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SUPERSYMMETRY AND CLASSICAL PARTICLE SPIN DYNAMICS

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SUMMARY:

A pseudoclassical mechanics of particle with any spin is constructed in terms of anticommuting Grassmann variables, in addition to space time variables, in the form of pseudovectors and pseudoscalars. A linear transformation (supersymmetry) group is defined over them and Lagrangian is required to be invariant under it. The singular Lagrangian thus obtained is handled by Dirac's method to construct Hamiltonian dynamics. On its quantization we obtain Bargmann and Wigner formulation for particle with spin. We find, however, that the general supersymmetry is required to be replaced by a lower 'supergauge' symmetry in order to eliminate the redundant variables and some of the Dirac brackets of anticommuting variables must correspond to commutators on quantization.

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I - INTRODUCTION. SUPERSYMMETRY:

In recent years the use of anticommuting c-numbers has drawn renewed interest. The formalism of Grassmann algebra[1)] became more exciting with the possibility of formulating supersymmetry[2)] between bosons and fermions over superspace[3)]. The applications to Hamilton's dynamics of the electron were considered recently[4),5),6),7)] using Dirac's formulation[8),7)] of quantizing constrained systems and supersymmetry as a guiding principle. The pseudomechanics-classical mechanics of systems described by usual c-number variables and by Grassmann variables and possibility of its quantization was developed systematically by Casalbuoni in a series of papers[5),9),10)]. We extend in this paper the above considerations to a relativistic particle with arbitrary spin by using Dirac's[8)] Hamiltonian dynamics for systems with constraints adapted to pseudomechanics. The formulation of Bargmann and Wigner[11)] is obtained on self consistent quantization of pseudomechanics. It may be worth remarking that some of the Dirac brackets of anticommuting variables are required to go over to commutators instead of anticommutators on quantization. Moreover, to realize the program of obtaining the dynamics purely in terms of the variables x_μ, P_μ and spin variables (with no corresponding canonical momenta) the general supersymmetry invariance must be replaced by the requirement of a more restricted supersymmetry invariance.

To formulate the pseudomechanics of a particle of spin s we introduce, following the discussion in refs. (6) and (7), 2s sets of real anticommuting variables (ξ^a_μ , ξ^a_5) $a = 1,\dots, 2s$ where ξ^a_μ are pseudo-vectors and ξ^a_5 are pseudoscalars. They commute with the spacetime variables x_μ. Guided by earlier success of similar nature [3, 12] we define a linear supersymmetry group over the space of these variables by

$$x'_\mu = x_\mu - \frac{i\beta^a}{mc} \varepsilon^a_\mu \xi^a_5 - \frac{i\gamma^a}{mc} \xi^a_\mu \varepsilon^a_5 + b_\mu ,$$

$$\xi'_\mu = \xi^a_\mu + \varepsilon^a_\mu ,$$

$$\xi'_5 = \xi^a_5 + \varepsilon^a_5 , \qquad (1)$$

where summation over repeated indices is understood and the ε's are real anticommuting[1] while β^a, γ^a, b_μ are real commuting constants. This group will be taken as the initial guideline for constructing a Lagrangian dynamics for the particle. These transformations leave the differential forms $d\xi^a_\mu$, $d\xi^a_5$ and

$$\Omega_\mu = d\left(x_\mu + \frac{i\gamma^a}{mc} \xi^a_\mu \xi^a_5\right) + \frac{i(\beta^a - \gamma^a)}{mc} \xi^a_\mu \, d\xi^a_5 \qquad (2)$$

invariant. Anticipating our interest in the quantized theory we may introduce the corresponding generating operators $\hat{G}^a_\mu$, $\hat{G}^a_5$, $\hat{P}_\mu$ by

$$\delta \hat{x}_\mu = - i \left[\varepsilon^a . \ \hat{G}^a + \varepsilon^a_5 \ \hat{G}^a_5 + b . \hat{P} , \hat{x}_\mu \right]_- \tag{3}$$

where the anticommuting parameters ε^a_μ, ε^a_5 will be assumed to anticommute with the generators $\hat{G}^a_\mu$, $\hat{G}^a_5$ and to commute with, say, the generators $\hat{P}_\mu$ corresponding to the commuting variables b_μ . From Jacobi's identity we may determine

$$\left[\hat{G}^a_\mu , \hat{G}^b_\nu \right]_+ = k \, g_{\mu\nu} \, \delta_{ab} ,$$

$$\left[\hat{G}^a_5 , \hat{G}^b_5 \right]_+ = \ell \, \delta_{ab} ,$$

$$\left[\hat{G}^a_\mu , \hat{G}^b_5 \right]_+ = \frac{\beta^a - \gamma^a}{mc} \delta_{ab} \hat{P}_\mu ,$$

$$\left[\hat{G} , \hat{P}_\mu \right]_- = 0 , \tag{4}$$

where k , ℓ are arbitrary constants and $\hat{G}$ stands for $\hat{G}^a_\mu$ or $\hat{G}^a_5$. The algebra of operators is a graded Lie algebra.

A realization of this algebra may be obtained over the phase space corresponding to a classical Lagrangian, which is invariant or quasi-invariant with respect to the supersymmetry transformations of Eqs. (1), through Poisson brackets, since in a quantized theory these go over to commutators or anticommutators of operators.

II - Pseudoclassical Lagrangian:

The points on the trajectory of the particle will be labelled by a monotonic parameter τ which does not change under Poincaré or supersymmetry transformations. We require that the Lagrangian be 'even' and homogeneous of first degree in the derivates with respect to τ and be quasi-invariant under supersymmetry transformations. The most general form is clearly given by

$$L = -i\,\alpha_1^a\,\xi_5^a\,\dot{\xi}_5^a - i\,\alpha_2^a\,\xi_\mu^a\,\dot{\xi}_a^\mu - mc\sqrt{(\dot{x}_\mu + V_\mu)^2} \tag{5}$$

where the dot denotes differentiation with respect to τ, α_1^a, α_2^a are parameters and

$$V_\mu = \frac{i\beta^a}{mc}\,\xi_\mu^a\,\dot{\xi}_5^a + \frac{i\gamma^a}{mc}\,\dot{\xi}_\mu^a\,\xi_5^a\,. \tag{6}$$

Under supersymmetries

$$\delta L = i\,\frac{d}{d\tau}\left(\alpha_1^a\,\varepsilon_5^a\,\xi_5^a + \alpha_2^a\,\varepsilon_\mu^a\,\xi_a^\mu\right), \tag{7}$$

that is, the transformations are canonical leaving the action

$$S = \int_{\tau_i}^{\tau_f} L\,d\tau. \tag{8}$$

invariant. The action is also invariant under the reparametrization $\tau' = \tau'(\tau)$ where τ' is a monotonic function. The infinitesimal generator of the (canonical) supersymmetry and Poincaré transformations[13] is

$$F = -\delta x_\mu \; P^\mu + \delta\,\xi^a_\mu\,\pi^\mu_a + \delta\,\xi^a_5\,\pi^a_5 - i\,(\,\alpha^a_1\,\varepsilon^a_5\,\xi^a_5 \;+ \alpha^a_2\,\varepsilon^a_\mu\,\xi^\mu_a\,)$$

$$\equiv \; b_\mu\, P^\mu \;+\; \varepsilon^a_\mu\, G^\mu_a + \varepsilon^a_5\, G^a_5 \;+\; \frac{1}{2}\; W_{\mu\nu}\, M^{\nu\mu} \tag{9}$$

where P^μ , π^μ_a and π^a_5 are the canonical momenta conjugate to x_μ , ξ^a_μ and ξ^a_5 respectively, $W_{\mu\nu}$ are infinitesimal parameters of Lorentz transformations and $M_{\mu\nu}$ corresponding generators. G^a_μ , G^a_5 are supersymmetry generators. Then

$$G^a_5 = -\,(\,\pi^a_5 \;+\; i\,\alpha^a_1\;\xi^a_5 \;-\; \frac{i\gamma^a}{mc}\;P.\,\xi^a\,)\;,$$

$$G^a_\mu = \;-\;(\,\pi^a_\mu \;+ i\;\alpha^a_2\;\xi^a_\mu \;+\; \frac{i\,\beta^a}{mc}\;P_\mu\,\xi^a_5\,)\;. \tag{10}$$

For the variation of any dynamical variable A under supersymmetry transformations we have $\delta A = -\{F,\,A\}$ where $\{\;\}$ indicates Poisson brackets[14]. By using the non-vanishing standard brackets

$$\{\,x_\mu\;,\;P_\nu\,\} \;=\; -\,g_{\mu\nu}\;,$$

$$\{\,\pi^a_\mu\;,\;\xi^b_\nu\,\} \;=\; -\,g_{\mu\nu}\;\;\delta_{ab}\;,$$

$$\{\,\pi^a_5\,,\,\xi^b_5\,\} \;=\; -\;\;\delta_{ab}\;, \tag{11}$$

we find

$$\{ G^a_5 , G^b_5 \} = - 2 i \alpha_1^{(a)} \delta_{ab} ,$$

$$\{ G^a_\mu , G^b_\nu \} = - 2 i \alpha_2^{(a)} \delta_{ab} g_{\mu\nu} ,$$

$$\{ G^a_\mu , G^b_5 \} = - \frac{i}{mc} (\beta^{(a)} - \gamma^{(a)}) P_\mu \delta_{ab} ,$$

$$\{ G , P_\mu \} = 0 , \tag{12}$$

which is a realization through Poisson brackets of the graded Lie algebra of Eqs.(4).

The Lagrangian above is singular and we will employ Dirac's method[8] to obtain the Hamiltonian dynamics. It is convenient first to simplify the Lagrangian by performing a cononical coordinate transformation suggested by Eq. (2):

$$\bar{x}_\mu = x_\mu + \frac{i\gamma^a}{mc} \xi^a_\mu \xi^a_5 ,$$

$$\bar{\xi}^a_\mu = \xi^a_\mu ,$$

$$\bar{\xi}^a_5 = \xi^a_5 \tag{13}$$

which is generated by[13)]

$$\psi' = - \bar{x}_\mu \bar{P}^\mu + \xi^a_\mu \bar{\pi}^\mu_a + \xi^a_5 \bar{\pi}^a_5 . \qquad (14)$$

It follows that

$$\bar{P}_\mu = P_\mu ,$$

$$\bar{\pi}^a_\mu = \pi^a_\mu + \frac{i\gamma^a}{mc} P_\mu \xi^a_5 ,$$

$$\bar{\pi}^a_5 = \pi^a_5 - \frac{i\gamma^a}{mc} P . \xi^a . \qquad (15)$$

Under supersymmetry $\bar{x}_\mu \longrightarrow \bar{x}_\mu - \frac{i}{mc}(\beta^a - \gamma^a)\varepsilon^a_\mu \xi^a_5$ and we may redefine ξ_5 so that $(\beta^a - \gamma^a) = -1$. We will drop the bar henceforth and work with the Lagrangian

$$L = - i \alpha^a_1 \xi^a_5 \dot{\xi}^a_5 - i \alpha^a_2 \xi^a_\mu \dot{\xi}^\mu_a - mc \sqrt{(\dot{x}_\mu - \frac{i}{mc} \xi^a_\mu \dot{\xi}^a_5)^2} \qquad (16)$$

while the supersymmetry transformation becomes

$$x'_\mu = x_\mu + \frac{i}{mc} \varepsilon^a_\mu \xi^a_5 ,$$

$$\xi^{a\prime}_\mu = \xi^a_\mu + \varepsilon^a_\mu ,$$

$$\xi^{a\prime}_5 = \xi^a_5 + \varepsilon^a_5 . \qquad (17)$$

The canonical momenta are found to be

$$P_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = mc \frac{\dot{x}_\mu + V_\mu}{\sqrt{(\dot{x}_\mu + V_\mu)^2}} ,$$

$$\pi^a_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu_a} = i\, \alpha^{(a)}_2 \, \xi^a_\mu ,$$

$$\pi^a_5 = \frac{\partial L}{\partial \dot{\xi}^a_5} = i\, \alpha^{(a)}_1 \, \xi^a_5 - \frac{i}{mc} P . \xi^a . \tag{18}$$

Here we enclose one of the repeated indices by parentheses to indicate that it is not summed and $V_\mu = -\frac{i}{mc} \xi^a_\mu \dot{\xi}^a_5$. Lagrange's equations are

$$\dot{P}_\mu = 0 ,$$

$$2\, \alpha^{(a)}_2 \, \dot{\xi}^a_\mu = \frac{1}{mc} P_\mu \, \dot{\xi}^a_5 ,$$

$$2\, \alpha^{(a)}_1 \, \dot{\xi}^a_5 = \frac{1}{mc} P . \dot{\xi}^a . \tag{19}$$

For non-vanishing $\dot{\xi}^a_\mu$ and $\dot{\xi}^a_5$ we require that

$$4\, \alpha^{(a)}_1 \, \alpha^a_2 = \frac{P^2}{m^2 c^2} = 1 , \tag{20}$$

the last equality following from Eqs. (18). In what follows we will assume $\alpha^a_1 = \alpha^a_2 = 1/2$. From Eqs. (18) we obtain the following primary constraints

$$\chi^a_\mu \equiv \pi^a_\mu - \frac{i}{2}\,\xi^a_\mu \,\tilde{\simeq}\, 0 \;,$$

$$\chi^a_5 \equiv \pi^a_5 - \frac{i}{2}\,\xi^a_5 + \frac{i}{mc}\,P.\,\xi^a \,\tilde{\simeq}\, 0 \;,$$

$$\chi \equiv P^2 - m^2 c^2 \,\tilde{\simeq}\, 0 \;, \tag{21}$$

where $\tilde{\simeq}$ is to remind us that, when calculating a Poisson bracket, these equalities must be made use of only at the end of the calculation.

For the constraints to be constants of motion we demand that

$(\dot{\chi}^a_\mu \,\tilde{\simeq}\, 0 \,,\; \dot{\chi}^a_5 \,\tilde{\simeq}\, 0)$

$$\dot{\pi}^a_\mu = \frac{i}{2}\,\dot{\xi}^a_\mu \;,$$

$$\dot{\pi}^a_5 = \frac{i}{2}\,\dot{\xi}^a_5 - \frac{i}{mc}\,P.\,\dot{\xi}^a \;, \tag{22}$$

which taken along with Lagrange's equations imply that

$$\chi'^a_5 = \pi^a_5 + \frac{i}{2}\,\xi^a_5$$

is a constant of motion. We will return to it later.

We may calculate formally the Hamiltonian

$$H_o \equiv - \dot{x}_\mu P^\mu + \dot{\xi}^a_\mu \pi^\mu_a + \dot{\xi}^a_5 \pi^a_5 - L =$$

$$\equiv \dot{\xi}^a_\mu \chi^\mu_a + \dot{\xi}^a_5 \chi^a_5 = 0 \qquad (24)$$

where we used Eqs. (18). To construct non-trivial Hamiltonian we use Dirac's procedure.

III.- HAMILTONIAN DYNAMICS:

Following Dirac[8] we calculate now Poisson brackets among our primary constraints:

$$\{ \chi, \chi \} = \{ \chi, \chi^a_5 \} = \{ \chi, \chi^a_\mu \} = 0 ,$$

$$\{ \chi^a_5, \chi^b_5 \} = i \, \delta_{ab} ,$$

$$\{ \chi^a_5, \chi^b_\mu \} = - \frac{i}{mc} P_\mu \, \delta_{ab} ,$$

$$\{ \chi^a_\mu , \chi^b_\nu \} = i \, g_{\mu\nu} \, \delta_{ab} . \qquad (25)$$

Thus χ is first class. By forming linear combinations of the set $\chi_A = (\chi_5^a, \chi_\mu^a)$ we look for any other first class constraints. For more first class constraints to exist

$$\det \left\| \{ \chi_A , \chi_B \} \right\| = \prod_{a=1}^{2s} \left(4 \alpha_1^{(a)} \alpha_2^a - \frac{P^2}{m^2 c^2} \right)$$

must vanish. This is already true in virtue of Eq. (20). Define

$$\chi_D^a = u^{(a)} \chi_5^a + v^{(a)} (P . \chi^a) . \tag{27}$$

We require that $\{ \chi_D^a , \chi_D^b \} \simeq 0$, $\{ \chi_D^a , \chi_5^b \} \simeq 0$ and $\{ \chi_D^a , \chi_\mu^b \} \simeq 0$ which gives $v^a = \frac{u^a}{mc}$. Thus

$$\chi_D^a \equiv \chi_5 + \frac{1}{mc} P . \chi^a \simeq 0 \tag{28}$$

are first class.

We will thus take χ, χ_D^a and χ_μ^a as a more convenient set of primary constraints and define the total Hamiltonian ($H_o = 0$)

$$H_T = u \chi + u_D^a \chi_D^a + u_a^\mu \chi_\mu^a , \tag{29}$$

where u is even and u_D^a, u_a^μ are odd to make H_T an even Grassmann function. The evolution of any dynamical variable is given by

$$\dot{A} = \{ A , H_T \} \tag{30}$$

and consistency conditions are obtained by requiring that $\dot{\chi} \approx 0$, $\dot{\chi}_D^a \approx 0$ and $\dot{\chi}_\mu^a \approx 0$. We find

$$\dot{\chi}_\mu^a \approx - u_b^\nu \{ \chi_\mu^a , \chi_\nu^b \} = - i u_\mu^a \approx 0 . \tag{31}$$

No secondary constraints arise and the total Hamiltonian is first class of the form

$$H_T = \rho_1 \chi + \rho_2^a \chi_D^a . \tag{32}$$

The constraints $\chi_\mu^a \approx 0$ are second class. They can be turned into strong relations $\chi_\mu^a = 0$, thus removing π_μ^a, if we use Dirac brackets[8] with respect to these constraints in place of Poisson brackets. Dirac's brackets are given by

$$\{ A, B \}^* = \{ A, B \} - \{ A, \chi_a^\mu \} C_{\mu\nu}^{ab} \{ \chi_b^\nu , B \} \tag{33}$$

where $(C_{\mu\nu}^{ab})$ is the inverse of the matrix $|| \{ \chi_a^\mu, \chi_b^\nu \} ||$, that is, $C_{\mu\nu}^{ab} = - i \delta_{ab} g_{\mu\nu}$. Hence

$$\{ A, B \}^* = \{ A, B \} + i \{ A, \chi_\mu^a \} \{ \chi_a^\mu, B \} . \tag{34}$$

The equations of motion are obtained from

$$\dot{A} = \{ A, H_T \}^* \simeq \{ A, H_T \} \tag{35}$$

and

$$\pi^a_\mu = \frac{i}{2} \xi^a_\mu . \tag{36}$$

Hamilton's equations are

$$\dot{x}_\mu \simeq -2 \rho_1 P_\mu - \frac{i}{mc} \rho^a_2 \xi^a_\mu ,$$

$$\dot{\xi}^a_\mu \simeq \frac{\rho^a_2}{mc} P_\mu ,$$

$$\dot{\xi}_5 \simeq \rho^a_2 ,$$

$$\dot{\Pi}^a_5 \simeq -\frac{i}{2} \rho^a_2 ,$$

$$\dot{P}_\mu \simeq 0 . \tag{37}$$

We observe that they contain Eq. (19). The τ- evolution mixes coordinate and spin degrees of freedom; the phase space of particle with spin is a 'super-space'. The second term in equation for $\dot{x}_\mu$ represents the classical analogue of 'zitterbewegung'. Considering spinless particle and a definite τ- gauge, say, proper-time or laboratory time gauge, we may fix ρ_1.

We proposed to built a pseudodynamics involving no π_μ^a and π_5^a . Supersymmetry invariance led us to remove π_μ^a but not π_5^a , We thus look for some other constraints which are of second class while leaving the surviving constraints χ , χ_D^a first class. We may then go over to new Dirac brackets using their iterative property[15) . A candidate is the constant of motion $\chi_5^{a'} \equiv \pi_5^a + \frac{i}{2} \xi_5^a$ and we may impose the constraints

$$\chi_5^{a'} \equiv \pi_5^a + \frac{i}{2} \xi_5^a \simeq 0 . \tag{38}$$

From

$$\begin{gathered} \{ \chi_5^{a'} , \chi_5^{b'} \} = - i \delta_{ab} , \\ \{ \chi_D^a , \chi_5^{b'} \} = \{ \chi_5^{a'} , \chi \} = \{ \chi_5^{b'} , \chi_\mu^a \} = 0 , \\ \{ \chi_5^b , \chi_5^{a'} \} = 0 , \end{gathered} \tag{39}$$

we see that they have the desired properties. Defining

$$\{ A, B \}^{**} = \{ A , B \}^* - i \{ A, \chi_5^{a'} \}^* \{ \chi_5^{a'} , B \}^* \tag{40}$$

we can use $\chi_5^{a'} = 0$ as strong relations if we use these new Dirac brackets.

Clearly

$$\{ A, B \}^{**} = \{ A, B \} + i \{ A, \chi^a_\mu \} \{ \chi^\mu_a , B \} - i\{ A, \chi^{a'}_5 \} \{\chi^{a'}_5 ,B\} \quad (41)$$

and

$$\dot{A} = \{ A, H_T \}^{**} \simeq \{ A, H_T \}^* \simeq \{ A, H_T \} . \quad (42)$$

We remark, however, that up to the level of Eqs. (34) we did maintain supersymmetry invariance but the imposition of Eqs. (38) may have destroyed it. We discuss the covariance of the theory below.

The new Dirac brackets of the surviving variables X_μ, P_μ, ξ^a_μ, ξ^a_5 can now be calculated. The non-vanishing ones are

$$\{ x_\mu , P_\nu \}^{**} = \{ x_\mu , P_\nu \} = - g_{\mu\nu} ,$$

$$\{ \xi^a_5 , \xi^b_5 \}^{**} = - i \{ \xi^a_5 , \pi^c_5 \} \{ \pi^c_5 , \xi^b_5 \} = - i \delta_{ab} ,$$

$$\{ \xi^a_\mu , \xi^b_5 \}^{**} = 0 ,$$

$$\{ \xi^a_\mu , \xi^b_\nu \}^{**} = i g_{\mu\nu} \delta_{ab} . \quad (43)$$

In addition we have

$$\chi_D^a = \chi_5^a = + (i/mc) (P . \xi^a - mc\, \xi_5^a) \approx 0 ,$$

$$\chi = P^2 - m^2 c^2 \approx 0 ,$$

$$\{ \chi_D^a , \chi_D^b \}^{**} \approx 0 ,$$

$$\{ \chi , \chi_D^a \}^{**} \approx 0 . \qquad (44)$$

IV - SELF-CONSISTENT QUANTIZATION:

This system may now be quantized by the correspondence

$$i \hbar \{ A , B \}^{**} \longleftrightarrow [\hat{A}, \hat{B}]_- \text{ or } [\hat{A} , \hat{B}]_+ \qquad (45)$$

and the requirement that the leftover first class constraints become now conditions on the states[8] (c = 1):

$$\hat{\chi}\, \psi \equiv (\hat{P}^2 - m^2)\, \psi = 0 , \qquad (46)$$

$$\hat{\chi}_D^a\, \psi \equiv (\hat{P} . \hat{\xi}^a - m\, \hat{\xi}_5^a)\, \psi = 0 . \qquad (47)$$

We will choose in Eq. (45) the commutator or anticommutator by requiring that Eqs. (45) - (47) be self-consistent instead of the prescription given in reference 9. (In any case, after quantization the graded Lie algebraic structure is not carried over for corresponding operators.)

We write

$$[\hat{\xi}^a_\mu , \hat{\xi}^b_\nu]_\pm = - \hbar \, g_{\mu\nu} \, \delta_{ab} ,$$

$$[\hat{\xi}^a_5 , \hat{\xi}^b_5]_\pm = \hbar \, \delta_{ab} ,$$

$$[\hat{\xi}^a_\mu , \hat{\xi}^b_5]_\pm = 0 . \qquad (48)$$

and for obvious reasons

$$[\hat{\xi}^a_\nu , \hat{P}_\mu]_- = [\hat{\xi}^a_5 , \hat{P}_\mu]_- = 0 ,$$

$$[\hat{x}_\mu , \hat{P}_\nu]_- = - i \hbar \, g_{\mu\nu} . \qquad (49)$$

The sign in the first three cases will be determined by self-consistency requirements. We must require also

$$[\hat{\chi} , \hat{\chi}^a_D]_\pm \, \psi = 0 ,$$

$$[\hat{X}_D^a , \hat{X}_D^b]_\pm \psi = \{ [\hat{\xi}_\mu^b , \hat{\xi}_\nu^a]_\pm \hat{P}^\mu \hat{P}^\nu - m [\hat{\xi}_\mu^b , \hat{\xi}_5^a]_\pm \hat{P}^\mu -$$

$$- m [\hat{\xi}_5^b , \hat{\xi}_\mu^a]_\pm \hat{P}^\mu + m^2 [\hat{\xi}_5^b , \hat{\xi}_5^a]_\pm \} \psi = 0 . \qquad (50)$$

From Eq. (47) we derive

$$m \hat{P}^\mu (\hat{\xi}_\mu^b \hat{\xi}_5^a \pm \hat{\xi}_\mu^a \hat{\xi}_5^b) \psi = [\hat{\xi}_\mu^b , \hat{\xi}_\nu^a]_\pm \hat{P}^\mu \hat{P}^\nu \psi ,$$

$$\hat{P}^\mu (\hat{\xi}_5^b \hat{\xi}_\mu^a \pm \hat{\xi}_5^a \hat{\xi}_\mu^b) \psi = m [\hat{\xi}_5^b , \hat{\xi}_5^a]_\pm \psi . \qquad (51)$$

Consider the case $a = b$. From (48) it follows that we must require (dropping the superscripts):

$$[\hat{\xi}_\mu , \hat{\xi}_\nu]_+ = - \hbar g_{\mu\nu} ,$$

$$[\hat{\xi}_5 , \hat{\xi}_5]_+ = \hbar . \qquad (52)$$

From (51) we obtain

$$2 \hat{P}^\mu \hat{\xi}_\mu \hat{\xi}_5 \psi \doteq - \hbar m \psi ,$$

$$2\,\hat{P}^{\mu}\;\hat{\xi}_5\,\hat{\xi}_{\mu}\,\psi = \hbar\, m\,\psi\;, \tag{53}$$

leading to

$$[\,\hat{\xi}_{\mu}\,,\,\hat{\xi}_5\,]_+ = 0\;. \tag{54}$$

Therefore $\hat{\xi}^a_{\mu}\,,\,\hat{\xi}^a_5$ generate a Clifford algebra C_5 . We obtain the usual Dirac-Pauli algebra for the γ' s on writing

$$\xi^a_{\mu} = \sqrt{\frac{\hbar}{2}}\;\gamma^{(a)}_5\;\gamma^a_{\mu}\;,\quad \xi^a_5 = \sqrt{\frac{\hbar}{2}}\;\gamma^a_5 \tag{55}$$

and deduce $\gamma_5 = i\,\gamma^{(a)}_0\,\gamma^a_1\,\gamma^a_2\,\gamma^a_3$. Eqs. (47) give

$$(\,\gamma^a\,.\,\hat{P} - m\,)\,\psi = 0\;,\quad a = 1, \ldots, 2s. \tag{56}$$

For these equations to be satisfied simultaneously we require for $a \neq b$ that

$$[\,\gamma^a_{\mu}\,,\;\gamma^b_{\nu}\,]_- = 0 \tag{57}$$

and consequently

$$[\,\gamma^a_5\;,\,\gamma^b_5\,]_- = 0 \tag{58}$$

even though the ξ^a_s are anticommuting numbers. The consistency requirements $[\hat{\chi}, \hat{\chi}^a_D]_\pm \psi = 0$ and $[\hat{\chi}^a_D, \chi^b_D]_\pm = 0$ do not lead to any new conditions and are satisfied identically. Thus we obtain the formulation of Bargmann and Wigner[9] for a particle with spin s.

V - POINCARÉ AND SUPERSYMMETRY INVARIANCE:

From Eq. (9) we identify

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{59}$$

where

$$L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \tag{60}$$

and

$$S_{\mu\nu} = \pi^a_\nu \xi^a_\mu - \pi^a_\mu \xi^a_\nu \tag{61}$$

which goes to

$$S_{\mu\nu} = -\frac{i}{2} (\xi^a_\mu \xi^a_\nu - \xi^a_\nu \xi^a_\mu) \tag{62}$$

on using the constraints. By making use of Eq. (43) we obtain the following non-vanishing brackets:

$$\{ L_{\rho\lambda} , L_{\mu\nu} \}^{**} = \{ L_{\rho\lambda} , L_{\mu\nu} \}^{*} = \{ L_{\rho\lambda} , L_{\mu\nu} \} =$$

$$= g_{\lambda\mu} L_{\rho\nu} - g_{\rho\mu} L_{\lambda\nu} + g_{\rho\nu} L_{\lambda\mu} - g_{\lambda\nu} L_{\rho\mu} ,$$

$$\{ S_{\rho\lambda} , S_{\mu\nu} \}^{**} = \{ S_{\rho\lambda} , S_{\mu\nu} \}^{*} = \{ S_{\rho\lambda} , S_{\mu\nu} \} =$$

$$= g_{\lambda\mu} S_{\rho\nu} - g_{\rho\mu} S_{\lambda\nu} + g_{\rho\nu} S_{\lambda\mu} - g_{\lambda\nu} S_{\rho\mu} ,$$

$$\{ M_{\mu\nu} , x_{\lambda} \}^{**} = \{ M_{\mu\nu} , x_{\lambda} \}^{*} = \{ M_{\mu\nu} , x_{\lambda} \} =$$

$$= x_{\mu} g_{\nu\lambda} - x_{\nu} g_{\mu\lambda} ,$$

$$\{ M_{\mu\nu} , P_{\lambda} \}^{**} = \{ M_{\mu\nu} , P_{\lambda} \}^{*} = \{ M_{\mu\nu} , P_{\lambda} \} =$$

$$= P_{\mu} g_{\nu\lambda} - P_{\nu} g_{\mu\lambda} ,$$

$$\{ M_{\mu\nu} , \xi^{a}_{\lambda} \}^{**} = \{ M_{\mu\nu} , \xi^{a}_{\lambda} \}^{*} = \xi^{a}_{\mu} g_{\nu\lambda} - \xi^{a}_{\nu} g_{\mu\lambda} ,$$

$$\{ M_{\mu\nu} , G^{a}_{\lambda} \}^{**} = \{ M_{\mu\nu} , G^{a}_{\lambda} \}^{*} = G^{a}_{\mu} g_{\nu\lambda} - G^{a}_{\nu} g_{\mu\lambda} ,$$

$$\{ G^a_\mu , G^b_\nu \}^{**} = - i (g_{\mu\nu} - \frac{P_\mu P_\nu}{m^2 c^2}) \delta_{ab} ,$$

$$\{ G^a_\mu , G^b_\nu \}^{*} = - i g_{\mu\nu} \delta_{ab} ,$$

$$\{ G_\mu{}^a , G^b_5 \}^{*} = \frac{i}{mc} P_\mu \delta_{ab} ,$$

$$\{ G^a_5 , G^b_5 \}^{*} = - i \delta_{ab} ,$$

$$\{ G^a_\mu , \chi^b_D \}^{**} = \{ G^a_\mu , \chi\}^{**} = 0 . \qquad (63)$$

At the level of $\{ \ \}^{**}$ the generators $G^a_5 = 0$ in virtue of the constraints $\chi^{a'}_5 = 0$, showing that $\chi^{a'}_5 = 0$ breaks the general (canonical) supersymmetry invariance, Eqs. (17) , we started with. The surviving supersymmetry invariance is that generated by $F = \eta^a (\tau) \frac{G^a . P}{mc}$ where we put $\varepsilon^a_\mu \equiv \eta^a (\tau) \frac{P_\mu}{mc}$. The corresponding transformations are given by

$$x'_\mu = x_\mu + i \frac{P_\mu}{(mc)^2} \eta^a (\tau) \xi^a_5 ,$$

$$\xi_\mu^{a'} = \xi_\mu^a + \frac{P_\mu}{mc} \eta^a (\tau) ,$$

$$\xi_5^{a'} = \xi_5^a + \eta^a (\tau) . \quad (64)$$

That it is possible to allow the pseudoscalar parameters η^a to be τ - dependent may be seen from Eqs. (19) . Quasi-invariance of the Lagrangian may also be verified. At the level of brackets $\{ \}^*$ the theory is covariant with respect to general supersymmetry as well as under the transformations just discussed. In fact the constraints χ , χ_D^a , χ_μ^a are invariant under general supersymmetry. However,

$$\delta \chi_5^{a'} = - \{ F , \chi_5^{a'} \} = - i (\varepsilon_5^a - \frac{\varepsilon^a . P}{mc}) \quad (65)$$

vanishes only if $\varepsilon_\mu^a = \varepsilon_5^a \frac{P_\mu}{mc}$. Thus the corresponding generator

$$G^a \equiv G_5^a + \frac{G^a . P}{mc} \quad (66)$$

leaves invariant all the constraints and equations of motion.

VI - GAUGE CONSTRAINT $\tau = x^0$:

We mentioned in Sec. II that our action is invariant under reparametrization of the invariant parameter τ . We may study another gauge by imposing the constraint

$$\chi_0 \equiv x_0(\tau) - \tau \approx 0 \quad . \tag{67}$$

In this gauge the Hamiltonian generates evolution in time coordinate x^o. The other remaining constraints are χ , χ^a_D . We have for the non-vanishing brackets

$$\{ \chi , \chi_0 \}^{**} \approx 2 P_0 \quad ,$$

$$\{ \chi^a_D , \chi_0 \}^{**} \approx \xi^a_0 \quad ,$$

$$\{ \chi^a_D , \chi^b_D \}^{**} = i \chi \delta_{ab} \approx 0 \quad , \tag{68}$$

that is, χ , χ_0 , χ^a_D are all second class. Let us try to build a first class $\chi^a_E \equiv \mu \chi + \mu_0 \chi_0 + \mu^{(a)}_D \cdot \chi^a_D$ out of them. We require that

$$2 \mu P_0 + \mu^{(a)}_D \xi^a_0 \approx 0 \quad ,$$

$$- 2 \mu_0 P_0 \approx 0 \quad ,$$

$$- \mu_0 \xi^a_0 + i \chi \mu^a_D \approx \mu_0 \xi^a_0 \approx 0 \tag{69}$$

implying $\mu_0 = 0$. A solution is $\mu = 0$, $\mu_D^a = \zeta_0^a$; the corresponding first class constraints are $\xi_0^{(a)} \chi_D^a$. In fact

$$\{ \xi_0^{(a)} \chi_D^a , \chi_0 \}^{**} = (\xi_0^a)^2 = 0 ,$$

$$\{ \xi_0^{(a)} \chi_D^a , \xi_0^{(b)} \chi_D^b \}^{**} = - i (\chi_D^{(a)})^2 \delta_{ab} \approx 0 ,$$

$$\{ \xi_0^{(a)} \chi_D^a , \chi \}^{**} = 0 . \tag{70}$$

We define then new Dirac brackets relative to the second class constraints χ and χ_0 , viz,

$$\{ A , B \}^{***} = \{ A , B \}^{**} + \frac{1}{2P_0} \left[\{A,P^2\}^{**} \{x_0,B\}^{**} - \{A, x_0\}^{**} \{P^2, B\}^{**} \right] \tag{71}$$

and we may set inside them

$$x_0 = \tau ,$$

$$P^2 = m^2 c^2 . \tag{72}$$

Also , for any dynamical variable A, it is clear

$$\{ A, P^2 \}^{***} = \{ A, x_0 \}^{***} = 0 .$$

We verify that the Poincaré algebra of P_μ, $M_{\mu\nu}$ is preserved as well as that of supersymmetry, Eq. (64),. Total Hamiltonian now is

$$H_T \doteq P_0 + \lambda^a \xi_0^a \chi_D^a \tag{73}$$

where we must add P_0 so that when all the constraints are satisfied the Hamiltonian reduces to the operator for evolution in x_o [17]. The equations of motion are given by

$$\dot{A} \equiv \frac{dA}{dx^o} = \frac{\partial A}{\partial x^o} + \{ A , H_T \}^{***} . \tag{74}$$

We find

$$\dot{x}^o = 1 + \{ x^o , H_T \}^{***} = 1 ,$$

$$\dot{P}_\mu \doteq 0 ,$$

$$\dot{\vec{x}} = \frac{\vec{P}}{P_0} - \lambda^a \xi_0^a \vec{\xi}^a ,$$

$$\dot{\xi}_0^a = - i \lambda^{(a)} (\vec{P} . \vec{\xi}^{\,a} + m c \xi_5^a) \approx - i \lambda^{(a)} P_0 \xi_0^a ,$$

$$\dot{\vec{\xi}}^a = - i \lambda^{(a)} \vec{P} \xi_0^a ,$$

$$\dot{\xi}_5^a = - i m c \lambda^{(a)} \xi_0^a ,$$

$$\dot{M}_{oi} = \lambda^a \left[P. \xi^a - m c \xi_5^a \right] \xi_i^a \approx 0 ,$$

$$\dot{M}_{ij} = 0 . \tag{75}$$

The fourth and fifth equations in Eq. (75) may be integrated to obtain ($x^o = t$)

$$\xi_0^a (t) = \xi_0^{(a)} (0) \, e^{- i \lambda^a P_0 t} , \tag{76}$$

$$\vec{\xi}^{\,a} (t) = \vec{\xi}^{\,a} (0) + \frac{\vec{P}}{P_0} \xi_0^{(a)} (0) \, (\bar{e}^{\, i \lambda^a P_0 t} - 1) \tag{77}$$

where $\xi_0^a (0)$, $\vec{\xi}^{\,a}(0)$ are the values of the corresponding variables at $t = 0$. It follows then

$$\dot{\vec{x}} (t) = \frac{\vec{P}}{P_o} - \lambda^a \xi_o^a (0) \, \vec{\xi}^{\,a} (0) \, \bar{e}^{i \lambda^a P_o t} \tag{78}$$

and its solution is given by

$$\vec{x} (t) = \vec{x} (0) + \frac{\vec{P}}{P_0} t - \frac{i}{P_0} \xi_0^a (0) \, \vec{\xi}^{\,a} (0) \, (\bar{e}^{\, i \lambda^a P_0 t} - 1) . \tag{79}$$

The particle thus moves with the classical velocity $\vec{P} / P_o$ but superimposed on to it is a pseudoclassical analogous, with real exponentials, of the quantum mechanical 'zitterbewegung'[16)]. On quantization for $\hat{H}_T$ to be hermitian, we must redefine $\lambda^{a'}$ s to be real. In addition we must ensure that $\hat{H}_T = \hat{P}_o$ when all the constraints are satisfied. We find $\hat{H}_T = \frac{1}{2s} \sum_a (\vec{P} . \vec{\alpha}^a + m c \beta^a)$.

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14. $\{ A, B \} = \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i} \frac{\partial B}{\partial q_i} \right) \pm \left(\frac{\partial A}{\partial \theta\alpha} \frac{\partial B}{\partial \pi^\alpha} + \frac{\partial A}{\partial \pi^\alpha} \frac{\partial B}{\partial \theta_\alpha} \right)$

where + or - sign is used in the second term according as A is even or odd See refs. 9, 10 for their algebraic properties.

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17. In Eq. (73) the constants λ^a are pure imaginary to ensure that H_T be real.