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Recently, the use of anticommuting c-numbers in describing physical systems and their symmetries has drawn much interest. Supersymmetry among bosons and fermions[1] can be given an elegant formulation[2] using them. Applications to Hamiltonian dynamics of electron[3,4,5,6] adapting Dirac's method[7] of handling singular Lagrangians were quite successful. We discuss in this paper an extension to particle of any spin following the systematic treatment[8] of Casalbuoni et.al[6]. Formulation of Bargmann and Wigner[9] for relativistic particle is obtained on quantization in self-consistent manner. It may be remarked that some of the Dirac brackets between anticommuting variables are required to go to commutators instead of anticommutators.

To formulate pseudomechanics of particles of spin 's' we introduce[5,6] 2s sets of real anticommuting variables (ξ_μ^a , ξ_5^a), a = 1, 2,...2s, where ξ_μ^a are pseudovectors and ξ_5^a are pseudoscalars. They commute with the space-time variables x_μ. Guided by earlier success[2,10] of similar nature we require that our Langrangian be (quasi-) invariant under a linear supersymmetry group of transformations

$$x'_\mu = x_\mu - \frac{i}{mc} \beta^a \varepsilon^a_\mu \xi^a_5 - \frac{i}{mc} \gamma^a \xi^a_\mu \varepsilon^a_5 + b_\mu$$

$$\xi^{a\prime}_\mu = \xi^a_\mu + \varepsilon^a_\mu$$

$$\xi^{a\prime}_5 = \xi^a_5 + \varepsilon^a_5 \qquad (1)$$

where summation over repeated indices is understood (we enclose one of the indices by a parenthesis when it is not to be summed) and ε's are anticommuting real while β, γ, b are real commuting constants. The transformations above leave the differential forms $d\,\xi^a_5$, $d\,\xi^a_\mu$ and

$$d\,(x_\mu + \frac{i\,\gamma^a}{mc}\,\xi^a_\mu\,\xi^a_5) + i\,\frac{(\beta^a - \gamma^a)}{mc}\,\xi^a_\mu\,d\,\xi^a_5 \qquad (2)$$

invariant. The geometry is flat in space-time but curved in other sectors.

The particle trajectory will be labelled by an invariant monotonic parameter τ. We also require that our Langrangian is an 'even' Grassmann function[11] and is homogeneous of first degree in derivatives w.r.t. τ, indicated in the following by a dot. The form of eq. (2) then suggests that it is convenient to perform a canonical coordinate transformation[8, 12] in x_μ and redefine ξ^a_5 so that the most general Lagrangian is given by

$$L = -\,i\,\alpha^a_1\,\xi^a_5\,\dot{\xi}^a_5 - i\,\alpha^a_2\,\xi^a_\mu\,\dot{\xi}^\mu_a - mc\,\sqrt{(\dot{x}_\mu + v_\mu)^2} \qquad (3)$$

where $V_\mu = -\frac{i}{mc}\,\xi^a_\mu\,\dot{\xi}^a_5$ and $x_\mu \longrightarrow x_\mu + \frac{i}{mc}\,\varepsilon^a_\mu\,\xi^a_5 + b_\mu$ under supersymmetry. The variation under it is given by

$$\delta L = i\,\frac{d}{d\tau}\,(\alpha^a_1\,\varepsilon^a_6\,\xi^a_5 + \alpha^a_2\,\varepsilon^a_\mu\,\xi^\mu_a) \qquad (4)$$

The canonical momenta are found to be

$$P_\mu = - \frac{\partial L}{\partial \dot{x}^\mu} = mc \frac{(\dot{x}_\mu + v_\mu)}{\sqrt{(\dot{x}_\mu + v_\mu)^2}}$$

$$\pi^a_\mu = \frac{\partial L}{\partial \dot{\xi}^a_\mu} = i\, \alpha^{(a)}_2 \quad \xi^a_\mu$$

$$\pi^a_5 = \frac{\partial L}{\partial \dot{\xi}^a_5} = i\, \alpha^{(a)}_1 \xi^a_5 - \frac{i}{mc} (P.\, \xi^a) \tag{5}$$

and Lagrange equations are

$$\dot{P}_\mu = 0$$

$$2\alpha^{(a)}_2 \quad \dot{\xi}^a_\mu = \frac{1}{mc} \; P_\mu \; \dot{\xi}^a_5$$

$$2\alpha^{(a)}_1 \quad \dot{\xi}^a_5 = \frac{1}{mc} \; (P.\, \dot{\xi}^a) \tag{6}$$

which give for nonvanishing velocities

$$4\alpha^{(a)}_1 \quad \alpha^a_2 = \frac{P^2}{(mc)^2} = 1 \tag{7}$$

We choose $\alpha^a_1 = \alpha^a_2 = 1/2$.

The Hamiltonian $H_o = - \dot{x}.P + \dot{\xi}^a.\pi^a + \dot{\xi}^a_5 \pi^a_5 - L$, is found to vanish. Eq. (5) shows that our Lagrangian is singular and dynamics in phase space is constrained. We use Dirac's[7] method to remove redundant variables so that the particle is described solely in terms of coordinates x_μ, ξ^a_μ, ξ^a_5 and momenta P_μ. We will see below that to achieve this goal we have to relax the invariance w.r.t the general supersymmetry but can still retain the invariance under a gauge supersymmetry (eq. (26)).

The primary constraints following from eq. (5) are

$$\chi^a_\mu \equiv (\pi^a_\mu - \frac{i}{2}\ \xi^a_\mu) \simeq o$$

$$\chi^a_5 \equiv (\pi^a_5 - \frac{i}{2}\ \xi^a_5 + \frac{i}{mc}\ P.\ \xi^a) \simeq o$$

$$\chi \equiv (P^2 - m^2\ c^2) \simeq o \tag{8}$$

Requiring $\dot{\chi}^a_\mu \approx o$ and $\dot{\chi}^a_5 \approx o$ and combining with eq. (6) we find

$$\dot{\chi}'^a_5 \equiv (\dot{\pi}^a_5 + \frac{i}{2}\ \dot{\xi}^a_5) \approx o \tag{9}$$

showing χ'^a_5 is a constant of motion. If we impose additional constraint $\chi'^a_5 \approx 0$ we are able to remove both π^a_5 and π^a_μ in virtue of eq. (8).

The nonvanishing Poisson brackets (8) among the constraints are found to be

$$\{ \chi^a_5, \chi^b_5 \} = - \{ \chi'^a_5, \chi'^b_5 \} = i\ \delta_{ab}$$

$$\{ \chi^a_\mu, \chi^b_\nu \} = i\ g_{\mu\nu}\ \delta_{ab}$$

$$\{ \chi^a_5, \chi^b_\mu \} = - \frac{i}{mc}\ P_\mu\ \delta_{ab} \tag{10}$$

We find χ is first class. Another first class constraint is given by the linear combination

$$\chi^a_D \equiv \chi^a_5 + \frac{1}{mc} (P.\ \chi^a) \approx 0 \tag{11}$$

We will use χ , χ^a_D , χ^a_μ , χ'^a_5 as a convenient set of constraints, the last two being second class, on phase space. To turn over the second class constraints into strong relations we define Dirac bracket[7] w.r.t. the constraints χ'^a_5 and χ^a_μ ,

$$\{ A, B \}^* = \{ A, B \} + i \{ A, \chi^a_\mu \} \{ \chi^\mu_a, B \} - i \{ A, \chi'^a_5 \} \{ \chi'^a_5, B\} \tag{12}$$

We find for the surviving variables

$$\{ \xi^a_\mu , \xi^b_\nu \}^* = i\, g_{\mu\nu}\ \delta_{ab}$$

$$\{ \xi^a_5 , \xi^b_5 \}^* = - i\, \delta_{ab}$$

$$\{ \xi^a_\mu , \xi^b_5 \}^* = 0$$

$$\{ x_\mu , P_\nu \}^* = - g_{\mu\nu} \tag{13}$$

Total Hamiltonian [7] is a combination of first class constraints and is given by

$$H = \rho_1 \chi + \rho^a_2\ \chi^a_D \tag{14}$$

Here ρ_1 is even while ρ^a_2 is odd Grassmann variable. Equations of motion are given by $\dot{A} = \{ A, H \}^*$ and in particular $\dot{P}_\mu = 0$.

The (unconstrained) Hamiltonian dynamics can now be quantized by formulating the correspondence (^ indicates operator)

$$i\hbar \{ A, B \}^* \longrightarrow [\hat{A}, \hat{B}]_- \text{ or } [\hat{A}, \hat{B}]_+ \qquad (15)$$

and requiring that the first class constraints become conditions on the states

$$\hat{\chi}\psi \equiv (\hat{P}^2 - m^2 c^2)\psi = 0$$

$$\hat{\chi}_D^a \psi \equiv (\hat{P}.\ \hat{\xi}^a - m c\ \hat{\xi}_5^a)\Psi = 0 \qquad (16)$$

The choice of commutator or anticommutator will be determined by requiring self-consistency of eqns. (15) and (16). Thus we write

$$[\hat{\xi}_\mu^a, \hat{\xi}_\nu^b]_\pm = -\hbar g_{\mu\nu} \delta_{ab}$$

$$[\hat{\xi}_5^a, \hat{\xi}_5^b]_\pm = \hbar \delta_{ab}$$

$$[\hat{\xi}_\mu^a, \hat{\xi}_5^b]_\pm = 0 \qquad (17)$$

while for obvious reasons $[\hat{x}_\mu, \hat{P}_\nu]_- = i\hbar g_{\mu\nu}$ and $[\hat{\xi}, \hat{P}_\mu] = 0$. For $a = b$ it is immediate in the first two relations and derived for the third by using eq. (16) that anticommutator must be used. $\hat{\xi}^a$, $\hat{\xi}_5^a$ generate Clifford algebra C_5. We obtain the usual Dirac-Pauli algebra for γ's on writing

$$\hat{\xi}_\mu^a = \sqrt{\frac{\hbar}{2}}\ \gamma_5^{(a)} \gamma_\mu^a \quad , \quad \hat{\xi}_5^a = \sqrt{\frac{\hbar}{2}}\ \gamma_5^a \qquad (18)$$

and deduce $\gamma_5^a = i\ \gamma_0^{(a)} \gamma_1^a \gamma_2^a \gamma_3^a$. Eq. (16) gives

$$(\gamma^a . P - m c)\psi = 0 \ , \ a = 1, \ldots 2s. \qquad (19)$$

For these equations to be satisfied simultaneously, we require for $a \neq b$

$$[\gamma^a_\mu \; , \gamma^b_\nu]_- = 0 \tag{20}$$

and consequently

$$[\gamma^a_5 \; , \gamma^b_5]_- = 0 \tag{21}$$

even though ξ^a_5 are anticommuting numbers. Consistency requirements $[\hat{\chi} , \hat{\chi}^a_D]_\pm \psi = 0$ and $[\hat{\chi}^a_D , \hat{\chi}^b_D]_\pm \psi = 0$ do not lead to new conditions and are satisfied identically. Thus we obtain the formulation of Bargmann and Wigner (9) for particle with spin 's'.

Finally, we discuss the covariance of the theory. The infinitesimal generator of (canonical) Pincarē and supersymmetry transformations(8) is

$$F = - \delta x^\mu \; P_\mu + \delta \xi^a_\mu \pi^\mu_a + \delta \xi^a_5 \; \pi^a_5 - \frac{i}{2} (\varepsilon^a_5 \xi^a_5 + \varepsilon^a . \xi^a)$$

$$\equiv b_\mu P^\mu + \frac{1}{2} w_{\mu\nu} M^{\nu\mu} + \varepsilon^a_\mu \; G^\mu_a + \varepsilon^a_5 \; G^a_5 \tag{22}$$

where $w_{\mu\nu} = - w_{\nu\mu}$ are infinitesimal parameters of Lorentz transformatios and $M_{\mu\nu}$ the corresponding generator. G^a_μ and G^a_5 are the generators of supersymmetry transformations. They may readily identified to be given by

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} = (x^\mu \; P^\nu - x^\nu \; P^\mu) - (\xi^\mu_a \pi^\nu_a - \xi^\nu_a \; \pi^\mu_a)$$

$$G^a_\mu = - (\pi^a_\mu + \frac{i}{2} \xi^a_\mu - \frac{i}{mc} \xi^a_5 \, P_\mu)$$

$$G^a_5 = - (\pi^a_5 + \frac{i}{2} \xi^a_5) \tag{23}$$

Under supersymmetry transformations we find that χ , χ^a_D , χ^a_μ remain invariant while

$$\delta \chi'^a_5 = - \{ F , \chi'^a_5 \} = - i (\varepsilon^a_5 - \frac{\varepsilon^a . P}{mc}) \qquad (24)$$

The constraint $\chi'^a_5 \approx 0$ thus breaks the supersymmetry invariance. We find, however, that the canonical generator ($\dot{P}_\mu = 0$)

$$G^a = (G^a_5 + \frac{G^a . P}{mc}) \qquad (25)$$

has vanishing Poisson brackets with all the constraints. Theory thus retains the invariance under the following supergauge transformations by(§) $F = \eta^a_5 (\tau) \; G^a$,

$$x'_\mu = x_\mu + \frac{i \, P_\mu}{(mc)^2} \eta^a_5 (\tau) \, \xi^a_5$$

$$\xi'_\mu = \xi^a_\mu + \frac{P_\mu}{mc} \eta^a_5 (\tau)$$

$$\xi'^a_5 = \xi^a_5 + \eta^a_5 (\tau) \qquad (26)$$

Poincaré invariance may be also verified. For example,

$$S^{\mu\nu} = - \frac{i}{2} (\xi^\mu_a \; \xi^\nu_a - \xi^\nu_a \; \xi^\mu_a) \qquad (27)$$

and

(§) Eqns. (5), (6) are left invariant even when parameter is τ-dependent. Quasi-invariance of Lagrangian may also be checked.

$$\{ S^{\mu\nu}, S^{\lambda\rho} \}^* = g^{\nu\rho} S^{\lambda\mu} - g^{\mu\rho} S^{\mu\lambda} - g^{\mu\lambda} S^{\lambda\rho} + g^{\nu\lambda} S^{\mu\rho} \quad (28)$$

Hamilton's equations of motion are

$$\dot{x}_\mu = -2 \rho_1 P_\mu - \frac{i}{mc} \rho_2^a \xi_\mu^a$$

$$\dot{\xi}_\mu^a = (P_\mu/mc) \rho_2^a$$

$$\dot{\xi}_5^a = \rho_2^a$$

$$\dot{P}_\mu = 0 \quad (29)$$

The time evolution mixes coordinate and spin degrees of freedom; the whole phase space of spinning particle is a 'superspace'. The second term in the equation for $\dot{x}_\mu$ is the classical analogue of Schroedinger's Ziterbewegung.

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REFERENCES:

1. J. Wess and B. Zumino: Nucl. Phys., 70 B, 39 (1974). See also B. Zumino: Proceedings of the XVII International Conference on High Energy Physics, London, 1974.
2. A. Salam and J. Strathdee: Nucl. Phys., 76 B, 77 (1974).
3. J. L. Martin: Proc. Roy. Soc. 251 A, 536 (1959); F. A. Berezin and M. S. Marinov: JETP Letters 21, 321 (1975).
4. R. Casalbuoni: Phys. Letters 62 B, 49 (1976).
5. F. A. Berezin and M. S. Marinov: Particle Spin Dynamics as the Grassmann Variant of Classical Mechanics, ITEP-43, 1976, Moscow.
6. A. Barducci, R. Casalbuoni and L. Lusanna: Supersymmetries and the Pseudoclassical Relativistic Electron, Firenze preprint, July 1976.
7. P. A. M. Dirac: Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University (New York), 1964.
8. R Casalbuoni: The Classical Mechanics for Bose-Fermi System, CERN TH-2139 (1976); Nuovo Cimento 33 A, 115 (1976).
9. V. Bargmann and E. P. Wigner: Proc. Nat. Acad. Sci. USA, 34, 211 (1948).
10. P. P. Srivastava: Lett. Nuovo Cimento, 12, 161 (1975); 13, 657 (1975).
11. F. A. Berezin: The Method of Second Quantization, Academic Press, N. Y., 1966.
12. See for example H. C. Corben and P. Stehle: Classical Mechanics, John Wiley, N. Y., 1965 and refs. 6, 8.