ISSN 0029-3865

CONSELHO NACIONAL DE DESENVOLVIMENTO CIENTÍFICO E TECNOLÓGICO - CNPq

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS - CBPF

Coordenação de Documentação e Informação Científica - CDI

Divisão de Publicações

CBPF-NF-033/83

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Rio de Janeiro
CBPF
1983

CRITICAL PROPERTIES OF THE D=3 BOND-MIXED QUANTUM HEISENBERG FERROMAGNET

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ABSTRACT

Within a Migdal-Kadanoff-like real-space renormalisation group procedure we treat critical properties of the quenched bond-mixed spin-$\frac{1}{2}$ Heisenberg ferromagnet in simple cubic lattice. We verify that it is possible, within a very simple framework, to obtain quite reliable results for the critical temperatures. In addition to that, a general method for renormalising arbitrary clusters of Heisenberg-coupled spins $\frac{1}{2}$ is outlined.

Real-space renormalisation group (RG) techniques are being extensively used to treat criticality (critical frontiers and exponents, specific heat, magnetisation, surface tension, correlation lengths) in all types of magnetic systems. If conveniently formulated, they prove to be an extremely useful tool to discuss a great variety of situations such as quenched or annealed, classical (percolation, Ising, standard and chiral Potts, Z(N), $S\to\infty$ Heisenberg models) or quantum (transverse field Ising, isotropic and anisotropic Heisenberg models), pure or bond- and site-mixed, ferromagnetic, antiferromagnetic and spin-glass systems. In particular, an overall analysis, within a Migdal-Kadanoff-like framework, of critical properties in quenched bond-dilute spin-$\frac{1}{2}$ Heisenberg and Ising magnets is already available (Stinchcombe 1979b). Herein we extend that treatment in order to cover the binary bond-mixed Heisenberg ferromagnetic case in simple cubic lattice. In spite of the simplicity of the procedure, the results (especially those for the critical temperatures) will turn out to be qualitatively (and to a reasonable extent quantitatively) reliable.

Let us consider the dimensionless Hamiltonian

$$\mathcal{H} = \sum_{\langle i,j\rangle} K_{ij}\, \vec{S}_i \cdot \vec{S}_j \qquad (1)$$

where $S_i=1/2$, $\forall i$, the sum runs over all pairs of first-neighbouring sites on a simple cubic lattice, and the factor $(-1/k_BT)$ has been incorporated into the Hamiltonian ($K_{ij}\equiv J_{ij}/k_BT$); furthermore K_{ij} is a random variable associated with the following probability law:

$$P(K_{ij}) = (1-p)\,\delta(K_{ij}-K_1) + p\delta(K_{ij}-K_2) \qquad (0\leq K_1\leq K_2) \tag{2}$$

The particular cases $\alpha\equiv K_1/K_2=1$ and $\alpha=0$ respectively correspond to the pure and diluted situations.

Before establishing the RG recursive relations on which our treatment will be based, let us consider a series array of two bonds with coupling constants K_{12} and K_{23}, respectively connecting the pairs of spins $(\vec{S}_1,\vec{S}_2)$ and $(\vec{S}_2,\vec{S}_3)$ (see Fig. 1.a). The preserval of the partition function under tracing on the intermediate spin $\vec{S}_2$, i.e.

$$e^{K_0' + K_s \vec{S}_1.\vec{S}_3} = \mathrm{Tr}_2\; e^{K_{12}\vec{S}_1.\vec{S}_2 + K_{23}\vec{S}_2.\vec{S}_3} \tag{3}$$

provides (see Appendix):

$$K_s = \frac{1}{4}\,\ell n\,\frac{4\lambda_2 e^{2\lambda_1} + 2\lambda_2\cosh 2\lambda_2 - \lambda_1 \sinh 2\lambda_2}{3(2\lambda_2\cosh 2\lambda_2 + \lambda_1\sinh 2\lambda_2)} \equiv K_s(K_{12},K_{13}) \tag{4}$$

where

$$\lambda_1 \equiv K_{12}+K_{23}$$

$$\lambda_2 \equiv \sqrt{K_{12}^2+K_{23}^2-K_{12}K_{23}}$$

This expression: (i) satisfies $K_s(K_{12},K_{23})=K_s(K_{23},K_{12})$, $\forall\, K_{12}, K_{23}$; (ii) satisfies $K_s(K_{12},0)=0$ for any finite K_{12}; (iii) recovers, for $K_{12}=K_{23}$, Eq. (4) in Stinchcombe 1979b; (iv) provides, in the high temperature limit,

$$K_s \sim K_{12}K_{23} \qquad (K_{12},K_{23}<<1), \tag{5}$$

which coincides with the high temperature limit of the series composition law ($\tanh K_s^{Ising}=\tanh K_1 \tanh K_2$) for spin-$\frac{1}{2}$ Ising couplings (see, for example, Young and Stinchcombe 1976); (v) provides, in the low temperature limit,

$$K_s \sim \frac{1}{2}(K_{12}+K_{23}-\sqrt{K_{12}^2+K_{23}^2-K_{12}K_{23}}) \qquad (K_{12},K_{23}>>1) \tag{6}$$

which can be compared with the low temperature limit series composition law for the classical Heisenberg model (Stinchcombe 1979a)

$$K_s^{class} \sim \frac{K_{12}K_{23}}{K_{12}+K_{23}} \qquad (K_{12},K_{23}>>1) \tag{7}$$

By using both Eqs. (6) and (7) we can verify that K_s/K_s^{class} monotonously increases from 3/4 ($h(S)$ for arbitrary spin size S, where $h(1/2)=3/4$ and $h(\infty)=1$) to 1 while K_{23}/K_{12} increases from 0 to 1.

An interesting quantum effect present in Eq. (4) is the following: while for classical systems such as the spin-$\frac{1}{2}$ Ising, $S\to\infty$ Heisenberg and q-state Potts (see Tsallis and Levy 1981) models, we verify that $\lim_{K_{23}\to\infty} K_s(K_{12},K_{23}) = K_{12}$, $\forall K_{12}$, for the $S=1/2$ Heisenberg case we verify that $\lim_{K_{23}\to\infty} K_s(K_{12},K_{23})=\eta K_{12}$, where η monotonously increases from 2/3 ($g(S)$ in general, with $g(1/2)=2/3$ and $g(\infty)=1$) to 3/4 ($h(S)$ in general) while K_{12} increases from 0 to infinity.

Let us now consider a <u>parallel</u> array of two bonds with coupling constants K_{12} and K_{13} (see Fig. 1.b). It is clear that the equivalent coupling constant is given by

$$K_p = K_{12} + K_{13} \equiv K_p(K_{12},K_{13}) \tag{8}$$

To construct the RG we follow along the lines of Stinchcombe 1979b by using the Migdal-Kadanoff procedure (Migdal 1976, Kadanoff 1976) which combines decimation and bond shifting as illustrated in Fig. 2 for an arbitrary set of coupling constants. Consequently the use of algorithms (4) and (8) leads to

$$\tilde{K} = K_s(K_1,K_2) + K_s(K_3,K_4) + K_s(K_5,K_6) + K_s(K_7,K_8) \tag{9}$$

which is the central formula for constructing the RG. If to every bond of Fig. 2.a we associate the probability law (2), then the renormalised bonds (indicated in Fig. 2.c) are to be associated with

$$P_G(K_{ij}) = \sum_{r=1}^{15} M_r (1-p)^{8-n_r} p^{n_r} \, \delta(K_{ij}-\tilde{K}_r) \tag{10}$$

where the exponents $\{n_r\}$, weights $\{M_r\}$ and coupling constants $\{\tilde{K}_r\}$ are given in Table I. It is clear that the present operations do not preserve the binary form of the initial distribution (2). It is possible, in principle, to follow the evolution of the distribution along the successive renormalisations. However a simpler, and nevertheless reasonable (see Stinchcombe 1979b, Levy et al 1980 and references therein) procedure can be adopted, namely to approximate distribution (10) by the following binary one:

$$P'(K_{ij}) = (1-p')\,\delta(K_{ij}-K_1') + p'\delta(K_{ij}-K_2') \qquad (11)$$

where p', K_1' and K_2' are chosen to preserve three convenient moments. By naturally extending Stinchcombe 1979b we impose

$$p' = 1-(1-p^2)^4 \qquad (\text{"zeroth" moment}) \qquad (12.a)$$

$$\langle K_{ij}\rangle_{P'} = \langle K_{ij}\rangle_{P_G} \qquad (\text{first moment}) \qquad (12.b)$$

$$\langle K_{ij}^2\rangle_{P'} = \langle K_{ij}^2\rangle_{P_G} \qquad (\text{second moment}) \qquad (12.c)$$

which completely determine the RG recursive relations in the (p,K_1,K_2)-space. The results are indicated in Fig. 3. The two physically distinct non trivial fixed points are the pure Heisenberg (located at $K_c=0.3439$, to be compared with the series result 0.30 (Ritchie and Fisher 1972)) and the bond percolation (located at $p_c = 0.2818$, to be compared with the series result 0.25 (Sykes et al 1976)) ones. The present values for K_c and p_c as well as the associated correlation length critical exponents ($\nu_T \cong 1.40$ and $\nu_p \cong 1.23$) reproduce those appearing in Stinchcombe 1979b. Within the critical surface (see Fig. 3.a), the RG flow is towards the pure Heisenberg fixed point, i.e. the criticality of the whole surface (excepting of course the strict percolation point) coincides with that of the pure case. However one should be precautious on these grounds as Eq. (12.a) strongly (and to a certain extent artificially) drives in that direction. Some other anomalies are present in the RG determined by Eqs. (12), such as small unphysical bumps in the RG flow in the neighbourhood of the percolation point. These and other minor points can be improved by (i) matching, instead of the "zeroth", first and second moments of K_{ij}, the first, second and third moments of a quantity $f(K_{ij})$ to be defined and satisfying $f(0)=0$ and $f(\infty)=1$; and (ii) using bigger and/or more convenient clusters (this is in principle feasible by following the general procedure outlined in the Appendix). This type of more sophisticated RG's have proved to be possible in other occasions (see, for instance, Levy et al 1980 for the Ising model); work along this line will be published elsewhere.

Let us conclude by stressing that, without introducing great mathematical complexities, the present treatment has provided reliable results concerning the critical temperatures (see Fig. 3.b) of a bond-mixed quantum Heisenberg ferromagnet in three-dimensional lattice (in particular the errors in the pure Heisenberg and bond percolation critical points are less than 15% and 13% respectively).

Two of us (CT and RBS) are indebted to R J Elliott for interesting remarks. CT also acknowledges relevant comments and valuable computational assistance from A O Caride, G Schwachheim and E M F Curado; he benefited as well from warm hospitality at the Theoretical Physics Department/Oxford and from the financial support of a Guggenheim Fellowship.

Appendix

Let us consider the array of bonds appearing in Fig. 1.c. Its dimensionless Hamiltonian is given by

$$\mathcal{H}_{123} = K_{12}\vec{S}_1.\vec{S}_2 + K_{23}\vec{S}_2.\vec{S}_3 + K_{13}\vec{S}_1.\vec{S}_3 \tag{A.1}$$

and can be rewritten as follows:

$$\mathcal{H}_{123} = 2\,H_{123} - (K_{12}+K_{23}+K_{13}) \tag{A.2}$$

where

$$H_{123} \equiv K_{12}H_{12} + K_{23}H_{23} + K_{13}K_{13} \tag{A.3}$$

with

$$H_{ij} \equiv \frac{1}{2}(1 + \vec{S}_i.\vec{S}_j) \tag{A.4}$$

For spins 1/2, H_{ij} is the exchange operator (it exchanges the states of the i-th and j-th spins) and its eigenvalues are ± 1.

$\mathcal{H}_{123}$ commutates with both S^2 and S^z where $\vec{S} \equiv \vec{S}_1+\vec{S}_2+\vec{S}_3$, therefore J and M are good quantum numbers (J=3/2, 1/2 and M=J,J-1,...,-J). The representation associated with J=3/2 appears only once and generates 4 projections ($M=\pm 3/2,\pm 1/2$); the representation associated with J=1/2 appears twice, and each of them generates 2 projections ($M=\pm 1/2$); hence the full representation is associated with $1\text{x}4+2\text{x}2=2^3$ states. In the present pro cedure we shall always work on the states subspace associated with M equal to the minimal value of J (M=1/2 for an odd number n of spins, and M=0 for an even number of spins) because every state is contained within. For the present n=3 case, J=3/2 contributes with a 1x1 matrix, and J=1/2 with a 2x2 one; as a whole we will have to deal with a 3x3 matrix. For the general case with n spins connected through n(n-1)/2 two-body coupling constants $\{K_{ij}\}$, the array will be a (n-1)-dimensional hypertetrahedron (the 2-dimensional hypertetrahedron being the triangle), and the Hamiltonian $\mathcal{H}_{12\ldots n}$ will commutate with S^2 and S^z where $\vec{S}\equiv\sum_{i=1}^{n}\vec{S}_i$ (J=n/2, n/2-1,..., 1/2 or 0; M=J, J-1,...,-J). The representation associated with a particular value of J appears N(n,J) times with

$$N(n,J) = \binom{n}{\frac{n}{2}-J}\frac{2(2J+1)}{n+2(J+1)} \qquad (A.5)$$

and each of them generates (2J+1) projections. We can verify that

$$\sum_{J=\frac{n}{2},\frac{n}{2}-1,\ldots}(2J+1)\,N(n,J) = 2^n \qquad (A.6)$$

Within the states subspace we mentioned before, a particular value of J contributes with a N(n,J) x N(n,J) matrix; consequently, we have to deal as a whole with a $\overline{N}\times\overline{N}$ matrix where

$$\overline{N}\equiv\sum_{J=\frac{n}{2},\frac{n}{2}-1,\ldots}N(n,J) = \begin{cases} n!/(\frac{n}{2}!)^2 & (n\text{ even}) \\ n!/(\frac{n+1}{2})!\,(\frac{n-1}{2})! & (n\text{ odd}) \end{cases} \qquad (A.7)$$

Let us go back to our present n=3 case, and introduce the notation $|S_1^z, S_2^z, S_3^z>$ to denote the states of the array. As we said before, it suffices to work in the subspace associated with M=1/2 and introduce

$$\phi_1 \equiv |\downarrow \uparrow \uparrow>$$

$$\phi_2 \equiv |\uparrow \downarrow \uparrow>$$

$$\phi_3 \equiv |\uparrow \uparrow \downarrow>$$

(if we had n spins, we would have to introduce $\bar{N}$ ϕ's). It is straightfoward to verify the following relations:

$$\begin{aligned} H_{12}\phi_1 &= \phi_2 \\ H_{12}\phi_2 &= \phi_1 \\ H_{12}\phi_3 &= \phi_3 \end{aligned} \qquad \text{(A.8)}$$

hence

$$H_{12}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_2 \\ e_1 \\ e_3 \end{pmatrix} \qquad (\forall\ e_1, e_2, e_3) \qquad \text{(A.8')}$$

Analogously we obtain

$$H_{23}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_3 \\ e_2 \end{pmatrix} \quad \text{and} \quad H_{13}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_2 \\ e_1 \end{pmatrix}$$

We can immediately verify that the symmetric form $\phi \equiv \phi_1+\phi_2+\phi_3$ satisfies

$$H_{123}\phi = \lambda_1\phi \tag{A.9}$$

$$\lambda_1 \equiv K_{12}+K_{23}+K_{13} \tag{A.9'}$$

and corresponds, assuming all $\{K_{ij}\}$ are positive, to the fundamental state (in general $\phi\equiv\sum_{k=1}^{\bar{N}}\phi_k$ satisfies $H_{12\ldots n}\phi=\lambda_1\phi$, with $\lambda_1\equiv\sum_{(ij)}K_{ij}$ where (ij) runs over the n(n-1)/2 couplings and where $H_{12\ldots n}\equiv\sum_{(ij)}K_{ij}H_{ij}$). The matrix $L\equiv\langle\phi_k|H_{12\ldots n}|\phi_{k'}\rangle=\sum_{(ij)}K_{ij}\langle\phi_k|H_{ij}|\phi_{k'}\rangle$ is given, in our n=3 case, by

$$L = \begin{pmatrix} K_{23} & K_{12} & K_{13} \\ K_{12} & K_{13} & K_{23} \\ K_{13} & K_{23} & K_{12} \end{pmatrix} \tag{A.10}$$

which is symmetric (hence the eigenvalues are real) and presents a cyclic structure. Its eigenvalues are λ_1 given by Eq. (A.9'), λ_2 given by

$$\lambda_2 = \sqrt{K_{12}^2+K_{23}^2+K_{13}^2-K_{12}K_{23}-K_{23}K_{13}-K_{13}K_{12}} \tag{A.11}$$

and $\lambda_3=-\lambda_2$. The corresponding eigenvectors are respectively

$$|\psi_1\rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \equiv \phi \tag{A.12.a}$$

$$|\psi_2\rangle \propto \begin{pmatrix} K_{12}-K_{13} \\ K_{13}-K_{23}+\lambda_2 \\ K_{23}-K_{12}-\lambda_2 \end{pmatrix} \qquad \text{(A.12.b)}$$

$$|\psi_3\rangle \propto \begin{pmatrix} K_{12}-K_{13} \\ K_{13}-K_{23}-\lambda_2 \\ K_{23}-K_{12}+\lambda_2 \end{pmatrix} \qquad \text{(A.12.c)}$$

Through Eq. (A.2) we see that the eigenvectors of $\mathcal{H}_{123}$ are those of H_{123} and its eigenvalues are given by

$$\varepsilon_1 = \lambda_1 \qquad \text{(A.13.a)}$$

$$\varepsilon_2 = -\lambda_1 + 2\lambda_2 \qquad \text{(A.13.b)}$$

$$\varepsilon_3 = -\lambda_1 - 2\lambda_2 \qquad \text{(A.13.c)}$$

(in general it will be $\varepsilon_1=\lambda_1\equiv \sum_{(ij)} K_{ij}$ and $\varepsilon_k=-\lambda_1+2\lambda_k$ $(k=2,3,\ldots,\bar{N})$).

Now that we have diagonalized $\mathcal{H}_{123}$ let us deal with $e^{\mathcal{H}_{123}}$. The expansion of this quantity leads to

$$e^{\mathcal{H}_{123}} = a_o + a_{12}\,\vec{S}_1.\vec{S}_2 + a_{23}\,\vec{S}_2.\vec{S}_3 + a_{13}\,\vec{S}_1.\vec{S}_3 \qquad \text{(A.14)}$$

where the a's are coefficients to be found as functions of $\{K_{ij}\}$. If we had n spins, the expansion would be

$$e^{\mathcal{H}_{12\ldots n}} = a_o + \sum_{(ij)} a_{ij}\, \vec{S}_i . \vec{S}_j + \sum_{(ij)\neq(k\ell)} a_{ij,k\ell} (\vec{S}_i . \vec{S}_j)(\vec{S}_k . \vec{S}_\ell)$$

$$+ \sum_{(ij)\neq(k\ell)\neq(m,n)} a_{ij,k\ell,mn} (\vec{S}_i . \vec{S}_j)(\vec{S}_k . \vec{S}_\ell)(\vec{S}_m . \vec{S}_n) + \ldots$$

For example

$$e^{\mathcal{H}_{1234}} = a_o + a_{12}\, \vec{S}_1 . \vec{S}_2 + a_{13}\, \vec{S}_1 . \vec{S}_3 + a_{14}\, \vec{S}_1 . \vec{S}_4 + a_{23}\, \vec{S}_2 . \vec{S}_3$$

$$+ a_{24}\, \vec{S}_2 . \vec{S}_4 + a_{34}\, \vec{S}_3 . \vec{S}_4 + a_{12,34} (\vec{S}_1 . \vec{S}_2)(\vec{S}_3 . \vec{S}_4)$$

$$+ a_{13,24} (\vec{S}_1 . \vec{S}_3)(\vec{S}_2 . \vec{S}_4) + a_{14,23} (\vec{S}_1 . \vec{S}_4)(\vec{S}_2 . \vec{S}_3) \tag{A.14"}$$

Let us go back to our n=3 case and rewrite Eq. (A.14) as follows:

$$e^{\mathcal{H}_{123}} = (a_o - a_{12} - a_{23} - a_{13}) + 2(a_{12}H_{12} + a_{23}H_{23} + a_{13}H_{13}) \tag{A.14'''}$$

where we have used definition (A.4). We intend to establish now 4 equations in order to determine a_o, a_{12}, a_{13} and a_{23}. By applying Tr on both sides of equality (A.14) we obtain

$$4\, e^{\lambda_1} + 2\, e^{-\lambda_1 + 2\lambda_2} + 2\, e^{-\lambda_1 - 2\lambda_2} = 8a_o$$

hence

$$a_o = \frac{1}{2}(e^{\lambda_1} + e^{-\lambda_1}\cosh 2\lambda_2) \tag{A.15}$$

Furthermore $[e^{\mathcal{H}_{123}}, \mathcal{H}_{123}]=0$, therefore (by using Eqs. (A.2) and (A.14'''))

$$a_{12}K_{23} - a_{23}K_{12} + a_{23}K_{13} - a_{13}K_{23} + a_{13}K_{12} - a_{12}K_{13} = 0 \tag{A.16}$$

where we have used the fact that $[H_{12},H_{23}]=[H_{23},H_{13}]=[H_{13},H_{12}]\neq 0$. We can also apply $<\phi|\ldots|\phi>$ on both sides of Eq. (A.14''') and obtain

$$e^{\lambda_1} = a_o + a_{12} + a_{23} + a_{13} \tag{A.17}$$

Finally we apply both sides of Eq. (A.14''') on any eigenvector different from ϕ, let us say $|\psi_2>$, and obtain equalities between the three components of both sides. We choose any of them, let us say the first one, and obtain

$$e^{-\lambda_1+2\lambda_2}(K_{12}-K_{13}) = (a_o-a_{12}+a_{23}-a_{13})(K_{12}-K_{13})$$
$$+ 2a_{13}(K_{13}-K_{23}+\lambda_2) + 2a_{13}(K_{23}-K_{12}-\lambda_2) \tag{A.18}$$

All the other equalities are automatically satisfied. Eq. (A.15) gives a_o, and Eqs. (A.16)-(A.18) completely determine the $\{a_{ij}\}$ as functions of the $\{K_{ij}\}$. For the particular case $K_{13}=0$ the solutions are given by Eq. (A.15) and

$$a_{12} = \frac{1}{6}(e^{\lambda_1} - e^{-\lambda_1}\cosh 2\lambda_2) + \frac{2K_{12}-K_{23}}{6\lambda_2} e^{-\lambda_1}\sinh 2\lambda_2 \tag{A.19a}$$

$$a_{23} = \frac{1}{6}(e^{\lambda_1} - e^{-\lambda_1}\cosh 2\lambda_2) + \frac{2K_{23}-K_{12}}{6\lambda_2} e^{-\lambda_1}\sinh 2\lambda_2 \qquad \text{(A.19.b)}$$

$$a_{13} = \frac{1}{6}(e^{\lambda_1} - e^{-\lambda_1}\cosh 2\lambda_2) - \frac{\lambda_1}{6\lambda_2} e^{-\lambda_1}\sinh 2\lambda_2 \qquad \text{(A.19.c)}$$

with

$$\lambda_1 = K_{12} + K_{23} \qquad \text{(A.20.a)}$$

$$\lambda_2 = \sqrt{K_{12}^2 + K_{23}^2 - K_{12}K_{23}} \qquad \text{(A.20.b)}$$

If we apply now Tr_2 on both sides of Eq. (A.14) we obtain

$$\mathrm{Tr}_2\, e^{\mathcal{H}_{123}} = 2(a_o + a_{13}\,\vec{S}_1.\vec{S}_3)$$

$$= 2(a_o - a_{13}) + 4\,a_{13}\,H_{13} \qquad \text{(A.21)}$$

where we have used definition (A.4). In the case of n spins, this equation is generalized into

$$\mathrm{Tr}_{2,3,\ldots n-1}\, e^{\mathcal{H}_{12\ldots n}} = 2^{n-2}(a_o + a_{13}\,\vec{S}_1.\vec{S}_3)$$

$$= 2^{n-2}(a_o - a_{13}) + 2^{n-1}a_{13}\,H_{13} \qquad \text{(A.21')}$$

where we have used expansion (A.14').

Let us now introduce the Hamiltonian

$$\mathcal{H}'_{13} = K_o' + K_{13}'\,\vec{S}_1.\vec{S}_3 \qquad \text{(A.22)}$$

which satisfies

$$e^{\mathcal{H}_{13}'} = \mathrm{Tr}_{2,3,\ldots,n-1}\; e^{\mathcal{H}_{12\ldots n}} \tag{A.23}$$

This equality can be rewritten (through use of Eqs. (A.4) and (A.21')) as follows:

$$e^{(K_o'-K_{13}') + 2K_{13}'H_{13}} = 2^{n-2}(a_o-a_{13}) + 2^{n-1}a_{13}H_{13} \tag{A.24}$$

which implies

$$e^{K_o' + K_{13}'} = 2^{n-2}(a_o+a_{13}) \tag{A.25.a}$$

$$e^{K_o' - 3K_{13}'} = 2^{n-2}(a_o-3a_{13}) \tag{A.25.b}$$

where we have used the fact that the eigenvalues of H_{13} are ± 1. Eqs. (A.25) lead to

$$e^{4K_o'} = 2^{4(n-2)}(a_o+a_{13})^3(a_o-3a_{13}) \tag{A.26.a}$$

$$e^{4K_{13}'} = \frac{a_o+a_{13}}{a_o-3a_{13}} \tag{A.26.b}$$

which close the problem. For the particular case n=3 and $K_{13}=0$ we can replace Eqs. (A.15) and (A.19.c) into Eqs. (A.26) and obtain

$$e^{4K_o'} = \frac{e^{-\lambda_1}}{81\,\lambda_2^4}\,(2\lambda_2 \cosh 2\lambda_2+\lambda_1 \sinh 2\lambda_2)$$

$$x(4\lambda_2 e^{\lambda_1}+2\lambda_2 e^{-\lambda_1}\cosh 2\lambda_2-\lambda_1 e^{-\lambda_1}\sinh 2\lambda_2)^3 \qquad \text{(A.27.a)}$$

and

$$e^{4K'_{13}} = \frac{4\lambda_2 e^{2\lambda_1}+2\lambda_2\cosh 2\lambda_2-\lambda_1\sinh 2\lambda_2}{3(2\lambda_2\cosh 2\lambda_2+\lambda_1\sinh 2\lambda_2)} \qquad \text{(A.27.b)}$$

This last expression provides, by identifying $K'_{13} \equiv K_s$, Eq. (4) of the body of the present paper.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - Simple arrays of spins 1/2 interacting through the Heisenberg coupling constants $\{K_{ij}\}$ and their equivalent bonds (O and ● respectively denote terminal and internal nodes; the partition function is preserved through tracing on the internal nodes).

Fig. 2 - Migdal-Kadanoff-like b=2 scaling for the simple-cubic lattice (we have ommited to indicate in (b) and (c) the constants K_3, K_4, K_5 and K_6).

Fig. 3 - Critical surface separating the para(P)- and ferro(F)-magnetic phases (● denotes the bond percolation fixed point). (a) This figure is only indicative and the axes $K_i/(K_i+1)$ have been freely chosen for the surface to be contained in the unitary cube; all three straight segments $K_1=K_2$, $\forall p$, $K_2=K_c$, $p=1$, $\forall K_1$, and $K_1=K_c$, $p=0$, $\forall K_2$, correspond to the pure Heisenberg fixed point, we recall that the present RG has been devised to describe only the region $K_2 \geq K_1$ (the region $K_2<K_1$ can be obtained through the transformation $(p,K_1,K_2) \rightleftarrows (1-p,K_2,K_1)$. (b) Critical lines (in scale) associated with typical values of $\alpha \equiv K_1/K_2=J_1/J_2$ ($\alpha=1$ corresponds to the pure Heisenberg fixed point).

Table I- Elements of the renormalised distribution law $P_G(K_{ij})$ (Eq.(10) of the text).

r	n_r	M_r	$\tilde{K}_r$
1	0	1	$4\ K_s(K_1,K_1)$
2	1	8	$3\ K_s(K_1,K_1) + K_s(K_1,K_2)$
3	2	4	$3\ K_s(K_1,K_1) + K_s(K_2,K_2)$
4	2	24	$2\ K_s(K_1,K_1) + 2\ K_s(K_1,K_2)$
5	3	24	$2\ K_s(K_1,K_1) + K_s(K_1,K_2) + K_s(K_2,K_2)$
6	3	32	$K_s(K_1,K_1) + 3\ K_s(K_1,K_2)$
7	4	6	$2\ K_s(K_1,K_1) + 2\ K_s(K_2,K_2)$
8	4	48	$K_s(K_1,K_1) + 2\ K_s(K_1,K_2) + K_s(K_2,K_2)$
9	4	16	$4\ K_s(K_1,K_2)$
10	5	32	$3\ K_s(K_1,K_2) + K_s(K_2,K_2)$
11	5	24	$K_s(K_1,K_1) + K_s(K_1,K_2) + 2\ K_s(K_2,K_2)$
12	6	24	$2\ K_s(K_2,K_2) + 2\ K_s(K_1,K_2)$
13	6	4	$K_s(K_1,K_1) + 3\ K_s(K_2,K_2)$
14	7	8	$K_s(K_1,K_2) + 3\ K_s(K_2,K_2)$
15	8	1	$4\ K_s(K_2,K_2)$

TABLE I

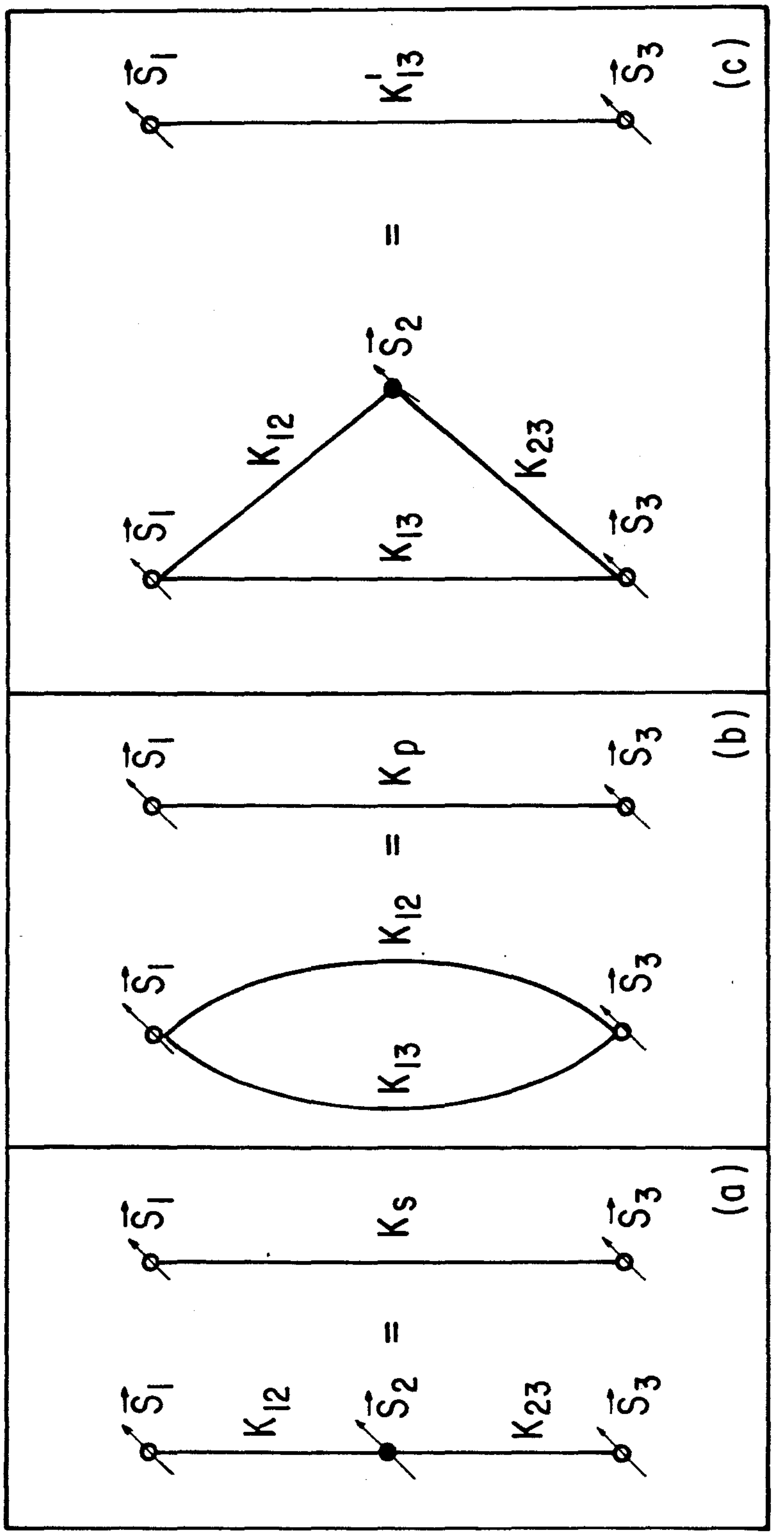

(a)
$\vec{S}_1$ K_{12} $\vec{S}_2$ K_{23} $\vec{S}_3$ = $\vec{S}_1$ K_S $\vec{S}_3$
(b)
$\vec{S}_1$ K_{12} K_{13} $\vec{S}_3$ = $\vec{S}_1$ K_p $\vec{S}_3$
(c)
$\vec{S}_1$ K_{12} $\vec{S}_2$ K_{23} $\vec{S}_3$ K_{13} = $\vec{S}_1$ K'_{13} $\vec{S}_3$

FIG.1

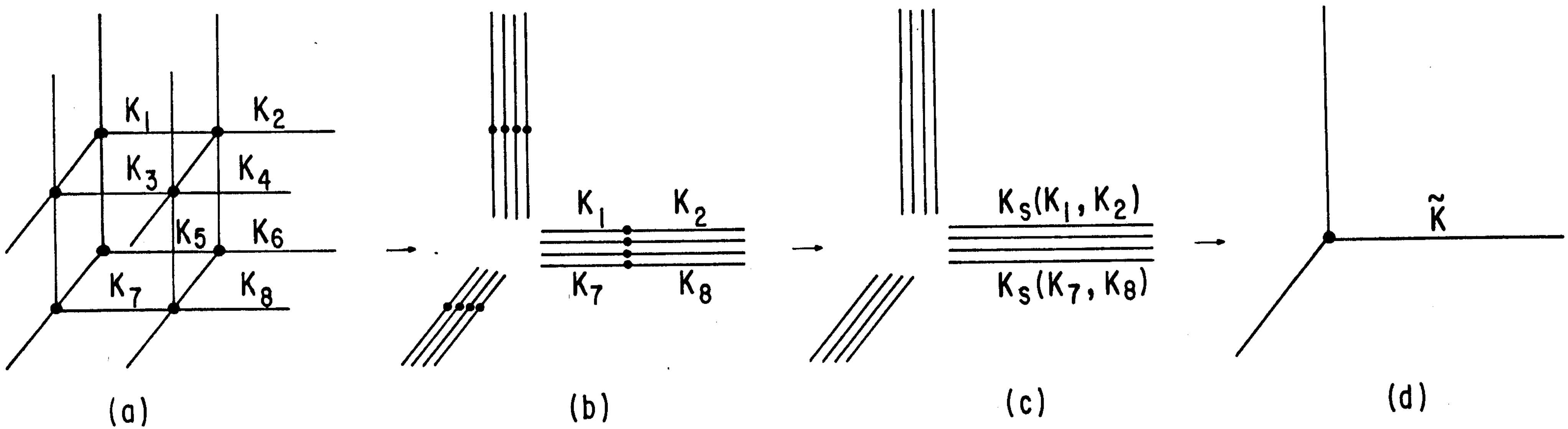
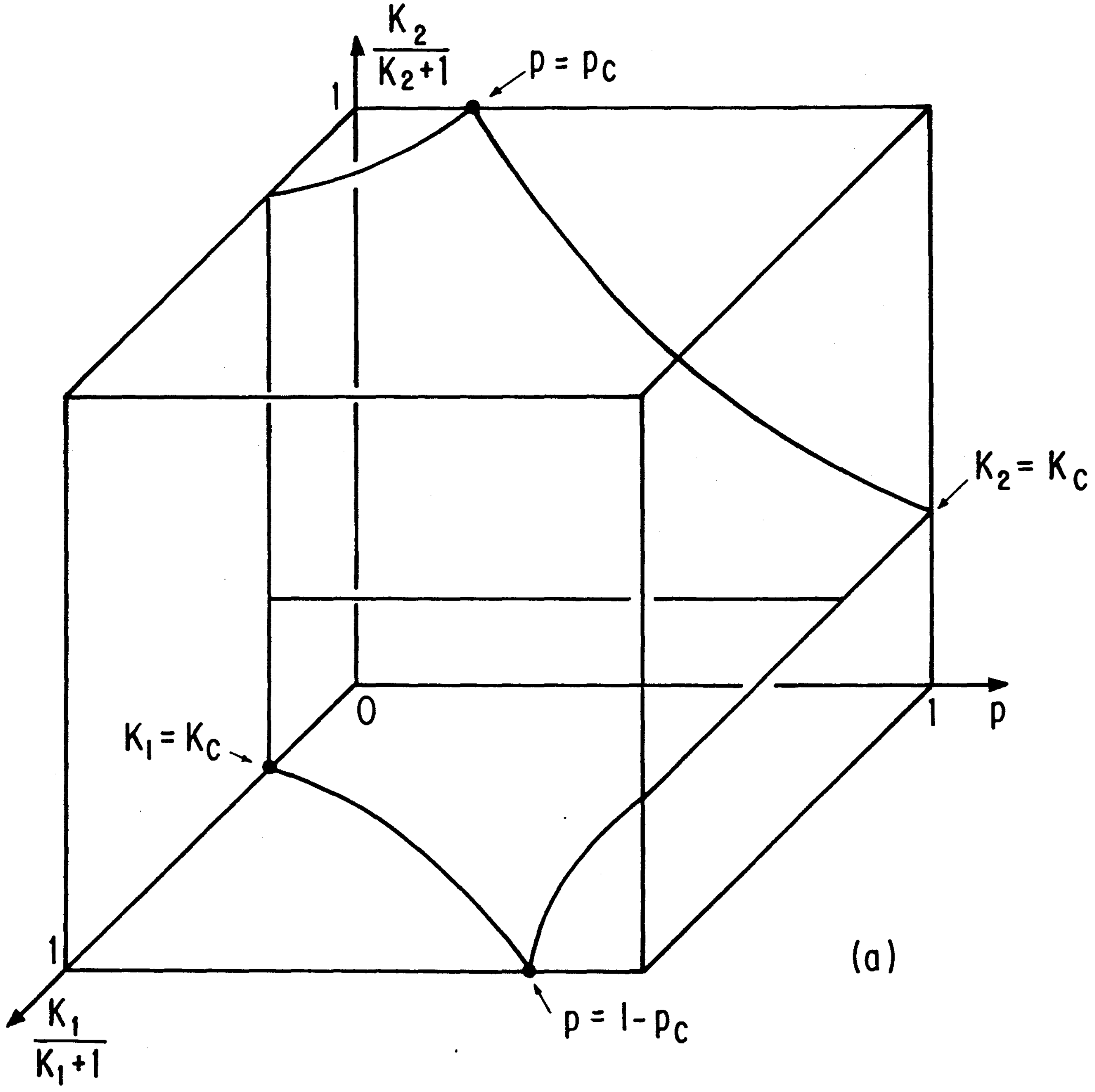


FIG. 2



$\frac{K_2}{K_2+1}$
$p = p_c$
1
$K_2 = K_c$
$K_1 = K_c$
0
1
p
1
(a)
$p = 1 - p_c$
$\frac{K_1}{K_1+1}$

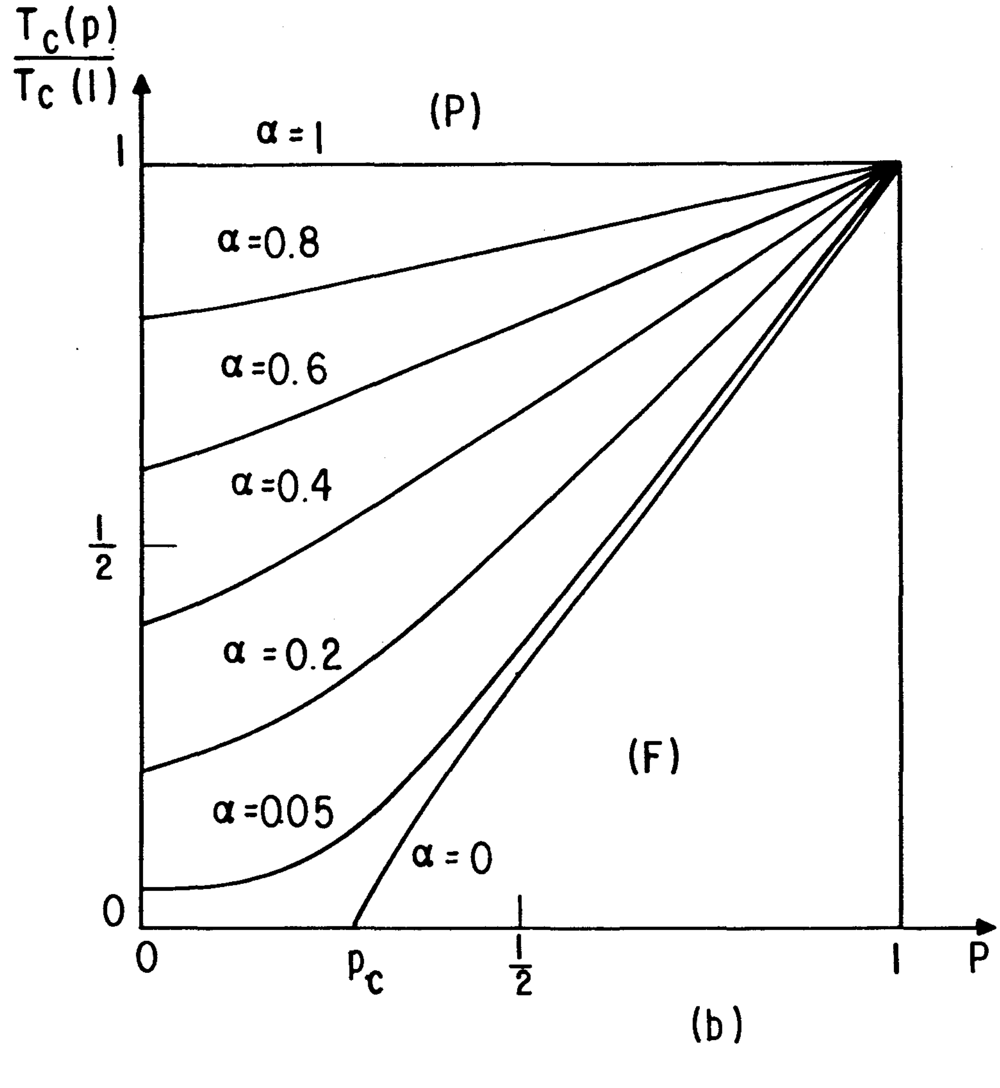


FIG. 3