CBPF
CENTRO BRASILEIRO
DE PESQUISAS FÍSICAS

A0034/78

DEZ,1978

# ON DUAL PROPERTIES OF WEYL TENSOR

M. Novello and J. Duarte de Oliveira

RIO DE JANEIRO
BRASIL

*General Relativity and Gravitation, Vol. 12, No. 11, 1980*



# On Dual Properties of the Weyl Tensor

M. NOVELLO
*Centro Brasileiro de Pesquisas Físicas*
and
J. DUARTE DE OLIVEIRA[1]
*Universidade Estadual de Maringá, Paraná*

*Received October 24, 1979*

*Abstract*

We present some new identities satisfied by double and triple products of the Weyl tensor $W_{\alpha\beta\mu\nu}$ and its dual $\overset{*}{W}_{\alpha\beta\mu\nu}$.

## §(1): *Introduction and Notation*

In this paper we present some new identities satisfied by the Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ and its dual $W^*{}_{\alpha\beta\mu\nu}$. Such identities generalize previous work by Lanczos [1] and Debever [2], who have derived two relations among the Weyl tensor and its dual. We complete here the task of finding all possible relations between $W_{\alpha\beta\mu\nu}$ and $W^*{}_{\alpha\beta\mu\nu}$ and analyze the similitude and dissimilitude to analogous relations which can be derived for the antisymmetric Maxwell tensor $F_{\mu\nu}$ (actually, for any second-order antisymmetric tensor).

In order to fix notation we review briefly, in this section, some formulas and results which will be used in the text.

Following Weyl, we define the tensor $W_{\alpha\beta\mu\nu}$ in terms of the curvature tensor $R_{\alpha\beta\mu\nu}$ and its contractions $R_{\beta\nu} = R_{\alpha\beta\mu\nu}g^{\alpha\mu}$ and $R = R_{\mu\nu}g^{\mu\nu}$:

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \tfrac{1}{2}(g_{\alpha\mu}R_{\beta\nu} + g_{\beta\nu}R_{\alpha\mu} - g_{\alpha\nu}R_{\beta\mu} - g_{\beta\mu}R_{\alpha\nu}) + \tfrac{1}{6}R(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) \quad (1.1)$$

[1]Partially supported by C.N.Pq.

This tensor has the same symmetries as the curvature tensor. Furthermore, it is trace free. Thus it has only ten degrees of freedom. For a given observer with four-velocity $V^\mu$ the Weyl tensor can be decomposed into its electric and magnetic parts by using as definitions of the electric $(E_{\mu\nu})$ and magnetic $(H_{\mu\nu})$ tensors

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu}V^\mu V^\nu \qquad (1.2a)$$

$$H_{\alpha\beta} = -W^*_{\alpha\mu\beta\nu}V^\mu V^\nu \qquad (1.2b)$$

The dual, represented by the symbol *, is defined for any antisymmetric tensor $f_{\mu\nu}$ by the expression

$$f^*_{\mu\nu} = \tfrac{1}{2}\,\eta_{\mu\nu}{}^{\rho\sigma}f_{\rho\sigma} \qquad (1.3)$$

in which $\eta_{\mu\nu}{}^{\rho\sigma}$ is given in terms of the Levi-Civita completely antisymmetric symbol $\epsilon_{\mu\nu}{}^{\rho\sigma}$ by

$$\eta_{\mu\nu\rho\sigma} = (-g)^{1/2}\,\epsilon_{\mu\nu\rho\sigma} \qquad (1.4)$$

Thus,

$$\eta^{\mu\nu\rho\sigma} = -(-g)^{-1/2}\,\epsilon^{\mu\nu\rho\sigma}$$

The Lorentz metric will be represented by $\eta_{\mu\nu}$. We have $\eta_{\mu\nu} = \mathrm{diag}\,(+\,-\,-\,-)$. Because of the symmetries of the Weyl tensor, we can write

$$W^*_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma}^{\;\;\;\;\;\;*} \equiv \overset{*}{W}_{\mu\nu\rho\sigma} \qquad (1.5)$$

It is trivial to see that $f^{**}_{\mu\nu} = -f_{\mu\nu}$. From the definition of $E_{\mu\nu}$ and $H_{\mu\nu}$ we obtain the following properties:

$$\begin{aligned} E_{\mu\nu} &= E_{\nu\mu} \\ E_{\mu\nu}V^\mu &= 0 \\ E_{\mu\nu}g^{\mu\nu} &= 0 \\ H_{\mu\nu} &= H_{\nu\mu} \\ H_{\mu\nu}V^\mu &= 0 \\ H_{\mu\nu}g^{\mu\nu} &= 0 \end{aligned} \qquad (1.6)$$

Thus, the Weyl tensor is given in terms of $E_{\mu\nu}$ and $H_{\mu\nu}$ through the expression

$$W_{\alpha\beta}{}^{\mu\nu} = 2V_{[\alpha}E_{\beta]}{}^{[\mu}V^{\nu]} + \delta^{[\mu}_{[\alpha}E^{\nu]}_{\beta]} - \eta_{\alpha\beta\lambda\sigma}V^\lambda H^{\sigma[\mu}V^{\nu]} - \eta^{\mu\nu\lambda\sigma}V_\lambda H_{\sigma[\alpha}V_{\beta]} \qquad (1.7)$$

The square brackets indicate antisymmetrization: $a_{[\mu\nu]} = a_{\mu\nu} - a_{\nu\mu}$; the parentheses indicate symmetrization: $a_{(\mu\nu)} = a_{\mu\nu} + a_{\nu\mu}$.

Sometimes it is convenient to write the Weyl tensor under another form, equivalent to (1.7), given by (cf. [4])

$$W^{\alpha\mu\beta\nu} = (\eta^{\alpha\mu\lambda\sigma}\eta^{\beta\nu\tau\epsilon} - g^{\alpha\mu\lambda\sigma}g^{\beta\nu\tau\sigma})\, V_\lambda V_\tau E_{\sigma\epsilon} + (\eta^{\alpha\mu\lambda\sigma}g^{\beta\nu\tau\epsilon} + g^{\alpha\mu\lambda\sigma}\eta^{\beta\nu\tau\epsilon}) V_\lambda V_\tau H_{\sigma\epsilon} \quad (1.8)$$

in which

$$g_{\alpha\mu\beta\nu} = g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}$$

The comparison of the electric and magnetic parts of the Weyl tensor which describe free gravitational field with analogous expressions for the electromagnetic field are given in Table I. From the definition of the electric ($E_\alpha$) and the magnetic ($H_\alpha$) parts of the Maxwell tensor we can write

$$F_{\alpha\beta} = -V_\alpha E_\beta + V_\beta E_\alpha + \eta_{\alpha\beta}{}^{\rho\sigma} V_\rho H_\sigma$$

## §(2): *Dual Identities: Old and New Relations*

In the study of the antisymmetric electromagnetic tensor $F_{\mu\nu}$ use is generally made of two important identities, relating $F_{\mu\nu}$ to its dual $\overset{*}{F}_{\mu\nu}$. They are

$$\overset{*}{F}_{\mu\lambda}\overset{*}{F}{}^{\lambda}{}_{\nu} - F_{\mu\lambda}F^{\lambda}{}_{\nu} = I_1 g_{\mu\nu} \quad (2.1)$$

$$\overset{*}{F}_{\mu\nu}F^{\nu\lambda} = F_{\mu\nu}\overset{*}{F}{}^{\nu\lambda} = -\tfrac{1}{2} I_2 \delta^{\lambda}{}_{\mu} \quad (2.2)$$

in which the two invariants $I_1$ and $I_2$ are defined by

$$I_1 = \tfrac{1}{2} F_{\mu\lambda}F^{\mu\lambda} = E^2 - H^2 \quad (2.3)$$

$$I_2 = \tfrac{1}{2} F_{\mu\nu}\overset{*}{F}{}^{\mu\nu} = 2E_\alpha H^\alpha \quad (2.4)$$

**Table I.** Electromagnetic and Gravitational Fundamental Quantities

| Field | Electromagnetic | Gravitational |
|---|---|---|
| Variables | $F_{\mu\nu}$ | $W_{\alpha\beta\mu\nu}$ |
| Symmetries | $F_{\mu\nu} = -F_{\nu\mu}$ | $W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta}$ |
| Electric part | $E_\alpha = F_{\alpha\mu}V^\mu$ | $E_{\alpha\beta} = -W_{\alpha\mu\beta\nu}V^\mu V^\nu$ |
| Magnetic part | $H_\alpha = \overset{*}{F}_{\alpha\mu}V^\mu$ | $H_{\alpha\beta} = -\overset{*}{W}_{\alpha\mu\beta\nu}V^\mu V^\nu$ |
| Quadratic invariants | $I_1 = \frac{1}{2} F_{\mu\nu}F^{\mu\nu} = E_\alpha E^\alpha - H_\alpha H^\alpha$ | $A = \frac{1}{8} W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu} = E_{\alpha\beta}E^{\alpha\beta} - H_{\alpha\beta}H^{\alpha\beta}$ |
| | $I_2 = \frac{1}{2} \overset{*}{F}_{\mu\nu}F^{\mu\nu} = 2E_\alpha H^\alpha$ | $B = \frac{1}{8} W_{\alpha\beta\mu\nu}\overset{*}{W}{}^{\alpha\beta\mu\nu} = 2E_{\alpha\beta}H^{\alpha\beta}$ |
| Cubic invariants | | $C = \frac{1}{16} W_{\alpha\beta\mu\nu}W^{\mu\nu\rho\sigma}W_{\rho\sigma}{}^{\alpha\beta} = -E_{\alpha\rho}E^{\rho\beta}E_\beta{}^\alpha + 3H_{\alpha\rho}H^{\rho\beta}E_\beta{}^\alpha$ |
| | | $D = \frac{1}{16} \overset{*}{W}_{\alpha\beta\mu\nu}W^{\mu\nu\rho\sigma}W_{\rho\sigma}{}^{\alpha\beta} = H_{\alpha\rho}H^{\rho\beta}H_\beta{}^\alpha - 3E_{\alpha\rho}E^{\rho\beta}H_\beta{}^\alpha$ |

Such relations allow great simplification of certain expressions. For instance, the stress-energy tensor of the electromagnetic field, given by

$$T_{\mu\nu} = F_{\mu\alpha}F^{\alpha}{}_{\nu} + \tfrac{1}{4}\, g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.5}$$

can be written using (2.1) under the equivalent manifestly dual symmetric form

$$2T_{\mu\nu} = F_{\mu\alpha}F^{\alpha}{}_{\nu} + \overset{*}{F}_{\mu\alpha}\overset{*}{F}{}^{\alpha}{}_{\nu} \tag{2.5$'$}$$

The great similarity of the properties of Weyl ($W_{\alpha\beta\mu\nu}$) and Maxwell ($F_{\mu\nu}$) tensors induces us to believe that analogous relations should exist relating $W^{\alpha\beta\mu\nu}$ and its dual $\overset{*}{W}{}^{\alpha\beta\mu\nu}$. There are, however, important formal differences between these two cases, related to the fact that the $W$ tensor has ten degrees of freedom and the $F$ tensor has only six. Thus, one should suspect that probably in the case of the Weyl tensor the number of such relations should be greater than 2. In addition, in the case of free gravitational field ($R_{\mu\nu} = 0$), which will be the case treated in this paper, the number of invariants we can construct with the $W$ tensor is twice the equivalent number formed with the electromagnetic field. There are two invariants constructed with products of three Weyl tensors, which have no correspondence in the electromagnetic case.

We will show that, indeed, this suspicion was correct and that we can form more identities with the $W$ tensor than in the analogous case of electromagnetic field.

We find, in the literature, that only two such identities are known. They are

$$W_{\alpha\rho\mu\sigma}\, W_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{W}_{\alpha\rho\lambda\sigma}\, \overset{*}{W}_{\alpha}{}^{\rho}{}_{\mu}{}^{\sigma} = A g_{\lambda\mu} g_{\alpha\beta} \tag{2.6}$$

$$W_{\alpha\rho\lambda\sigma}\, W_{\beta}{}^{\rho\lambda\sigma} = 2A g_{\alpha\beta} \tag{2.7}$$

The first one is quoted by Debever [2] and the second one has been presented by Lanczos [1]. Identity (2.7) can be obtained by contraction of expression (2.6) although the proof of (2.6) given by Debever makes explicit use of (2.7). We remark that Lanczos has shown (2.7) independently of (2.6). Contracting (2.7) once more we obtain the definition of the invariant $A$ (see Table I).

Using expression (2.6) twice and interchanging the role of $\rho$ and $\lambda$ we obtain the identity

$$W^{(\alpha}{}_{\rho\mu\sigma}\, W^{\beta)\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{W}{}^{(\alpha}{}_{\rho\mu\sigma}\, \overset{*}{W}{}^{\beta)\rho}{}_{\lambda}{}^{\sigma} = 2A g_{\lambda\mu} g^{\alpha\beta} \tag{2.8}$$

We have developed a direct proof of this identity which does not make any reference to the previous relations. We have decided to give this demonstration here since some results, like the values of $W^{\alpha}{}_{\rho\mu\sigma}\, W^{\beta\rho}{}_{\lambda}{}^{\sigma}$ and $\overset{*}{W}{}^{\alpha}{}_{\rho\mu\sigma}\, \overset{*}{W}{}^{\beta\rho}{}_{\lambda}{}^{\sigma}$ given in terms of the electric and the magnetic tensors $E_{\mu\nu}$ and $H_{\mu\nu}$, are of great interest and help.

In order to make this proof we use the decomposition of the Weyl tensor into its electric and magnetic parts. The evaluation of $\overset{*}{W}{}^{\alpha}{}_{\rho\mu\sigma}\, \overset{*}{W}{}^{\beta\rho}{}_{\lambda}{}^{\sigma}$ is greatly simplified by noting that the dual operation is nothing but the transformation of the pair $(E_{\mu\nu}, H_{\mu\nu})$ into $(H_{\mu\nu}, -E_{\mu\nu})$, respectively:

$$
\begin{aligned}
W^{\alpha\mu\beta\nu}W_{\alpha\epsilon\beta\tau} = {} & [(\eta^{\alpha\mu pq}\eta^{\beta\nu rs} - g^{\alpha\mu pq}g^{\beta\nu rs})V_pV_rE_{qs} + (\eta^{\alpha\mu pq}g^{\beta\nu rs} \\
& + g^{\alpha\mu pq}\eta^{\beta\nu rs})V_pV_rH_{qs}]\,[(\eta_{\alpha\epsilon ab}\eta_{\beta\tau cd} - g_{\alpha\epsilon ab}g_{\beta\tau cd})V^aV^cE^{bd} \\
& + (\eta_{\alpha\epsilon ab}g_{\beta\tau cd} + g_{\alpha\epsilon ab}\eta_{\beta\tau cd})V^aV^cH^{bd}]
\end{aligned}
\tag{2.9}
$$

Collecting all these terms we obtain

$$
\begin{aligned}
W^{\alpha\mu\beta\nu}W_{\alpha\epsilon\beta\tau} = {} & E^2\delta^\mu_\epsilon\delta^\nu_\tau - \delta^\mu_\epsilon E_{\alpha\tau}E^{\alpha\nu} - \delta^\mu_\epsilon H_{\alpha\tau}H^{\alpha\nu} - V_\tau V^\nu\delta^\mu_\epsilon(E^2 + H^2) \\
& - E_{\epsilon\alpha}E^{\mu\alpha}\delta^\nu{}_\tau + 2E_{\epsilon\tau}E^{\mu\nu} + 2V_\tau V^\nu(E_{\alpha\epsilon}E^{\alpha\mu} + H_{\alpha\epsilon}H^{\alpha\mu}) \\
& - V_\epsilon V^\mu\delta^\nu_\tau(E^2 + H^2) + 2V_\epsilon V^\mu(E_{\alpha\tau}E^{\alpha\nu} + H_{\alpha\tau}H^{\alpha\nu}) \\
& - 2V_\epsilon V_\tau E_\alpha{}^\mu E^{\alpha\nu} - V_\epsilon V_\tau g^{\mu\nu}E^2 + 2V^\mu V^\nu V_\tau(E^2 + H^2) \\
& - V^\mu V^\nu g_{\epsilon\tau}E^2 + 2V^\mu V^\nu E_{\alpha\tau}E^\alpha{}_\epsilon + 2H_\epsilon{}^\nu H^\mu{}_\tau - V_\epsilon V^\nu\delta^\mu_\tau H^2 \\
& + 2V_\epsilon V^\nu H^\mu{}_\beta H^\beta{}_\tau + 2V^\mu V_\tau H^\alpha{}_\epsilon H_\alpha{}^\nu - V^\mu V_\tau\delta^\nu_\epsilon H^2 - \delta^\nu_\tau H^{\mu\alpha}H_{\alpha\epsilon} \\
& + \eta^{\beta\nu r\gamma}\delta^\mu_\epsilon V_\tau V_\epsilon E^\alpha{}_\gamma H_{\alpha\beta} - \eta^{\beta\nu r\gamma}V_\epsilon V^\mu V_\tau V_r E^\alpha{}_\gamma H_{\alpha\beta} \\
& - \eta^{\beta\nu r\gamma}V_rV_\tau E_{\epsilon\gamma}H^\mu{}_\beta + \eta^{\alpha\mu pq}\delta^\nu_\tau V_\epsilon V_p H_{\alpha\gamma}E_q{}^\gamma \\
& - V_\epsilon V_\tau V^\nu V_\lambda H_{\alpha\gamma}E_q{}^\gamma\eta^{\alpha\mu\lambda q} - V_\epsilon V_\lambda H^{\alpha\nu}E_{\gamma\tau}\eta_\alpha{}^{\mu\lambda\gamma} \\
& + \eta_{\alpha\epsilon ab}V^\mu V^a E^{\alpha\nu}H^b_\tau + \eta_{\alpha\epsilon ab}V^\mu V^\nu V^a V_\tau E^{\alpha\gamma}H_\gamma{}^b \\
& + V^\nu V^\lambda E^{\mu\beta}H_\epsilon{}^\gamma\eta_{\beta\tau\lambda\gamma} + V^\mu V^\nu V_\epsilon V^\lambda E^{\alpha\beta}H_\alpha{}^\gamma\eta_{\beta\tau\lambda\gamma} \\
& + \eta_{\beta\tau\lambda\gamma}V^\nu V^\lambda H^{\alpha\beta}E_\alpha{}^\gamma\delta^\mu_\epsilon - V^\nu V^\lambda H_\epsilon{}^\beta E^{\mu\gamma}\eta_{\beta\tau\lambda\gamma} \\
& - V_\epsilon V^\mu V^\nu V^\lambda H^{\gamma\beta}E_\gamma{}^\sigma\eta_{\beta\tau\lambda\sigma} + \eta^{\alpha\mu\lambda\gamma}V_\epsilon V_\lambda H_\gamma{}^\nu E_{\alpha\tau} \\
& + \eta^{\alpha\mu\lambda\gamma}V_\epsilon V^\nu V_\tau H^\sigma{}_\gamma E_{\sigma\alpha}V_\lambda + \eta_{\alpha\epsilon\rho\sigma}V^\rho V^\mu H^{\alpha\gamma}E^\sigma{}_\gamma\delta^\nu{}_\tau \\
& - \eta_{\alpha\epsilon\lambda\gamma}V^\lambda V^\mu V^\nu V_\tau H^{\alpha\sigma}E^\gamma{}_\sigma - \eta_{\alpha\epsilon\lambda\gamma}V^\lambda V^\mu H^\alpha{}_\tau E^{\gamma\nu} \\
& - \eta^{\beta\nu\lambda\gamma}V_\lambda V_\tau H^\mu{}_\gamma E_{\epsilon\beta} + V^\mu V_\epsilon V_\tau V_\sigma\eta^{\beta\nu\sigma\gamma}H_\gamma{}^\rho E_{\rho\beta}
\end{aligned}
\tag{2.10}
$$

From this expression we can obtain by the substitution of $(E_{\mu\nu}, H_{\mu\nu})$ by $(H_{\mu\nu}, -E_{\mu\nu})$ the product of the duals $\overset{*}{W}{}^{\alpha\mu\beta\nu}\overset{*}{W}_{\alpha\epsilon\beta\tau}$. Then adding these terms gives the desired identity (2.8):

$$
W^{\alpha(\mu}{}_{\beta\nu}W_\alpha{}^{\epsilon)\beta}{}_\tau - \overset{*}{W}{}^{\alpha(\mu}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\epsilon)\beta}{}_\tau = 2(E^2 - H^2)g^{\mu\epsilon}g_{\nu\tau}
\tag{2.8$'$}
$$

Such an identity has no analog in the electromagnetic case, as the product $F_{\mu\alpha}F^\alpha{}_\nu$ is already symmetric in the $\mu$, $\nu$ indices. Contracting indices $\mu\epsilon$ in (2.8) we obtain the Lanczos identity (2.7).

The above relations are similar to the relation (2.1) of the electromagnetic tensor. We will now show that expression (2.2) has a gravitational analog.

Let us start by computing $\overset{*}{W}{}^{\alpha\mu}{}_{\beta\nu} W_{\alpha}{}^{\epsilon\beta}{}_{\tau}$. Unfortunately, we cannot guess the value of such expression by comparison with the product of the Weyl tensors which we previously evaluated. Thus, let us evaluate it by direct calculation. We have

$$\begin{aligned}\overset{*}{W}{}^{\alpha\mu\beta\nu} W_{\alpha\epsilon\beta\tau} = {} & [(\eta^{\alpha\mu pq}\eta^{\beta\nu\rho\sigma} - g^{\alpha\mu pg} g^{\beta\nu\rho\sigma}) V_p V_\rho H_{q\sigma} \\ & - (\eta^{\alpha\mu pq} g^{\beta\nu\rho\sigma} + g^{\alpha\mu pq}\eta^{\beta\nu\rho\sigma}) V_p V_\rho E_{q\sigma}] \; [(\eta_{\alpha\epsilon ab}\eta_{\beta\tau cd} \\ & - g_{\alpha\epsilon ab} g_{\beta\tau cd}) V^a V^c E^{bd} + (\eta_{\alpha\epsilon ab} g_{\beta\tau cd} + g_{\alpha\epsilon ab}\eta_{\beta\tau cd}) V^a V^c H^{bd}]\end{aligned} \tag{2.11}$$

Notice that the Latin and Greek alphabet are used to denote the same kind of tensor indices. We follow the procedure used in the previous case and will give only the final results of the product. Collecting all terms we obtain

$$\begin{aligned}\overset{*}{W}{}^{\alpha\mu\beta\nu} W_{\alpha\epsilon\beta\tau} = {} & \delta^{\mu}{}_{\epsilon}\delta^{\nu}{}_{\tau} E_{\rho\sigma} H^{\rho\sigma} + H_{\epsilon\tau} E^{\mu\nu} - \eta^{\alpha\nu r\sigma} V_\tau V_r H_{\epsilon\sigma} H^{\mu}{}_{\alpha} \\ & - \eta^{\alpha\mu pq} V_p V_\epsilon H_{q\tau} H_{\alpha}{}^{\nu} + H^{\mu\nu} E_{\epsilon\tau} + V^{\nu} V_{\tau} H^{\mu\alpha} E_{\alpha\epsilon} \\ & - E_{\epsilon}{}^{\nu} H^{\mu}{}_{\tau} - V_\epsilon V^{\nu} E_{\alpha\tau} H^{\alpha\mu} - V^{\nu} V_\tau H_{\epsilon\alpha} E^{\alpha\mu} \\ & - V_\epsilon V^{\nu} \delta^{\mu}_{\tau} E_{\rho\sigma} H^{\rho\sigma} - V_\epsilon V^{\nu} E^{\alpha\mu} H_{\alpha\tau} + V_\tau V^{\mu} \delta^{\nu}{}_{\epsilon} E_{\rho\sigma} H^{\rho\sigma} \\ & - V_\tau V^{\mu} E_{\epsilon\alpha} H^{\alpha\nu} - V_\tau V^{\mu} E^{\alpha\nu} H_{\alpha\epsilon} - E^{\mu}{}_{\tau} H_{\epsilon}{}^{\nu} + V_\epsilon V_\tau H^{\alpha\nu} E_{\alpha}{}^{\mu} \\ & - V_\epsilon V_\tau g^{\mu\nu} E_{\rho\sigma} H^{\rho\sigma} + V_\epsilon V_\tau H^{\alpha\mu} E^{\nu}_{\alpha} + V^{\mu} V^{\nu} H_{\alpha\epsilon} E^{\alpha}{}_{\tau} \\ & - V^{\mu} V^{\nu} g_{\epsilon\tau} E_{\rho\sigma} H^{\rho\sigma} + V^{\mu} V^{\nu} H_{\tau\alpha} E^{\alpha}{}_{\epsilon} + \eta_{\alpha\epsilon ab} V^{\mu} V^{a} H^{\alpha\nu} H^{b}{}_{\tau} \\ & + \eta_{\beta\tau cd} V^{c} V^{\nu} H^{\mu\beta} H_{\epsilon}{}^{d} + \eta_{\beta\tau cd} V^{\nu} V^{c} E_{\epsilon}{}^{\beta} E^{\mu d} \\ & - \eta^{\alpha\mu pq} V_p V_\epsilon E_q{}^{\nu} E_{\alpha\tau} + \eta_{\alpha\epsilon\lambda\beta} V^{\lambda} V^{\mu} E^{\alpha}{}_{\tau} E^{\beta\nu} \\ & - \eta^{\beta\nu r\sigma} V_\tau V_r E^{\mu}{}_{\sigma} E_{\epsilon\beta}\end{aligned} \tag{2.12}$$

Symmetrizing the above expression in $\epsilon, \mu$ a straightforward calculation gives to the new identity

$$W^{\alpha(\mu}{}_{\beta\nu} \overset{*}{W}_{\alpha}{}^{\epsilon)\beta}{}_{\tau} = B g^{\mu\epsilon} g_{\nu\tau} \tag{2.13}$$

Contracting (2.13) in $\mu, \epsilon$ we obtain

$$\overset{*}{W}{}^{\alpha\mu\beta}{}_{\nu} W_{\alpha\mu\beta}{}^{\tau} = 2B\delta^{\tau}{}_{\nu} \tag{2.14}$$

or, due to the dual property

$$W^{\alpha\mu\beta}{}_{\nu} \overset{*}{W}_{\alpha\mu\beta}{}^{\tau} = 2B\delta^{\tau}{}_{\nu} \tag{2.14$'$}$$

Which generalizes the Lanczos identity (2.7). Let us show now a simple consequence of such identity. We have

$$W^{\alpha\epsilon}{}_{\beta\tau} \overset{*}{W}_{\alpha}{}^{\mu\beta}{}_{\nu} + W^{\alpha\mu}{}_{\beta\tau} \overset{*}{W}_{\alpha}{}^{\epsilon\beta}{}_{\nu} = B g^{\mu\epsilon} g_{\nu\tau} \tag{2.15}$$

and also

$$W^{\alpha\epsilon}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\mu\beta}{}_\tau + W^{\alpha\mu}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\epsilon\beta}{}_\tau = Bg^{\mu\epsilon}g_{\nu\tau} \tag{2.16}$$

Adding (2.15) and (2.16) we obtain

$$\begin{aligned}\overset{*}{W}{}^{\alpha\mu}{}_{\beta\nu}W_\alpha{}^{\epsilon\beta}{}_\tau + W^{\alpha\epsilon}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\mu\beta}{}_\tau &= -W^{\alpha\mu}{}_{\beta\tau}\overset{*}{W}_\alpha{}^{\epsilon\beta}{}_\nu - W^{\alpha\mu}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\epsilon\beta}{}_\tau + 2Bg^{\mu\epsilon}g_{\nu\tau}\\ &= -W^{\mu\alpha}{}_{\tau\beta}\overset{*}{W}{}^\epsilon{}_{\alpha\nu}{}^\beta - W^{\mu\alpha}{}_{\beta\nu}\overset{*}{W}{}^\epsilon{}_{\alpha\tau}{}^\beta + 2Bg^{\mu\epsilon}g_{\nu\tau}\end{aligned}$$

Using the identity (2.13) we obtain a new relation which is given by

$$\overset{*}{W}{}^{\alpha\mu}{}_{\beta\nu}W_\alpha{}^{\epsilon\beta}{}_\tau + W^{\alpha\epsilon}{}_{\beta\nu}\overset{*}{W}_\alpha{}^{\mu\beta}{}_\tau = Bg^{\mu\epsilon}g_{\nu\tau} \tag{2.17}$$

From these identities one can obtain a very useful corollary. Subtracting (2.15) and (2.17) we have

$$\overset{*}{W}_{\alpha\mu\beta\nu}W^{\alpha\rho\beta\sigma} = W_{\alpha\mu\beta\nu}\overset{*}{W}{}^{\alpha\rho\beta\sigma} \tag{2.18}$$

Now, from the dual property we have identically

$$\overset{*}{W}_{\alpha\beta\mu\nu}W^{\alpha\beta\rho\sigma} = W_{\alpha\beta\mu\nu}\overset{*}{W}{}^{\alpha\beta\rho\sigma} \tag{2.19}$$

These two identities (2.18) and (2.19) allow us to enunciate the following result (which can be understood as a rule). Given the product of a Weyl tensor and its dual (represented symbolically by $W\,W^*$) and if two indices which belong to any pair are contracted, then it is always possible to interchange the asterisk symbol. Symbolically, this rule can be represented by $W\overset{*}{W} = \overset{*}{W}W$ if there are any two contractions not necessarily in the same pair.

## §(3): *Dual Identities of Third Order*

In the preceding section we have investigated some identities satisfied by the Weyl tensor, which have analogies for the electromagnetic field. We will now show that the Weyl tensor satisfies further identities, which have no counterpart for the Maxwell tensor $F_{\mu\nu}$. The ultimate reason for such dissimilitude is related to the existence of four invariants ($A, B, C$, and $D$) in the free gravitational field although only two invariants can be constructed with the Maxwell tensor.

In this section, contrary to the procedure we used before, we will apply a slightly different technique to arrive at the desired relations.

Let us evaluate the triple product $\overset{*}{W}_{\mu\alpha\epsilon\beta}W^{\epsilon\tau\lambda\sigma}\overset{*}{W}_{\lambda\rho\alpha\gamma}$. We have

$$\begin{aligned}\overset{*}{W}_{\mu\alpha\epsilon\beta}W^{\epsilon\tau\lambda\sigma}\overset{*}{W}_{\lambda\rho\alpha\gamma} &= -\overset{*}{W'}_{\mu\alpha\epsilon\beta}\overset{**}{W'}{}^{\epsilon\tau\lambda\sigma}\overset{*}{W'}_{\lambda\rho\alpha\gamma}\\ &= -\tfrac{1}{16}\,\eta_{\epsilon\beta ab}\eta^{\epsilon\tau cd}\eta^{mn\lambda\sigma}\eta_{\lambda\rho kl}W'^{ab}{}_{\mu\alpha}W'_{cdmn}W'^{kl}{}_{\alpha\gamma}\\ &= -\tfrac{1}{16}\,\delta^{\tau cd}_{\beta ab}W'^{ab}{}_{\mu\alpha}W'_{cdmn}\delta^{mn\sigma}_{\rho kl}W'^{kl}{}_{\alpha\gamma}\\ &= -\tfrac{1}{16}\,(\delta^\tau_\beta\delta^c_a\delta^d_b - \delta^\tau_\beta\delta^c_b\delta^d_a - \delta^\tau_b\delta^c_a\delta^d_\beta + \delta^\tau_a\delta^c_b\delta^d_\beta + \delta^\tau_b\delta^c_\beta\delta^d_a\end{aligned}$$

$$- \delta^{\tau}_{a}\delta^{c}_{\beta}\delta^{d}_{b}) (\delta^{m}_{\rho}\delta^{n}_{k}\delta^{\sigma}_{l} - \delta^{m}_{\rho}\delta^{n}_{l}\delta^{\sigma}_{k} - \delta^{m}_{l}\delta^{n}_{k}\delta^{\sigma}_{\rho} + \delta^{m}_{k}\delta^{n}_{l}\delta^{\sigma}_{\rho}$$

$$- \delta^{m}_{k}\delta^{n}_{\rho}\delta^{\sigma}_{l} + \delta^{m}_{l}\delta^{n}_{\rho}\delta^{\sigma}_{k}) W^{ab}{}_{\mu\alpha} W_{cdmn} W^{kl}{}_{\alpha\gamma}$$

$$= - \tfrac{1}{8} (W^{ab}{}_{\mu\alpha} W_{abmn} \delta^{\tau}_{\beta} + 2 W^{\tau b}{}_{\mu\alpha} W_{b\beta mn}) (\delta^{m}_{\rho}\delta^{n}_{k}\delta^{\sigma}_{l}$$

$$- \delta^{m}_{\rho}\delta^{n}_{l}\delta^{\sigma}_{k} - \delta^{m}_{l}\delta^{n}_{k}\delta^{\sigma}_{l} + \delta^{m}_{k}\delta^{n}_{l}\delta^{\sigma}_{\rho} - \delta^{m}_{k}\delta^{n}_{\rho}\delta^{\sigma}_{l}$$

$$+ \delta^{m}_{l}\delta^{n}_{\rho}\delta^{\sigma}_{k}) W^{kl}{}_{\alpha\gamma}$$

$$= - \tfrac{1}{4} \delta^{\tau}_{\beta} (2 W^{ab}{}_{\mu\alpha} W_{abk\rho} W^{\sigma k\alpha}{}_{\gamma} + W^{ab}{}_{\mu\alpha} W_{abkl} W^{kl\alpha}{}_{\gamma} \delta^{\sigma}_{\rho})$$

$$- W^{\tau b}{}_{\mu\alpha} W_{b\beta\rho k} W^{k\sigma}{}_{\alpha\gamma} - \tfrac{1}{2} W^{\tau b}{}_{\mu\alpha} W_{b\beta kl} W^{kl}{}_{\alpha\gamma} \delta^{\sigma}_{\rho} \quad (3.1)$$

This is the fundamental expression which will be used to prove the identities we are looking for.

Contracting (3.1) in the $\sigma, \gamma$ indices we obtain

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\tau\lambda\gamma} \overset{*}{W}_{\lambda\rho\alpha\gamma} = - \tfrac{1}{4} W^{ab}{}_{\mu\alpha} W_{abkl} W^{kl\alpha}{}_{\rho} \delta^{\tau}_{\beta} - \tfrac{1}{2} W^{\tau b}{}_{\mu\alpha} W_{b\beta kl} W^{kl}{}_{\alpha\rho} \quad (3.2)$$

Contracting once more, in $\beta, \tau$ we find

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\beta\lambda\gamma} \overset{*}{W}_{\lambda\rho\alpha\gamma} = - W^{ab}{}_{\mu\alpha} W_{abkl} W^{kl\alpha}{}_{\rho} - \tfrac{1}{2} W^{\beta b}{}_{\mu\alpha} W_{b\beta kl} W^{kl}{}_{\alpha\rho} \quad (3.3)$$

Now,

$$W^{\epsilon\beta\lambda\gamma} W_{\lambda\rho\alpha\gamma} = \tfrac{1}{2} W^{\epsilon\beta\lambda\gamma} (W_{\lambda\rho\alpha\gamma} - W_{\gamma\rho\alpha\lambda})$$

$$= \tfrac{1}{2} W^{\epsilon\beta\lambda\gamma} (W_{\lambda\rho\alpha\gamma} + W_{\lambda\alpha\gamma\rho})$$

and the cyclic identity satisfied by the Weyl tensor gives

$$W^{\epsilon\beta\lambda\gamma} W_{\lambda\rho\alpha\gamma} = - \tfrac{1}{2} W^{\epsilon\beta\lambda\gamma} W_{\lambda\gamma\rho\alpha} \quad (3.4)$$

Thus, equation (3.3) turns into

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\beta\lambda\gamma} \overset{*}{W}_{\lambda\gamma\rho\alpha} = - 2 W_{\mu\alpha ab} W^{abkl} W_{kl\rho}{}^{\alpha} + W_{\mu\alpha ab} W^{abkl} W_{kl\rho}{}^{\alpha}$$

or, finally

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\beta\lambda\gamma} \overset{*}{W}_{\lambda\gamma\rho}{}^{\alpha} = - W_{\mu\alpha\epsilon\beta} W^{\epsilon\beta\lambda\gamma} W_{\lambda\gamma\rho}{}^{\alpha}$$

which is a trivial identity.

Let us go back to expression (3.2) and contract it in $\mu, \rho$. We obtain

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\tau\lambda\gamma} \overset{*}{W}_{\lambda}{}^{\mu\alpha}{}_{\gamma} = - \tfrac{1}{4} W^{\rho\sigma}{}_{\mu\alpha} W_{\rho\sigma\epsilon\lambda} W^{\epsilon\lambda\alpha\mu} \delta^{\tau}_{\beta} - \tfrac{1}{2} W^{\tau\rho}{}_{\mu\alpha} W_{\rho\beta\epsilon\lambda} W^{\epsilon\lambda\alpha\mu} \quad (3.5)$$

or, using (3.4),

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\tau\lambda\gamma} \overset{*}{W}_{\lambda\gamma\mu\alpha} = - 8 C \delta^{\tau}_{\beta} + W_{\mu\alpha\tau\epsilon} W^{\epsilon}{}_{\beta\lambda\gamma} W^{\lambda\gamma\alpha\mu} \quad (3.6)$$

or

$$- \overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\epsilon\tau\lambda\gamma} \overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} + W_{\mu\alpha\epsilon\tau} W^{\epsilon}{}_{\beta\lambda\gamma} W^{\lambda\gamma\mu\alpha} = 8 C \delta^{\tau}_{\beta}$$

**Table II.** Dual Identities for the Weyl Tensor (Double Product)

$$W_{\alpha\rho\mu\sigma}W_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{W}_{\alpha\rho\lambda\sigma}\overset{*}{W}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} = Ag_{\lambda\mu}g_{\alpha\beta}$$

$$W^{(\alpha}{}_{\rho\mu\sigma}W^{\beta)\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{W}{}^{(\alpha}{}_{\rho\mu\sigma}\overset{*}{W}{}^{\beta)\rho}{}_{\lambda}{}^{\sigma} = 2Ag_{\lambda\mu}g^{\alpha\beta}$$

$$\overset{*}{W}{}^{\alpha(\mu}{}_{\beta\nu}W_{\alpha}{}^{\epsilon)\beta}{}_{\tau} = Bg^{\mu\epsilon}g_{\nu\tau}$$

$$W_{\alpha\mu\beta\nu}\overset{*}{W}{}^{\alpha\epsilon\beta\rho} = \overset{*}{W}_{\alpha\mu\beta\nu}W^{\alpha\epsilon\beta\rho}$$

$$W_{\alpha\beta\mu}{}^{\nu}W^{\alpha\beta\mu}{}_{\tau} = 2A\delta^{\nu}_{\tau}$$

$$W_{\alpha\beta\mu}{}^{\nu}\overset{*}{W}{}^{\alpha\beta\mu}{}_{\tau} = 2B\delta^{\nu}_{\tau}$$

$$\overset{*}{W}{}^{\alpha\mu}{}_{\beta\nu}W_{\alpha}{}^{\epsilon\beta}{}_{\tau} + W^{\alpha\epsilon}{}_{\beta\nu}\overset{*}{W}_{\alpha}{}^{\mu\beta}{}_{\tau} = Bg^{\mu\epsilon}g_{\nu\tau}$$

Finally, we arrive at the desired identity

$$W_{\mu\alpha\epsilon\beta}W^{\mu\alpha\lambda\gamma}W_{\lambda\gamma}{}^{\epsilon}{}_{\tau} = 4Cg_{\beta\tau} \tag{3.7}$$

This expression has the same formal structure as the Lanczos identity (2.7), and is in this way its generalization. The generalization of (2.15) to the case of a triple product of the Weyl tensor can be obtained along the same lines. Indeed, let us evaluate the expession $\overset{*}{W}_{\mu\alpha\epsilon\beta}W^{\epsilon\tau\lambda\gamma}W_{\lambda\gamma\rho\alpha}$.

We have

$$\begin{aligned}
\overset{*}{W}_{\mu\alpha\epsilon\beta}W^{\epsilon\tau\lambda\gamma}W_{\lambda\gamma\rho\alpha} &= -\overset{*}{W}_{\mu\alpha\epsilon\beta}\overset{**}{W}{}^{\epsilon\tau\lambda\gamma}W_{\lambda\gamma\rho\alpha} \\
&= -\tfrac{1}{4}\,\eta_{\epsilon\beta\lambda\gamma}W^{\lambda\gamma}{}_{\mu\alpha}\eta^{\epsilon\tau\sigma\nu}W_{\sigma\nu\delta\omega}\overset{*}{W}{}^{\delta\omega}{}_{\rho}{}^{\alpha} \\
&= +\tfrac{1}{4}\,\delta^{\tau\sigma\nu}_{\beta\lambda\gamma}W^{\lambda\gamma}{}_{\mu\alpha}W_{\sigma\nu\delta\omega}\overset{*}{W}{}^{\delta\omega}{}_{\rho}{}^{\alpha} \\
&= -\tfrac{1}{2}\,(W_{\mu\alpha\sigma\epsilon}W^{\sigma\epsilon\lambda\gamma}g_{\beta\tau} - 2W_{\mu\alpha\tau\rho}W_{\beta}{}^{\rho\lambda\gamma})\,\overset{*}{W}{}^{\lambda\gamma}{}_{\rho}{}^{\alpha}
\end{aligned} \tag{3.8}$$

Then,

$$\overset{*}{W}_{\mu\alpha\epsilon\beta}W^{\epsilon}{}_{\tau}{}^{\lambda\gamma}W_{\lambda\gamma\rho}{}^{\alpha} + W_{\mu\alpha\epsilon\tau}W^{\epsilon}{}_{\beta}{}^{\lambda\gamma}\overset{*}{W}_{\lambda\gamma\rho}{}^{\alpha} = \tfrac{1}{2}\,W_{\mu\alpha\sigma\epsilon}W^{\sigma\epsilon\lambda\gamma}\overset{*}{W}_{\lambda\gamma\rho}{}^{\alpha}g_{\beta\tau} \tag{3.9}$$

Contracting $\mu, \rho$ in (3.9) we obtain

$$\overset{*}{W}_{\alpha\mu\epsilon\beta}W^{\epsilon}{}_{\tau}{}^{\lambda\gamma}W_{\lambda\gamma}{}^{\mu\alpha} + \overset{*}{W}_{\mu\alpha\epsilon\tau}W^{\epsilon}{}_{\beta}{}^{\lambda\gamma}W_{\lambda\gamma}{}^{\mu\alpha} = 8Dg_{\beta\tau} \tag{3.10}$$

**Table III.** Dual Identities for the Weyl Tensor (Triple Product)

$$W_{\mu\alpha\epsilon\beta}W^{\epsilon}{}_{\tau}{}^{\lambda\gamma}W_{\lambda\gamma}{}^{\mu\alpha} - \overset{*}{W}_{\mu\alpha\epsilon\tau}\overset{*}{W}{}^{\epsilon}{}_{\beta}{}^{\lambda\gamma}W_{\lambda\gamma}{}^{\mu\alpha} = 8Cg_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta}W^{\mu\alpha\lambda\gamma}W_{\lambda\gamma}{}^{\epsilon}{}_{\tau} = 4Cg_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta}\overset{*}{W}{}^{\mu\alpha\lambda\gamma}W_{\lambda\gamma}{}^{\epsilon}{}_{\tau} = 4Dg_{\beta\tau}$$

$$W_{\mu\alpha\epsilon\beta}W^{\epsilon}{}_{\tau}{}^{\lambda\gamma}\overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} + W_{\mu\alpha\epsilon\tau}W^{\epsilon}{}_{\beta}{}^{\lambda\gamma}\overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} = 8Dg_{\beta\tau}$$

or, finally, we obtain the identity

$$\overset{*}{W}_{\mu\alpha\epsilon\beta} W^{\mu\alpha\lambda\gamma} W_{\lambda\gamma}{}^{\epsilon}{}_{\tau} = 4Dg_{\beta\tau} \tag{3.11}$$

We collect all these identities in Table II and Table III for double and triple Weyl products, respectively.

### *References*


1. Lanczos, C. (1938). *Ann. Math.*, 39, 842.
2. Debever, R. (1958). *Bull. Soc. Math. Belgique*, **10**, 112.
3. Bel, L. (1962). *Cah. Phys.*, **16**, 59.
4. Ellis, G. F. R. (1971). *Relativistic Cosomolgy*, Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi", XLVII Corso.