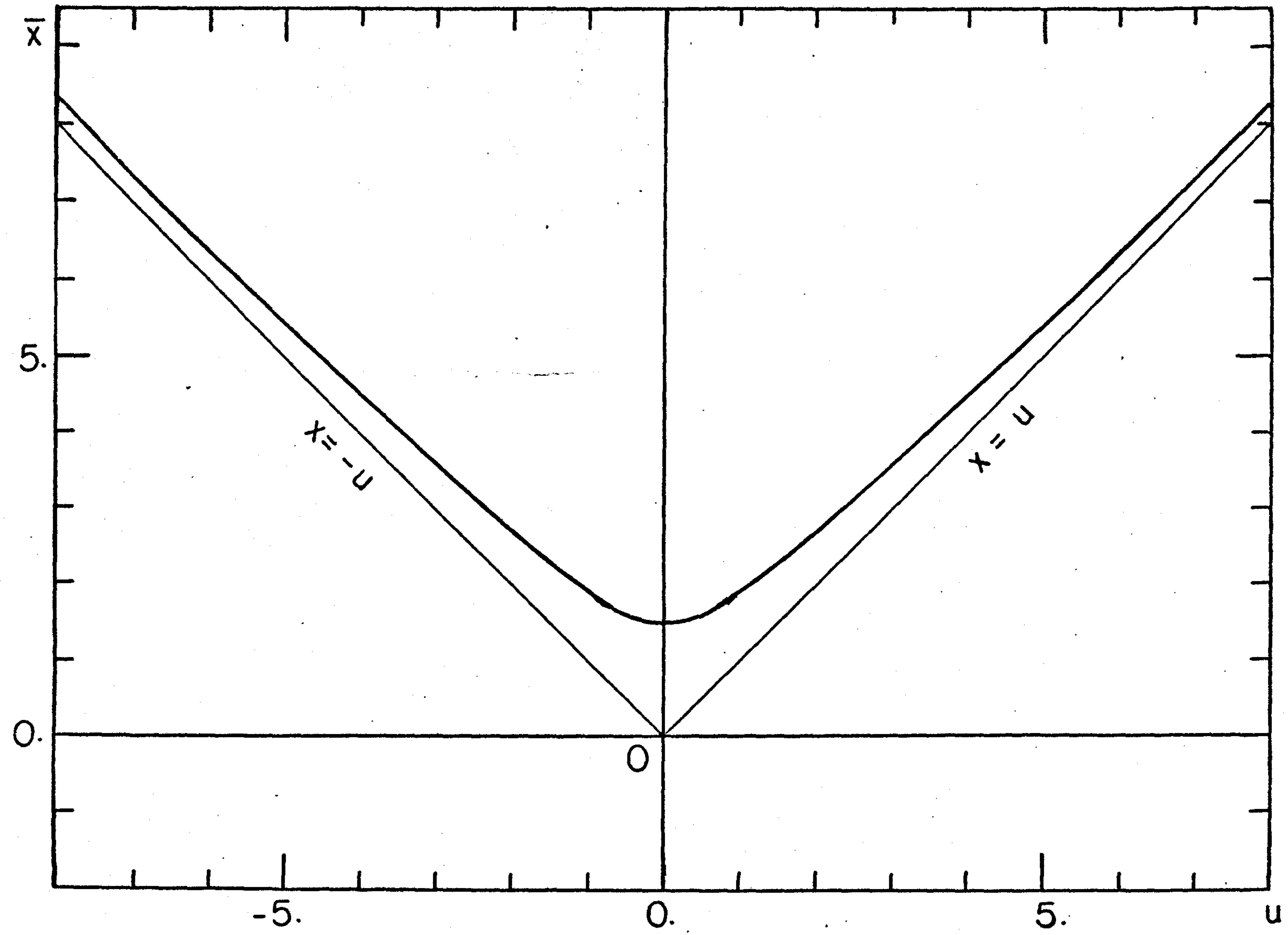
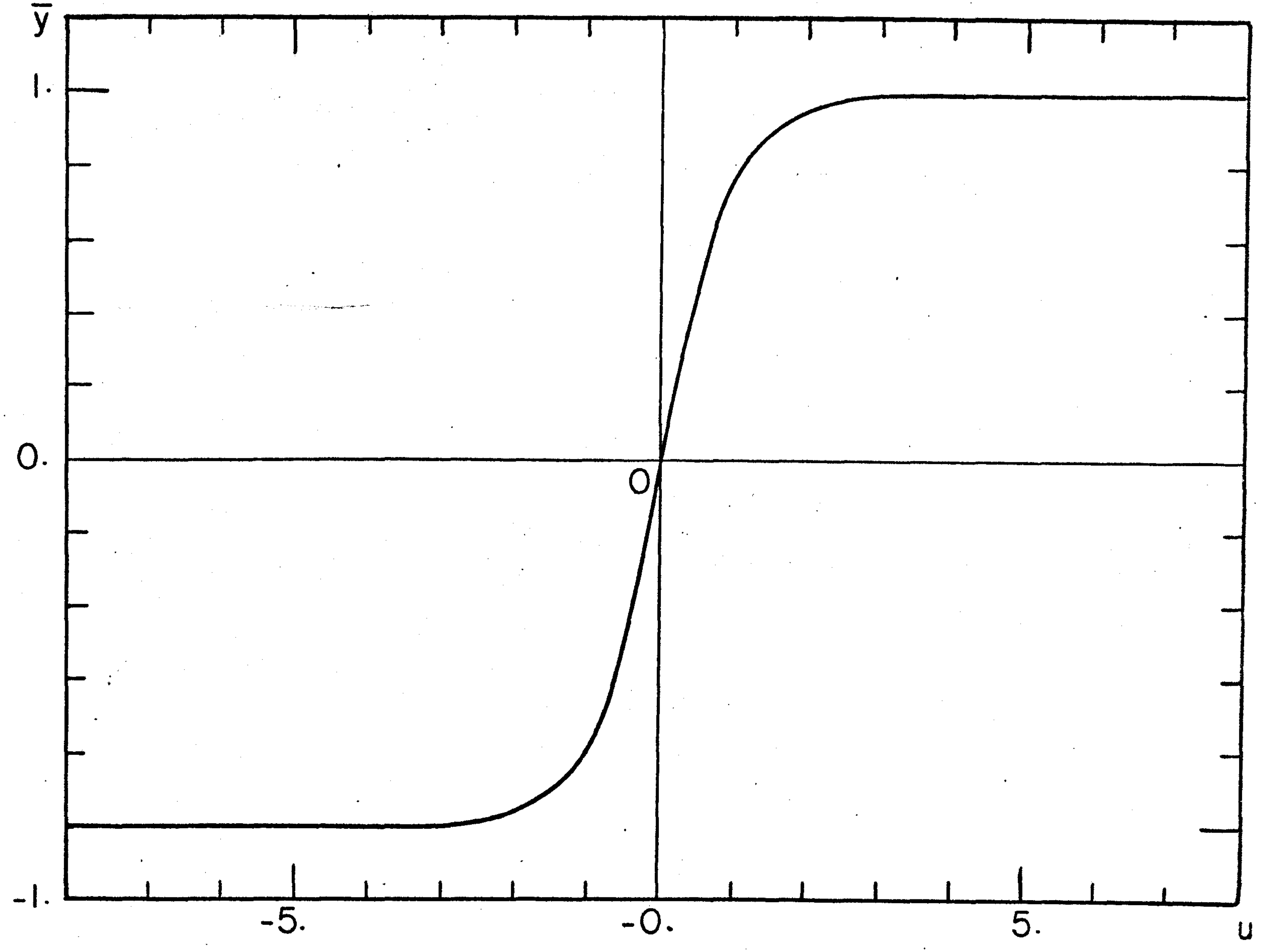


Fig 1-a



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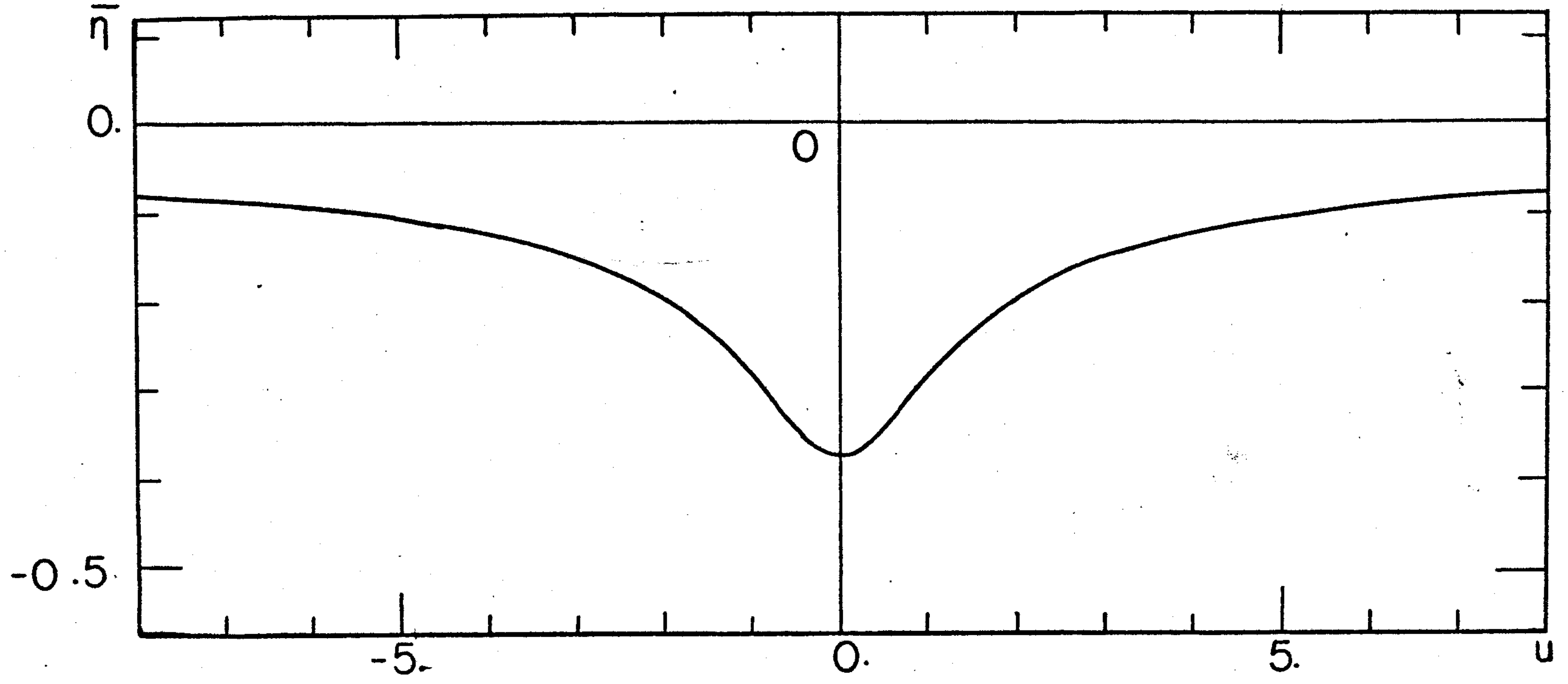

Fig 1-C

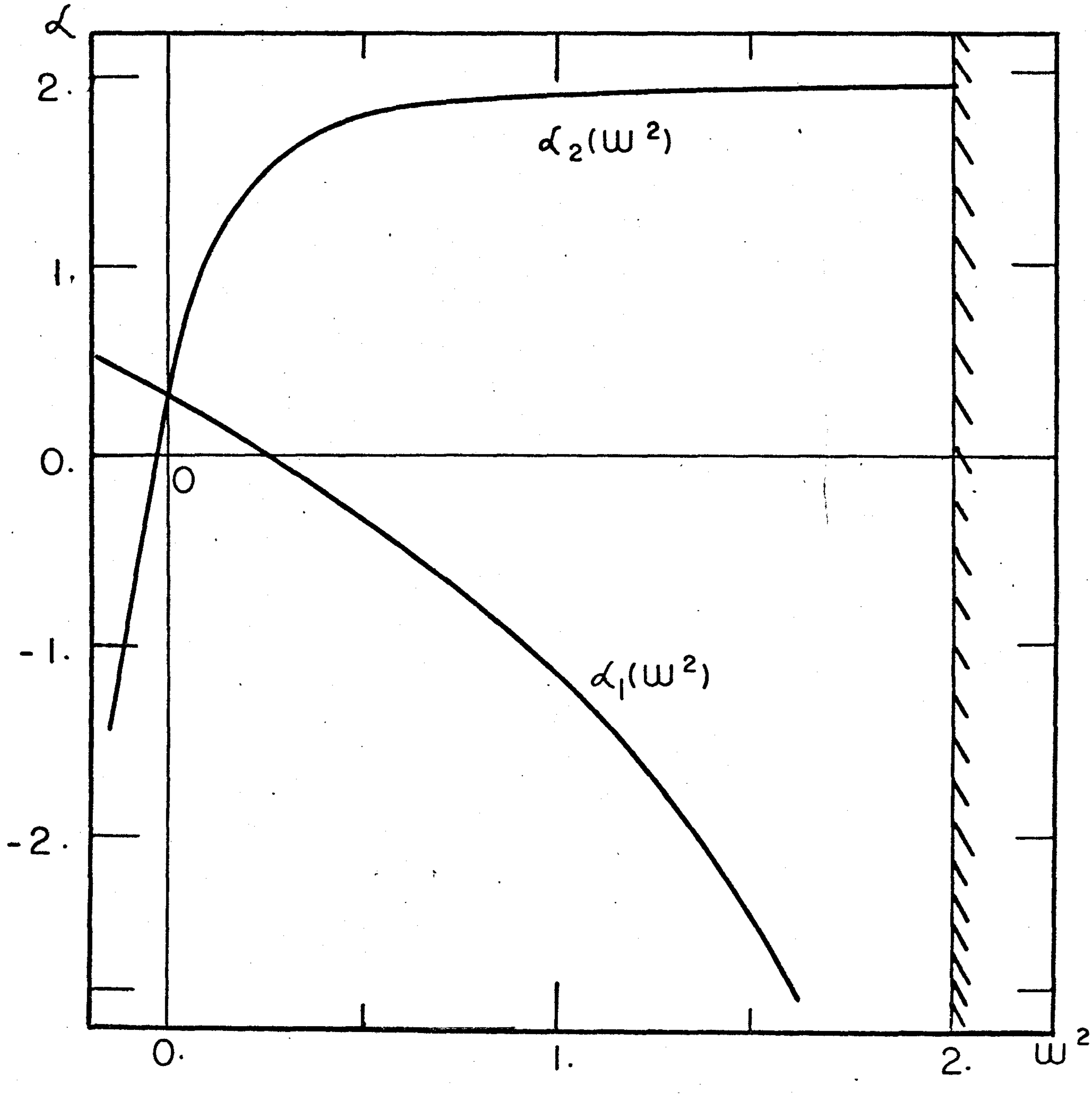

Fig 2

$$W_{\mu\alpha\epsilon\beta} \, W^{\epsilon\tau\lambda\gamma} \, W_{\lambda\gamma\mu\alpha} - \overset{*}{W}_{\mu\alpha\epsilon\tau} \, \overset{*}{W}{}^{\epsilon}{}_{\beta}{}^{\lambda\gamma} \, W_{\lambda\gamma}{}^{\mu\alpha} = 8 \, C \, g_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta} \, W^{\mu\alpha\lambda\gamma} \, W_{\lambda\gamma\epsilon\tau} = 4 \, C \, g_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta} \, \overset{*}{W}{}^{\mu\alpha\lambda\gamma} \, W_{\lambda\gamma\epsilon\tau} = 4 \, D \, g_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta} \, W^{\epsilon\tau\lambda\gamma} \, \overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} + W_{\mu\alpha\epsilon\tau} \, W^{\epsilon}{}_{\beta}{}^{\lambda\gamma} \, \overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} = 8 \, D \, g_{\beta\tau}$$

<u>Table III</u> : Dual identities for the Weyl tensor (triple product)