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ABSTRACT

It is shown that the potential (and field) of a non-abelian gauge theory is not well determined when it has a singular point. When this is the case, it is important to specify the regularization procedure used to give a precise definition of physical quantities at the singularity at any stage of the computation. We discuss the fact that a certain A_μ (associated with the given regularization) represents the vacuum when $F_{\mu\nu}$ is a zero distribution not only on the global space but also in all its projectionsto arbitrary subspaces.The example used as a base for the discussion is $\underset{\sim}{A} = i \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2}$. For this example we show that different regularizations give the same field in the global space but they give different distributions when projected to subspaces containing the singular point.

1- INTRODUCTION

We have shown in a previous paper [1] that whenever the potential of a non-abelian gauge field is singular in a two dimensional surface, the field itself is not well defined unless a regularization procedure is given. Different regularization schemes may lead to pysically different potentials and fields.

A similar situation appears when the singular surface shrinks to a point. Although in this case we will show that the ambigüities are hidden in subspaces of lesser number of dimensions.

For example, it is usually considered in gauge theories (and for sound reasons), that the potential

(1) $$A_\mu = g^{-1}\, \partial_\mu\, g \qquad (g = g(x) \text{ is a group element})$$

corresponds to a null field and consequently it represents the vacuum. However this is not strictly true when g has singular points, which is the case, for instance, when

(2) $$g = e^{-i\, \mathrm{arcotg}\, \frac{t}{r}\, \sigma_r} = \frac{1}{x}\, (t - i\underset{\sim}{\sigma} \cdot \underset{\sim}{r}),\ x^2 = r^2+t^2,$$
$$\sigma_r = \frac{1}{r}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{r} \quad .$$

The corresponding vacuum potential (1) is :

(3) $$A_\mu = -2i\, \frac{\sigma_{\mu\nu} x_\nu}{x^2} \quad ; \qquad (x_4 \equiv t)$$

with

(4) $$\sigma_{ij} = \frac{1}{2}\, \varepsilon_{ijk}\, \sigma_k\ ,\ \sigma_{i4} = \frac{1}{2}\, \sigma_i .$$

The potential (3) has a point singularity which is related to the lack of definition of σ_r at the origin (cf. (2)).

We shall discuss the fact that as a consequence of this singularity, the resulting field $F_{\mu\nu}$ cannot, strictly speaking (see also Reference [2], §III) be considered to represent the vacuum unless the singularity at the origin is conveniently dealt with.

In order to simplify the discussion, we shall take the three dimensional projection ($t \to 0$) of (2), namely:

$$(5) \qquad g = e^{-i\frac{\pi}{2}\sigma_r} = -i\,\sigma_r \ ,$$

which gives the "vacuum" potential:

$$(6) \qquad \underset{\sim}{A} = \sigma_r \underset{\sim}{\nabla} \sigma_r = i\,\frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2} \ .$$

We shall show in different ways, that it is possible to exhibit a peculiar singularity of the field at the origin. In §2 we do it through naïve application of Stokes theorem. In §3 and 4 we extend the discussion applying methods of distribution theory, in particular analytic regularization. In §5 we use another way to regularize, which we call "instanton-like" method. Neither of the regularizations presented in the above mentioned sections give a true vacuum. In §6 we study a regularized version of (6) which is a true vacuum potential. The implication of these results for the four dimensional fields (2) and (3) is obvious, as (5) and (6) are respectively their values in the hyperplane t=0.

2- **Use of Stokes Theorem**

If we compute $\underset{\sim}{\nabla} \wedge \underset{\sim}{A}$ and $\underset{\sim}{A} \wedge \underset{\sim}{A}$ from (6), outside the origin, we find:

$$(7) \qquad \underset{\sim}{F} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A} + \underset{\sim}{A} \wedge \underset{\sim}{A} = 0 \qquad\qquad \underset{\sim}{r} \neq 0 \ .$$

But both $\underset{\sim}{\nabla} \wedge \underset{\sim}{A}$ and $\underset{\sim}{A} \wedge \underset{\sim}{A}$ present problems at the origin which require further discussion.

Let us first take the vector field:

(8) $$\underset{\sim}{B} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A}.$$

If we consider a circle of radius R in the Z=0 plane, with center at the origin, the flux of $\underset{\sim}{B}$ through that surface is given by:

(9) $$\int \underset{\sim}{B} \cdot \underset{\sim}{ds} = \int (\underset{\sim}{\nabla} \wedge \underset{\sim}{A})_3 \, ds = \oint \underset{\sim}{A} \cdot \underset{\sim}{dl}$$

where the last step follows from a naïve application of Stokes theorem and the line integral is extended to the circunference of radius R.

It is easy to see that

$$\underset{\sim}{A} \cdot \underset{\sim}{dl} = i \, \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2} \cdot r \, \underset{\sim}{d\phi} = i \, \sigma_3 \, d\phi$$

Then, from (9)

(10) $$\int \underset{\sim}{B} \cdot \underset{\sim}{ds} = i \, \sigma_3 \oint d\phi = 2\pi \, i \, \sigma_3 \quad .$$

Equation (10) shows that when $\underset{\sim}{B}$ is considered as a distribution in a two dimensional subspace, it has a δ - type singularity at the origin, which is not obvious when looking at $\underset{\sim}{B}$ as a three dimensional function.

Let us now consider the term $\underset{\sim}{A} \wedge \underset{\sim}{A}$. Formally, we could say that we have

$$\underset{\sim}{A} \wedge \underset{\sim}{A} = -2i \, \frac{\sigma_r}{r^3} \, \underset{\sim}{r} \quad \text{and}$$

(11) $$(\underset{\sim}{A} \wedge \underset{\sim}{A})_3 = 0 \qquad (Z=0) \quad .$$

So $\underset{\sim}{A} \wedge \underset{\sim}{A}$ will not contribute to the flux of $\underset{\sim}{F}$ across the above mentioned circle. On the other hand, in the plane Z=0, the product of

(6) times itself is not well defined (in two dimensions) according to the rules of generalized functions.

We see that unless $\underset{\sim}{A} \wedge \underset{\sim}{A}$ has a δ-singularity at the origin, we cannot compensate the singularity in $\underset{\sim}{\nabla} \wedge \underset{\sim}{A}$ and F does not vanish everywhere.

So, the whole problem has to be looked at from the point of view of distribution theory[3] not only in the global space, but also in projections to smaller number of dimensions.

3- Distribution methods

The use of Stoke's theorem in the precedent paragraph is not canonically correct, as the integrand is singular at the origin. So, we shall compute the third component in the plane Z=0 using the rules for derivatives of distributions.

Let $\phi(x,y)$ be a trial function in the plane z=0. The curl of the distribution $\underset{\sim}{A}$ is defined by[3].

$$(12) \qquad (\underset{\sim}{\nabla} \wedge \underset{\sim}{A}, \phi) = (\underset{\sim}{A} \wedge, \underset{\sim}{\nabla} \phi) = \int \underset{\sim}{A} \wedge \underset{\sim}{\nabla} \phi \, dx \, dy$$

and we have, specifically for the 3rd component.

$$\left((\underset{\sim}{\nabla} \wedge \underset{\sim}{A})_3, \phi\right) = (\underset{\sim}{A} \wedge, \underset{\sim}{\nabla}\phi)_3 = i \int \left[\frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2} \wedge \underset{\sim}{\nabla} \phi\right] dx \, dy$$

$$\left((\underset{\sim}{\nabla} \wedge \underset{\sim}{A})_3, \phi\right) = i \int dx \, dy \frac{1}{r^2} \left[\underset{\sim}{r} (\underset{\sim}{\sigma} \cdot \underset{\sim}{\nabla}) \phi - \underset{\sim}{\sigma} (\underset{\sim}{r} \cdot \underset{\sim}{\nabla}) \phi\right]$$

$$= -i\sigma_3 \int dx \, dy \frac{\underset{\sim}{r} \cdot \underset{\sim}{\nabla} \phi}{r^2} \quad .$$

as in Z=0 the term proportional to $\underset{\sim}{r}$ has its third component equal to zero. The last integral contains only the radial derivative of the trial function. Using, for simplicity, axially simmetric ϕ, we have (polar coordinates ρ, φ)

$$(13) \qquad \left((\underset{\sim}{\nabla} \wedge \underset{\sim}{A})_3,\phi\right) = -2\pi\, i\, \sigma_3 \int_0^\infty \rho\, d\rho\, \frac{1}{\rho} \frac{d\phi}{d\rho} = 2\pi\, i\, \sigma_3\, \phi(0)$$

Which explicitly exhibits the singularity:

$$(14) \qquad B_3 = (\underset{\sim}{\nabla} \wedge \underset{\sim}{A})_3 = 2\pi\, i\, \sigma_3\, \delta(x)\, \delta(y) \quad (\text{in } Z=0) .$$

Or, for an arbitrary plane through the origin with normal $\underset{\sim}{n}$, with $\rho = (x',y',0)$:

$$(15) \qquad \underset{\sim}{n} \cdot \underset{\sim}{\nabla} \wedge \underset{\sim}{A} = i\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n}\, \frac{\delta(\rho)}{\rho} = 2\pi\, i\, \sigma.n\, \delta(x')\, \delta(y') .$$

$x'y'$ being the coordinates in that plane .

4- Use of analytic regularization

In the previous paragraph we discussed the singularity of curl $\underset{\sim}{A}$. In order to treat also the product $\underset{\sim}{A} \wedge \underset{\sim}{A}$, it is better to analyze simultaneously both terms contained in the definition of the field $\underset{\sim}{F}$. For that aim it is convenient to use the analytic regularization method (see references [4] [5] [6]) in the definition of the potential. Thus, instead of (6) we may write:

$$(16) \qquad \underset{\sim}{A} = i\, \underset{\sim}{\sigma} \wedge \underset{\sim}{r}\;\; r^{2\alpha}$$

which coincides with (6) for $\alpha = -1$, but is regular at the origin for positive α.

Of course (16) is not a pure gauge potential, but we may expect that the regularized limit for $\alpha \to -1$ will be a vacuum.

From (7) and (16) we have:

$$(17) \qquad \underset{\sim}{F} = 2i \left[(1+\alpha) r^2\, \underset{\sim}{\sigma} - \underset{\sim}{\sigma}.\underset{\sim}{r}\, (r^{4\alpha} + \alpha\, r^{2\alpha-2})\, \underset{\sim}{r}\right] .$$

We will now consider a plane through the origin (with normal $\underset{\sim}{n}$). On this plane, the normal component of $\underset{\sim}{F}$ is a two-dimensional distribution given by:

$$(18) \qquad \underset{\sim}{F} \cdot \underset{\sim}{n} = 2i(1+\alpha)\, \rho^{2\alpha}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n} \quad ; \underset{\sim}{\rho} = (x',y',0).$$

Note that the contribution of $\underset{\sim}{A} \wedge \underset{\sim}{A}$ drops out due to $\underset{\sim}{r} \cdot \underset{\sim}{n} = Z' = 0$.

Equation (18) is not zero for $\alpha = -1$, as the bidimensional distribution $\rho^{2\alpha}$(see reference [6]) has a simple pole at $\alpha = -1$, with residue $\pi\, \delta(x')\, \delta(y')$. The correct limit is then

$$(19) \qquad \underset{\sim}{F} \cdot \underset{\sim}{n} \Big|_{\alpha=-1} = 2\pi i\, \delta(x')\, \delta(y')\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n}\, \frac{\delta(\rho)}{\rho} \quad ; \quad (Z'=0)$$

Therefore the singularity of $\underset{\sim}{F} \cdot \underset{\sim}{n}$ (in the above mentioned plane) is entirely due to $\underset{\sim}{n} . \underset{\sim}{\nabla} \wedge \underset{\sim}{A}$ (cf. (15)), confirming again (10) and (11). Thus we have proved that the regularized potential (16) is not an actual vacuum potential for $\alpha = -1$.

5- <u>**Instanton-like regularization**</u>

The discussion of the previous sections can be repeated if we define $\underset{\sim}{A}$ as the limit for $\lambda \to 0$, of the instanton vector potential at the surface $x_4 = 0$. In fact the instanton four-potential is: [7] [8] (see (3) and (4))

$$(20) \qquad A_\mu = -\,2\, i\, \frac{\sigma_{\mu\nu}\, x_\nu}{x^2 + \lambda^2} \quad .$$

The "space" components of (20), at the surface $x_4=0$, give the vector potential:

$$(21) \qquad \underset{\sim}{A} = i\, \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2 + \lambda^2} .$$

From which we obtain:

$$(22) \qquad \underset{\sim}{F} = 2\, i \, \frac{\lambda^2 \, \underset{\sim}{\sigma}}{(r^2 + \lambda^2)^2} \, .$$

Indeed (22) seems to show that the limit $\lambda^2 \to 0$ corresponds to the vacuum. However (21) and (22) lead to a situation similar to that of the preceding paragraphs. As a matter of facts, it is well known that:

$$(23) \qquad \lim_{\lambda \to 0} \frac{\lambda^2}{(x^2 + y^2 + \lambda^2)^2} = \pi\, \delta(x)\, \delta(y) = \frac{\delta(\rho)}{2\rho} ; \quad \rho = (x^2 + y^2)^{1/2}$$

Then again, the normal component of (22), in any plane through the origin, reduces to (19) for $\lambda^2 \to 0$.

6- A true vacuum potential

If we take the group element given by (2), in which we choose $t = \lambda = \text{constant} \neq 0$ (instead of $t=0$, as in (5)), we get:

$$(24) \qquad g = \frac{\lambda - i\, \underset{\sim}{\sigma} . \underset{\sim}{r}}{\sqrt{r^2 + \lambda^2}}$$

From which it follows

$$(25) \qquad \underset{\sim}{A} = g^{-1}\, \underset{\sim}{\nabla}\, g = i\, \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2 + \lambda^2} - i\lambda \, \frac{\underset{\sim}{\sigma}}{r^2 + \lambda^2}$$

$$(26) \qquad \underset{\sim}{F} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A} + \underset{\sim}{A} \wedge \underset{\sim}{A} \equiv 0 \quad \text{(ie, vaccum for any } \lambda \text{)}$$

However, (6) is not-strictly speaking - the limit of (25) for $\lambda \to 0$ as

$$(27) \qquad \lim_{\lambda \to 0} \frac{\lambda}{x^2 + \lambda^2} = \pi\, \delta(x) \, .$$

This means that along any line passing through the origin, the second term in (25) will give (for $\lambda \to 0$) a distribution of the form

$$(28) \qquad -i\, \pi \vec{\sigma}\, \delta(x)$$

where x is the coordinate along the line. Thus

$$\int_{\infty}^{\infty} A_x dx = -i\pi \sigma_x \quad .$$

Nevertheless the field $\underset{\sim}{F}$ derived from (25) is identically zero for any $\lambda \neq 0$. Of course, the limit for $\lambda \to 0$ is then also a vacuum.

Note that the additional term in (25) (cf. with 21) responsible of (28) is necessary to garantee $\underset{\sim}{F}=0$ at any stage of the process and "a fortiori" at $\lambda \to 0$.

DISCUSSION

The previous paragraphs show that the potential (6) does not represent the vacuum, in a strict sense, as its projection on a plane through the origin presents singularities.

The restrictions (to sub-spacesof lesser number of dimensions) appear in a natural way when one wants to compute fluxes or line integrals.

In particular, when one is interested in a potential of the vacuum, it is clear that not only the field intensity (in the global space) must be zero, but at the same time this must also happen with all its restrictions (or projections) to arbitrary sub-spaces. The potential (6) does not satisfy this criterium. Furthermore, whenever a singularity is present a method should be given on how to handle it; ie a regularization procedure should be included. It so happens that different regularization schemes lead to different type of singularities.

In order to show clearly the relation between the different methods, we shall give a table where we show the projections to two and one dimension (containing the origin) for the regularizations introduced in sections 4,5 and 6. For all of them the potential for three dimensions reduces to $i \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2}$ and the field $\underset{\sim}{F}$ to zero.

Method I (§4) $\underset{\sim}{A} = i\, \underset{\sim}{\sigma} \wedge \underset{\sim}{r}\ \ r^{2\alpha}$ $\quad (\alpha \to -1)$

a) In 2-dimensions (Z=0) , $A = i\, \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{\rho}}{\rho^2}$

$$\left.\begin{aligned} F_1 &= -\, i\, \frac{\pi\, \sigma_1}{2(\alpha+1)}\, (2\alpha+1)\ \ \delta(x)\,\delta(y) \\ F_2 &= -\, i\, \frac{\pi\, \sigma_2}{2(\alpha+1)}\, (2\alpha+1)\ \ \delta(x)\,\delta(y) \end{aligned}\right\}\ \text{singular.}$$

$$F_3 = 2i\,\pi\,\delta(x)\ \ \delta(y)\,\sigma_3$$

b) In 1-dimension $\left(\left[\frac{1}{x}\right] = Vp\ \frac{1}{x} \right)$

$$A_1 = 0\ ,\ A_2 = i\ \sigma_3\ \left[\frac{1}{x}\right],\ A_3 = -i\ \sigma_2\ \left[\frac{1}{x}\right],\ \underset{\sim}{F} = 0\ .$$

Method II(§5)

$$\underset{\sim}{A} = \frac{i\ \underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2 + \lambda^2} \qquad (\lambda \to 0)\ .$$

a) d=2

$$\underset{\sim}{A} = i\, \frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{\rho}}{\rho^2}\ ,\ \ \underset{\sim}{F} = 2i\ \pi\ \delta(x)\,\delta(y)\ \underset{\sim}{\sigma}\ .$$

b) d=1

$$A_1 = 0\ ,\ A_2 = i\ \sigma_3\ \left[\frac{1}{x}\right],\ A_3 = -i\ \sigma_2\ \left[\frac{1}{x}\right]\ .$$

$$\underset{\sim}{F} = \frac{i\pi}{\lambda}\,\delta(x)\,\sigma \quad \text{(singular for } \lambda=0) \; .$$

Method III (§6)

$$A = i\,\frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2 + \lambda^2} - i\,\frac{\lambda\,\underset{\sim}{\sigma}}{r^2+\lambda^2}$$

a) d=2

$$\underset{\sim}{A} = i\,\frac{\underset{\sim}{\sigma} \wedge \underset{\sim}{\rho}}{\rho^2} \;, \qquad \underset{\sim}{F}=0$$

b) d=1

$$A_1 = -i\,\pi\,\sigma_1\,\delta(x)\;, \quad A_2 = i\,\sigma_3\left[\frac{1}{x}\right] - i\,\pi\,\sigma_2\,\delta(x)\;;$$

$$A_3 = -\,i\,\sigma_2\left[\frac{1}{x}\right] - i\,\pi\,\sigma_3\,\delta(x)\;, \qquad F \equiv 0 \; .$$

For these results we have used, when appropiate, the formulae:

$$\lim_{\lambda\to} \frac{\lambda^{2m-n}}{(r^2+\lambda^2)^m} = \pi^{\frac{m}{2}}\,\frac{\Gamma\left(m-\frac{n}{2}\right)}{\Gamma(m)}\,\delta(\vec{r}),\; \left(m > \frac{n}{2}\right)$$

$$r^\alpha \xrightarrow[\alpha\to\, -n-2k]{} \frac{\Omega_n\,\Delta^k\,\delta(\vec{r})}{2^k\,k!\;1.n.(n+2)\ldots(n+2k-2)(\alpha+n+2k)} + \left[r^{-n-2k}\right]$$

where n is the number of dimensions and $\Omega_n = \dfrac{2\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n}{2}\right)}$

Thus we found that method III (§6) is the only regularization procedure among those examined here, that produces a true vacuum ($\underset{\sim}{F} \equiv 0$) .

We also found that although

$$\underset{\sim}{A} = \lim_{\lambda\to 0} \frac{i\,\underset{\sim}{\sigma} \wedge \underset{\sim}{r}}{r^2+\lambda^2} \qquad \text{and}$$

$$\underset{\sim}{A} = \lim_{\alpha\to} i\,\underset{\sim}{\sigma} \wedge \underset{\sim}{r}\;\, r^{2\alpha}$$ coincide in three, two and one

dimensions, the corresponding fields $\underset{\sim}{F}$ coincide only in three dimensions.

It should also be mentioned that these problems are not exclusive of "vacuum" solutions but appear whenever the potential has a singular point.

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