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A Classification of Generalized Supersymmetries*

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Abstract

In this talk we present a division-algebra classification of the generalized supersymmetries admitting (contrary to the standard supersymmetries falling into the HŁS scheme) the presence of tensorial central charges. Division-algebra compatible constraint leading to the different classes of complex and quaternionic hermitian and holomorphic generalized supersymmetries are fully classified. Useful tables are here presented for the first time. Possible applications to M-theory related dynamical systems are briefly mentioned.

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1 Introduction

In this talk we present a work in progress, based on several papers of the author and his coworkers, concerning the division algebra classification of the generalized supersymmetries.

This is an extremely important issue. We recall in fact that in the seventies the HŁS scheme [1] was a cornerstone providing the supersymmetric extension of the Coleman-Mandula no-go theorem. However, in the eighties [2] and especially in the nineties, the generalized space-time supersymmetries going beyond the HŁS scheme (and admitting, in particular, a bosonic sector of the Poincaré or conformal superalgebra which could no longer be expressed as a tensor product $B_{geom} \oplus B_{int}$, where B_{geom} describes space-time Poincaré or conformal algebras, while the remaining generators spanning B_{int} are scalars) found widespread recognition [3, 4] in association with the dynamics of extended objects like branes (see [5, 6]). The eleven-dimensional M-algebra underlying the M-theory as a possible "Theory Of Everything" (TOE), admitting 32-real component spinors and maximal number ($= 528$) of saturated bosonic generators [3, 4] falls into this class of generalized supersymmetries. This is the reason why a lot of attention has been recently devoted to the problem of classifying generalized supersymmetries, see e.g. [7] and [8]. A step towards this classification was provided in [9]. Based on the available classification of Clifford algebras and spinors in terms of division algebras [10, 11, 12], it was there shown that, in the complex and quaternionic cases, a division-algebra compatible constraint, leading to the two big classes of hermitian and holomorphic generalized supersymmetries, could be consistently imposed. In this talk we review the main ingredients entering the mathematical classification of generalized supersymmetries. We will be able to prove that division algebra-compatible constrained generalized supersymmetries split in further subclasses. We will also emphasize some peculiar aspects of the relation of the real generalized supersymmetries with the so-called complex holomorphic supersymmetries, which are of importance when making the analytic continuation between different signature spacetimes. Some of the new results here presented have been elaborated for a work in preparation [13].

Due to the context of this school, the mathematical aspects of the classification of generalized supersymmetries have been stressed, on the other hand some possible applications to the characterization of dynamical systems broadly related with both the M-algebra and the M-theory investigations are briefly mentioned in the Conclusions.

The present paper is so conceived. In order to make it self-consistent, at first the division-algebra classification of Clifford algebras and fundamental spinors is recalled. The notion of "maximal Clifford algebras", essential for later developments, is introduced. It is explained how to recover all real, complex and quaternionic realizations in any given space-time from the set of fundamental maximal Clifford algebras which can be iteratively constructed. In the following, the notion of generalized supersymmetry is introduced in association with their division algebra properties. It is further explained how to implement various division algebra-compatible constraints, as well as their combinations. This amounts to introduce hermitian versus holomorphic constraints in the complex and quaternionic cases, as well as reality conditions implemented on bosonic generators. Some concrete examples of these division-algebra compatible constrained generalized supersymmetries are explicitly constructed. A series of tables with the main ingredients of the classification are presented. Finally, in the Conclusions, as already recalled, we will briefly mention the possible physical applications to supersymmetric dynamical systems (and their relation with the M-theory) of the above construction.

2 On the division-algebra classification of Clifford algebras.

This section is devoted to review the basic features of the division-algebra classification of Clifford algebras. To make the paper self-consistent we briefly recall at first the main, needed properties of division algebras.

The four division algebras of real ($\mathbf{R}$) and complex ($\mathbf{C}$) numbers, quaternions ($\mathbf{H}$) and octonions ($\mathbf{O}$) possess respectively 0, 1, 3 and 7 imaginary elements e_i satisfying the relations

$$e_i \cdot e_j = -\delta_{ij} + C_{ijk} e_k, \quad (1)$$

(i, j, k are restricted to take the value 1 in the complex case, $1, 2, 3$ in the quaternionic case and $1, 2, \ldots, 7$ in the octonionic case; furthermore, the sum over repeated indices is understood).

C_{ijk} are the totally antisymmetric division-algebra structure constants. The octonionic division algebra is the maximal, since quaternions, complex and real numbers can be obtained as its restriction. The totally antisymmetric octonionic structure constants can be expressed as

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{2}$$

(and vanishing otherwise).

The octonions are the only non-associative, however alternative (see [14]), division algebra.

For our later purposes it is of particular importance the notion of division-algebra principal conjugation. Any element X in the given division algebra can be expressed through the sum

$$X = x_0 + x_i e_i, \tag{3}$$

where x_0 and x_i are real, the summation over repeated indices is understood and the positive integral i are restricted up to 1, 3 and 7 in the **C**, **H** and **O** cases respectively. The principal conjugate X^* of X is defined to be

$$X^* = x_0 - x_i e_i. \tag{4}$$

It allows introducing the division-algebra norm through the product X^*X. The normed-one restrictions $X^*X = 1$ select the three parallelizable spheres S^1, S^3 and S^7 in association with **C**, **H** and **O** respectively.

For what concerns the main properties of Clifford algebras and their relation with the associative division algebras **R**,**C**, **H** it is convenient to follow [12] and [15].

The most general irreducible *real* matrix representations of the Clifford algebra

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}, \tag{5}$$

with $\eta^{\mu\nu}$ being a diagonal matrix of (p,q) signature (i.e. p positive, $+1$, and q negative, -1, diagonal entries)[1] can be classified according to the property of the most general S matrix commuting with all the Γ's ($[S,\Gamma^\mu] = 0$ for all μ). If the most general S is a multiple of the identity, we get the normal (**R**) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, **C** case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the **H** case). According to [12] the *real* irreducible representations are of **R**, **C**, **H** type, according to the following table, whose entries represent the values $p-q$ *mod* 8

R	**C**	**H**
$0,2$		$4,6$
1	$3,7$	5

(6)

The real irreducible representation is always unique unless $p-q$ *mod* $8 = 1,5$. In these signatures two inequivalent real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma^\mu \mapsto -\Gamma^\mu$).

Let us denote as $C(p,q)$ the Clifford irreps corresponding to the (p,q) signatures. The normal (**R**), almost complex (**C**) and quaternionic (**H**) type of the corresponding Clifford irreps can also be understood as follows. While in the **R**-case the matrices realizing the irrep have necessarily real entries, in the **C**-case matrices with complex entries can be used, while in the **H**-case the matrices can be realized with quaternionic entries.

Let us see how this works in a simple example. Let us take the **H**-type $C(0,3)$ Clifford algebra. It can be realized by associating the three Euclidean gamma matrices with the three imaginary quaternions e_i. The reason for that lies on the fact that the antisimmetry of the C_{ijk} (1) structure constants make the anticommutators $e_ie_j + e_je_i$ satisfy the relation

$$e_ie_j + e_je_i = -\delta_{ij}, \tag{7}$$

reproducing the three dimensional Euclidean Clifford algebra (5) with negative signs.

It is worth mentioning that in the given signatures $p-q$ *mod* $8 = 0,4,6,7$, without loss of generality, the Γ^μ matrices can be chosen block-antidiagonal (generalized Weyl-type matrices), i.e. of the form

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{8}$$

[1]Throughout this paper it will be understood that the positive eigenvalues are associated with space-like directions, the negative ones with time-like directions.

Since the generalized Lorentz algebra can be recovered from the algebra of the commutators $\Sigma^{[\mu\nu]} = [\Gamma^\mu, \Gamma^\nu]$, in those particular signatures the matrices $\Sigma^{[\mu\nu]}$ are of block-diagonal type and it is therefore possible to introduce Weyl-projected spinors, whose number of components is half of the size of the corresponding Γ-matrices (this notion of Weyl spinors, which is convenient for our purposes, has been introduced in [15]).

Most of these features and the notion of "maximal Clifford algebra" are conveniently analyzed by presenting the irreducible representations of Clifford algebras through an algorithm allowing to single out, in each arbitrary signature space-time, a representative (up to, at most, the sign flipping $\Gamma^\mu \leftrightarrow -\Gamma^\mu$) in each irreducible class of representations of Clifford's gamma matrices, see [15]. For completeness, this construction is here reported. Starting from a given D spacetime-dimensional representation of Clifford's Gamma matrices, one can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, if γ_i's denotes the d-dimensional Gamma matrices of a $D = p+q$ spacetime with (p,q) signature (namely, providing a representation for the $C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) &\mapsto (p+1, q+1). \end{aligned} \tag{9}$$

or

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) &\mapsto (q+2, p). \end{aligned} \tag{10}$$

All Clifford algebras of $\mathbf{R}$-type are obtained by recursively applying the algorithms (9) and (10) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0, 7+8m)$ (with m non-negative integer), which must be previously known. Similarly, all Clifford algebras of $\mathbf{H}$-type are obtained by recursively applying the algorithms to the Clifford algebras $C(0, 3+8m)$. The construction of these sets of "primitive" Clifford algebras, in terms of their division algebra presentation and their *mod* 8 Bott's multiplicity of Clifford algebras, has been explicitly presented in [15]. The notion of maximal Clifford algebra can be introduced as follows. "Maximal Clifford algebras" correspond to the irreps which can accommodate the maximal number of Gamma matrices for the corresponding size of the matrices. Non-maximal Clifford algebras are simply recovered after deleting a certain number of Gamma matrices from a given maximal one (a procedure which parallels the dimensional reduction).

Let us now briefly comment about the octonionic realization of the (5) relation, through matrices admitting octonionic entries. Since the octonions are non-associative, this realization presents peculiar features. In [16] and [17] it was shown how it could be associated with an octonionic version of the M algebra and its associated superconformal algebra. Throughout this paper we will limit ourselves to consider only standard, associative, Clifford algebras representations.

3 On fundamental spinors.

Fundamental spinors carry a representation of the generalized Lorentz group with a minimal number of real components in association with the maximal, compatible, allowed division-algebra structure.

It is worth reminding that the division-algebra character of fundamental spinors does not necessarily (depending on the given space-time) coincide with the division-algebra type of the corresponding Clifford irreps (this can be easily understood by recalling that in some given spacetimes the fundamental spinors are of Weyl type, see the previous section).

The following table, taken from the results of [7], [12], [15] and [9], presents the comparison between division-algebra properties of Clifford irreps (Γ) and fundamental spinors (Ψ), in different space-times parametrized by $\rho = s - t \mod 8$. We have

ρ	Γ	Ψ
0	**R**	**R**
1	**R**	**R**
2	**R**	**C**
3	**C**	**H**
4	**H**	**H**
5	**H**	**H**
6	**H**	**C**
7	**C**	**R**

(11)

It is clear from the above table that, for $\rho = 2, 3$, the fundamental spinors can accommodate a larger division-algebra structure than the corresponding Clifford irreps. Conversely, for $\rho = 6, 7$, the Clifford irreps accommodate a larger division-algebra structure than the corresponding spinors. In several cases this mismatch of division-algebra structures plays an important role. For instance in [16] a method was introduced to construct superconformal algebras based on the minimal division algebra structure common to both Clifford irreps and fundamental spinors. This method can be straightforwardly modified to produce extended superconformal algebras based on the largest division-algebra structure. The price to be paid, in this case, would imply the introduction, for $\rho = 2, 3$, of reducible Clifford representations and, conversely, for $\rho = 6, 7$ of non-minimal spinors.

For several purposes it is of capital importance to explicitly construct both the Clifford irreps and the fundamental spinors, in terms of their associated maximal Clifford algebra. Some specific cases, for real and quaternionic spacetimes, have been listed in [15]. The most general table, taken from [13], is here presented for the first time. The maximal Clifford irreps exist in $(p-q) = 1 \quad mod \quad 8$ (for the real case) and in $p - q = 5 \quad mod \quad 8$ (for the quaternionic case). For $p - q \neq 1, 5 \quad mod \quad 8$, real, complex and quaternionic Clifford irreps (Γ) and fundamental spinors (Ψ) are recovered through the following erasing procedure:

	$1 \quad mod \quad 8 \quad (\mathbf{R})$	$5 \quad mod \quad 8 \quad (\mathbf{H})$
$0 \quad mod \quad 8$	$\Gamma, \Psi : (p,q) \stackrel{W}{\rightarrow} (p, q-1)$	
$4 \quad mod \quad 8$		$\Gamma, \Psi : (p,q) \stackrel{W}{\rightarrow} (p-1, q)$
$2 \quad mod \quad 8$	$\Gamma : (p,q) \rightarrow (p, q-1)$	$\Psi : (p,q) \stackrel{*}{\rightarrow} (p-2, q) \stackrel{W}{\rightarrow} (p-3, q)$
$3 \quad mod \quad 8$		$\Gamma : (p,q) \stackrel{*}{\rightarrow} (p-2, q), \Psi : (p,q) \stackrel{W}{\rightarrow} (p-2, q)$
$6 \quad mod \quad 8$		$\Gamma : (p,q) \stackrel{*}{\rightarrow} (p, q-1), \Psi : (p,q) \stackrel{*}{\rightarrow} (p, q-2) \stackrel{W}{\rightarrow} (p-1, q-2)$
$7 \quad mod \quad 8$	$\Psi : (p,q) \stackrel{W}{\rightarrow} (p-2, q)$	$\Gamma : (p,q) \stackrel{*}{\rightarrow} (p, q-2)$

(12)

Some remarks are in order. The arrows denote which gamma matrices (spacetime or timelike) and how many of them have to be deleted. The symbols above the arrows specify the following. The "W" is in association with a Weyl projection, which is required to introduce fundamental spinors in their associated spacetimes. The "$*$" symbol corresponds to a "squeezing" of the original Clifford gamma matrices, after two of the three Clifford gamma matrices expressed in terms of imaginary quaternions have been eliminated. This amounts to half the size of the corresponding set of gamma matrices. A very convenient illustrative example of this feature concerns the imaginary unit i that, as a real matrix, can be realized as a 2×2 matrix, while the full set of three imaginary quaternions requires a presentation in terms of 4×4 real matrices).

The above table consents to recover a given Clifford irrep, as well as the construction of fundamental spinors, out of their associated maximal Clifford algebra. Almost all Clifford algebras and fundamental spinors can be recovered from the (12) table. There are just few exceptions, for very special space-time signatures, in association with the "squeezing". It requires a maximal Clifford algebra admitting, either in the space-like or in the time-like sectors, at least three Clifford matrices associated with the imaginary quaternions. For instance, the squeezing $(p,q) \stackrel{*}{\rightarrow} (p, q-2)$ is only allowed for $q \geq 3$ (as an example, the passage $(7,2) \stackrel{*}{\rightarrow} (7,0)$ is not allowed). These relatively few exceptional cases have to be treated separately, but for all the cases of actual physical interest the construction presented in the (12) table can be applied.

4 On generalized supersymmetries.

Let us introduce now the notion of generalized supersymmetries as an extension and generalization of the standard supertranslation algebra (in some cases, like the F-algebra presentation in a $(10,2)$ spacetime of the M-algebra [9], the bosonic sector admits no translation at all, but still it is convenient to refer to generalized supersymmetries as "generalized supertranslations"). Generalized supertranslations can be used as building blocks to construct superconformal algebras (by simply taking two separate copies of generalized supertranslations and then imposing the closure of the super-Jacobi identities on all generators, [16]). Once obtained a generalized superconformal algebra, generalized superPoincaré algebras admitting, besides the generalized supertranslations, also the generalized Lorentz generators, can be recovered through an Inonü-Wigner contraction procedure. Throughout this paper we will focus just on the building blocks, namely the generalized supertranslations.

At first we need to recall some basic ingredients and conventions. Three matrices, denoted as A, B, C, have to be introduced in association with the three conjugations (hermitian, complex and transposition) acting on Gamma matrices (see [18]). Since only two of the above matrices are independent we choose here, following [15] and [9], to work with A and C. A plays the role of the time-like Γ^0 matrix in the Minkowskian space-time and is used to introduce barred spinors. C, on the other hand, is the charge conjugation matrix. Up to an overall sign, in a generic (s,t) space-time, A and C are given by the products of all the time-like and, respectively, all the symmetric (or antisymmetric) Gamma-matrices[2]. The properties of A and C immediately follow from their explicit construction, see [18] and [15].

In a representation of the Clifford algebra realized by matrices with real entries, the conjugation acts as the identity, see (4). In this case the space-like gamma matrices are symmetric, while the time-like gamma matrices are antisymmmetric, so that A can be identified with the charge conjugation matrix C_A.

For our purposes the importance of A and the charge conjugation matrix C lies on the fact that, in a D-dimensional space-time ($D = s+t$) spanned by $d \times d$ Gamma matrices, they allow to construct a basis for $d \times d$ (anti)hermitian and (anti)symmetric matrices, respectively. It is indeed easily proven that, in the real and the complex cases (the quaternionic case is different, see [9]), the $\begin{pmatrix} D \\ k \end{pmatrix}$ antisymmetrized products of k Gamma matrices $A\Gamma^{[\mu_1 \cdots \mu_k]}$ are all hermitian or all antihermitian, depending on the value of $k \leq D$. Similarly, the antisymmetrized products $C\Gamma^{[\mu_1 \cdots \mu_k]}$ are all symmetric or all antisymmetric.

By considering a set of n-component real spinors Q_a, $a = 1, \ldots, n$ (real spinors can enter the following construction in two ways, either because of the **R**-character of fundamental spinors, or because we are representing **C** and **H** spinors as real column vectors), the most general supersymmetry algebra is represented by

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab}, \tag{13}$$

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n \times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space-time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of k-Gamma matrices, namely

$$\mathcal{Z}_{ab} = \sum_k (A\Gamma_{[\mu_1 \cdots \mu_k]})_{ab} Z^{[\mu_1 \cdots \mu_k]}, \tag{14}$$

where the values k entering the sum in the r.h.s. are restricted by the symmetry requirement for the $a \leftrightarrow b$ exchange and are specific for the given spacetime. The coefficients $Z^{[\mu_1 \cdots \mu_k]}$ are the rank-k abelian tensorial central charges.

When the fundamental spinors are complex or quaternionic (in accordance with table (11)) they can be organized in complex (for the **C** and **H** cases) and quaternionic (for the **H** case) multiplets, whose entries are respectively complex numbers or quaternions.

The real generalized supersymmetry algebra (13) can now be replaced [9] by the most general complex or quaternionic supersymmetry algebras, given by the anticommutators among the fundamental spinors Q_a and their conjugate ${Q^*}_{\dot{a}}$ (where the conjugation refers to the principal conjugation

[2] Depending on the given space-time (see [18] and [15]), there are at most two charge conjugations matrices, C_S, C_A, given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special space-time signatures they collapse into a single matrix C.

in the given division algebra, see (4)). We have in this case

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab} \quad , \quad \{{Q^*}_{\dot a}, {Q^*}_{\dot b}\} = {\mathcal{Z}^*}_{\dot a\dot b}, \tag{15}$$

together with

$$\{Q_a, {Q^*}_{\dot b}\} \;=\; \mathcal{W}_{a\dot b}, \tag{16}$$

where the matrix $\mathcal{Z}_{ab}$ (${\mathcal{Z}^*}_{\dot a\dot b}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{W}_{a\dot b}$ is hermitian.

The maximal number of allowed bosonic components (real counting) in the r.h.s. for both complex and quaternionic supersymmetries associated to n-component fundamental spinors can be easily computed [9]. Some remarks are in order. We can expand the r.h.s. of (15) and (16) in terms of the antisymmetrized product of Gamma matrices only when the division-algebra character of the Gamma matrices coincides with the division-algebra character of spinors. As discussed at length in Section **3**, for the spacetimes presenting a spinor-versus-Clifford mismatch, the supersymmetry algebra (15) and (16) still makes sense by letting either non-fundamental spinors enter the l.h.s. or, conversely, reducible representations of the Clifford algebra enter the r.h.s. .

As they have been written, the formulas (13), (15) and (16) are referring to supersymmetries algebras realized in terms of spinors which are not Weyl-projected. The modifications to accommodate generalized supersymmetry algebras with Weyl-type of spinors are straightforward [9].

Two big classes of subalgebras, respecting the Lorentz-covariance, can be obtained from (15) and (16) in both the complex and quaternionic cases, by setting identically equal to zero either $\mathcal{Z}$ or $\mathcal{W}$, namely

I) $\mathcal{Z}_{ab} \equiv {\mathcal{Z}^*}_{\dot a\dot b} \equiv 0$, so that the only bosonic degrees of freedom enter the hermitian matrix $\mathcal{W}_{a\dot b}$ or, conversely,

II) $\mathcal{W}_{a\dot b} \equiv 0$, so that the only bosonic degrees of freedom enter $\mathcal{Z}_{ab}$ and its conjugate matrix ${\mathcal{Z}^*}_{\dot a\dot b}$.

Accordingly, in the following we will refer to the (complex or quaternionic) generalized supersymmetries satisfying the *I)* constraint as "hermitian" generalized supersymmetries, while the (complex or quaternionic) generalized supersymmetries satisfying the *II)* constraint will be referred to as "holomorphic" generalized supersymmetries. [3]

Further refinements in the classification of division algebra constrained generalized supersymmetries can be produced by allowing a reality constraint in the bosonic sector of the holomorphic supersymmetries. This type of classification is here presented for the first time. It is convenient to illustrate it by discussing, at first, some specific examples of interest, for later producing general results.

Let us start describing the generalized supersymmetries associated with the $(4,1)$ space-time. Its fundamental spinors are quaternionic (11) and admit 8 real components. We are in the position to classify all real and complex saturated generalized supersymmetries associated to this spacetime (the quaternionic supersymmetries are introduced in the next section). There are five separated cases that we are able to consider, depending on whether in the complex case either the holomorphicity or the hermiticity conditions have been to be taken into account. One should also notice that, in the holomorphic case, a further reality condition on the bosonic sector, halving the number of bosonic degrees of freedom, can be imposed. The complete class of generalized supersymmetries can therefore be given as follows:

i) Real generalized supersymmetry with 36 bosonic components. This real generalized supersymmetry can also be expressed in the complex spinor formalism, the 36 bosonic components being recovered from $36 = 20 + 16$, the sum (in the real counting) of the holomorphic and hermitian sectors of the bosonic r.h.s.,

ii) The holomorphically constrained complex generalized supersymmetry with 20 bosonic components in the real counting,

iii) The holomorphically constrained complex generalized supersymmetry with reality condition on the bosonic r.h.s., leading to $\frac{1}{2} \times 20 = 10$ bosonic components,

iv) The hermitian complex generalized supersymmetry with 16 bosonic components (real counting) and, finally,

v) the complex supersymmetry with reality constraint on the holomorphic sector, leading to $10 + 16 = 26$ bosonic components.

[3] In [19] it was proven that the analytical continuation of the M-algebra can be carried out to the Euclidean, the corresponding Euclidean algebra being a complex holomorphic supersymmetry.

The generalized supersymmetries for $(4,1)$ allow us to immediately construct the generalized supersymmetries in the standard Minkowskian $(3,1)$ space-time, which can be obtained as a Weyl-type dimensional reduction from $(4,1)$, see table (12). The corresponding generalized supersymmetries in this case admit a total number of bosonic generators, whose counting is based on 4-component spinors (due to the Weyl condition), given by the following list

i) 10 in the real case ($10 = 6 + 4$, in the complex presentation),

ii) 6 in the complex holomorphic case,

iii) 3 in the complex holomorphic case supplemented by the reality constraint,

iv) 4 in the complex hermitian case and, finally,

v) $7 = 4 + 3$ when the reality constraint is imposed on the holomorphic sector.

The above classes of supersymmetries are general, they are present in all cases when the fermionic generators are realized through complex spinors.

Both the $(4,1)$ and the $(3,1)$ spacetimes are not maximal Clifford algebras. The maximal Clifford algebras associated to them are recovered from the (12) table. The above list of generalized supersymmetries finds immediate application in the construction of all possible constrained dynamical systems arising from dimensional reduction of one given system associated to the maximal Clifford spacetime (examples of such systems are the particle models admitting tensorial central charges, briefly discussed in the Conclusion). This explains the importance of both the (12) table (for the derivation of maximal Clifford algebras) and of the above constraints in the classification of generalized supersymmetries.

The above results can be extended to any kind of generalized supersymmetries admitting n-component complex spinors (i.e. $2n$ distinct components in the real counting). In the following table the associated generalized supersymmetries are listed, as well as the total number of bosonic (real-counting) degrees of freedom. We have

i	*Real supersymmetry*	$2n^2+n$ *bosonic components*
ii	*complex holomorphic supersymmetry*	n^2+n *bosonic components*
iii	*complex holomorphic with bosonic reality constraint*	$\frac{1}{2}(n^2+n)$ *bosonic components*
iv	*complex hermitian*	n^2 *bosonic components*
v	*reality constraint on holomorphic sector*	$\frac{1}{2}(3n^2+n)$ *bosonic components*

(17)

5 Real, complex and quaternionic generalized supersymmetries.

In this section we present a series of tables, taken from [9] and [13], listing the main properties of real, complex and quaternionic generalized supersymmetries. Several other tables can be produced, but cannot be reported here for lack of space. In any case the essential ingredients entering all these classifications will be outlined.

It is convenient to symbolically denoted as "M_k" the space of $\begin{pmatrix} D \\ k \end{pmatrix}$-component, totally antisymmetric rank-k tensors of a D-dimensional spacetime, associated to either the basis provided by the hermitian $A\Gamma^{[\mu_1\cdots\mu_k]}$ or the symmetric matrices $C\Gamma^{[\mu_1\cdots\mu_k]}$ (the context will clarify which is which).

In the case of generalized real supersymmetries, depending on the dimensionality D of the spacetime (and independently from its signature, provided that the spinors admit the same minimal number of components), the bosonic sector, together with its number of bosonic components, is reported in the following table. Since maximal Clifford algebras are odd-dimensional, without loss of generality only odd dimensions D enter the table below

spacetime	bosonic sectors	bosonic components
$D = 1$	M_0	1
$D = 3$	M_1	3
$D = 5$	M_2	10
$D = 7$	$M_0 + M_3$	$1 + 35 = 36$
$D = 9$	$M_0 + M_1 + M_4$	$1 + 9 + 126 = 136$
$D = 11$	$M_1 + M_2 + M_5$	$11 + 55 + 462 = 528$
$D = 13$	$M_2 + M_3 + M_6$	$78 + 286 + 1716 = 2080$

(18)

Generalized supersymmetries in even dimensional spacetime can be obtained from the previous list via a dimensional reduction (by erasing some Gamma matrices, as explained in Section **3**). We obtain that the dimensional reduction $D \to D - 1$ corresponding to the signature passage $(p, q) \to (p, q - 1)$ (here $D = p + q$) is expressed through

spacetime	bosonic sectors	bosonic components
$D = 3$	$M_1 \to \overline{M}_1 + \overline{M}_0$	$3 = 2 + 1$
$D = 5$	$M_2 \to \overline{M}_2 + \overline{M}_1$	$10 = 6 + 4$
$D = 7$	$M_0 + M_3 \to \overline{M}_0 + \overline{M}_3 + \overline{M}_2$	$36 = 1 + 20 + 15$
$D = 9$	$M_0 + M_1 + M_4 \to 2 \times \overline{M}_0 + \overline{M}_1 + \overline{M}_4 + \overline{M}_3$	$136 = 2 + 8 + 70 + 56$
$D = 11$	$M_1 + M_2 + M_5 \to \overline{M}_0 + \overline{M}2 \times \overline{M}_1 + \overline{M}_2 + \overline{M}_4 + \overline{M}_5$	$528 = 1 + 20 + 45 + 210 + 252$
$D = 13$	$M_2 + M_3 + M_6 \to \overline{M}_1 + 2 \times \overline{M}_2 + \overline{M}_3 + \overline{M}_5 + \overline{M}_6$	$2080 = 12 + 2 \times 66 + 220 + 792 + 924$

(19)

The overlined quantities $\overline{M}_k$ are referred to the totally antisymmetric k-tensors in the $D-1$-dimensional spacetime.

It is also convenient to illustrate the dimensional reduction leading from the $(p, q) \to (p - 1, q)$ spacetime. The difference w.r.t. the previous case lies on the fact that now the $(p - 1, q)$ spacetime is of Weyl type (confront the discussion in Section **2**). Only the subclass of totally antisymmetric bosonic k-tensors entering the upper left diagonal block will survive from the Weyl projection and enter the generalized supersymmetry. The corresponding symbols are marked in boldface ($\mathbf{M}_k$) in the table below, corresponding to the even-dimensional Weyl case

spacetime	bosonic sectors	bosonic components
$D = 2$	$M_0 + \frac{1}{2}\mathbf{M}_1$	1
$D = 4$	$\frac{1}{2}\mathbf{M}_2 + M_1$	3
$D = 6$	$M_0 + \frac{1}{2}\mathbf{M}_3 + M_2$	10
$D = 8$	$\mathbf{M}_0 + M_1 + M_3 + \frac{1}{2}\mathbf{M}_4$	$36 = 1 + 35$
$D = 10$	$M_0 + \mathbf{M}_1 + M_2 + M_4 + \frac{1}{2}\mathbf{M}_5$	$136 = 10 + 126$
$D = 12$	$M_1 + \mathbf{M}_2 + M_3 + M_5 + 2 + \frac{1}{2}\mathbf{M}_6$	$528 = 66 + 462$

(20)

In the above table the factor $\frac{1}{2}$ has been inserted to remind that $\mathbf{M}_{\frac{D}{2}}$ is self-dual, so that its total number of components has to be halved in order to fulfill the selfduality condition.

For what concerns the complex and quaternionic cases several tables can be produced. We reproduce here the most significant ones. Our main motivation is not presenting the complete classification, rather to furnish a list of selected examples illustrating all features of the general classification. Since we have already introduced the notion of hermitian (holomorphic) supersymmetry, let us present some tables illustrating both cases.

The first one corresponds to the hermitian quaternionic supersymmetry, whose fermionic generators are quaternionic spinors (the corresponding spacetimes supporting such spinors and the associated supersymmetry can be read from (11)). In this particular case the corresponding table is given by

spacetime	bosonic sectors	bosonic components
$D=3$	M_0	1
$D=4$	M_0	1
$D=5$	M_0+M_1	$1+5=6$
$D=6$	M_1	6
$D=7$	M_1+M_2	$7+21=28$
$D=8$	M_2	28
$D=9$	M_2+M_3	$36+84=120$
$D=10$	M_3	120
$D=11$	$M_0+M_3+M_4$	$1+165+330=496$
$D=12$	M_0+M_4	$1+495=496$
$D=13$	$M_0+M_1+M_4+M_5$	$1+13+715+1287=2016$

(21)

As an example of holomorphic supersymmetry we produce a table corresponding to the complex holomorphic supersymmetry for quaternionic spacetime, i.e. carrying a quaternionic structure, however expressing spinors only through their complex structure. This implies that the reality condition on the bosonic sector is automatically implemented. Some similarities should be observed between the table (18) and the table below. They correspond however to different cases, real versus complex holomorphic supersymmetries, associated to spacetimes with different signatures and different number of spinorial components (in the complex holomorphic case the number of spinor components are double than in the real case, for D-dimensional spacetimes). Their similarities on the other hand have a very deep physical meaning. They imply, e.g., that the complex holomorphic supersymmetry can be used to perform the analytic continuation of real supersymmetries to different signatures (the Euclideanized version of the M-algebra, see [19], corresponds to the analytical continuation of the real M algebra). We have now

spacetime	bosonic sectors	bosonic components
$D=3$	M_1	3
$D=4$	$\widetilde{M}_2$	3
$D=5$	M_2	10
$D=6$	$\widetilde{M}_3$	10
$D=7$	M_0+M_3	$1+35=36$
$D=8$	$M_0+\widetilde{M}_4$	$1+35=36$
$D=9$	$M_0+M_1+M_4$	$1+9+126=136$
$D=10$	$M_1+\widetilde{M}_5$	$10+126=136$
$D=11$	$M_1+M_2+M_5$	$11+55+462=528$
$D=12$	$M_2+\widetilde{M}_6$	$66+462=528$
$D=13$	$M_2+M_3+M_6$	$78+286+1716=2080$

(22)

The classification of the (full) quaternionic holomorphic supersymmetry, which presents peculiar features, has been given and discussed in [9]. The results can be summarized as follows

$-$	$D=0,6,7 \quad \mathrm{mod} \quad 8$
M_0	$D=1 \quad \mathrm{mod} \quad 8$
M_1	$D=4,5 \quad \mathrm{mod} \quad 8$
M_0+M_1	$D=2,3 \quad \mathrm{mod} \quad 8$

(23)

The above results can be interpreted as follows. Quaternionic holomorphic supersymmetries only arise in D-dimensional quaternionic space-times, where $D=2,3,4,5 \quad mod \quad 8$. No such supersymmetry exists in $D=0,6,7 \quad mod \quad 8$ D-dimensional spacetimes.

In $D=1 \quad mod \quad 8$ dimensions it only involves a single bosonic charge and falls into the class of quaternionic supersymmetric quantum mechanics, rather than supersymmetric relativistic theories. Finally, this supersymmetry algebra only admits at most a scalar bosonic central charge, found in D-dimensional quaternionic spacetimes for $D=2,3 \quad mod \quad 8$.

It must be said that so far no dynamical system supporting such a supersymmetry has been investigated.

6 Conclusions.

This paper was devoted to perform a division algebra classification of the generalized supersymmetries. Besides the notion of hermitian (complex and quaternionic) and holomorphic (complex and quaternionic) supersymmetries, already presented in [9], a further distinction of division-algebra constrained generalized supersymmetries, given by table (17), has been presented. This set of constrained supersymmetries corresponds to certain classes of division algebra constraints that can be consistently imposed (e.g., a reality condition on the bosonic sector of complex holomorphic supersymmetries). The sets of constraints can even be combined together, as discussed in Section **4**.

Another issue that we have here clarified consists in the explicit construction, see table (12), of the non-maximal Clifford algebras and their associated spinors, in terms of their associated maximal Clifford algebras. The two main new results here presented allow to classify and put in a single framework (via dimensional reduction), showing their web of inter-related dualities, a whole class of generalized supersymmetries.They can be combined to produce, on a physical side, the largest "oxydized" dynamical system which can be regarded as the generator of all reduced and constrained lowest dimensional models.

Some of the mathematical issues here discussed have already been employed to, e.g., performing the analytic continuation of the M algebra [19] (it corresponds to an eleven-dimensional complex holomorphic supersymmetry and in [9] it was further shown that the same algebra also admits a 12-dimensional Euclidean presentation in terms of Weyl-projected spinors). These two examples of Euclidean supersymmetries can find application in the functional integral formulation of higher-dimensional supersymmetric models.

There is an interesting class of models which nicely fits in the framework here described and is currently under intense investigation. It is the class of superparticle models, introduced at first in [20] and later studied in [21], whose bosonic coordinates correspond to tensorial central charges. It was shown in [22] that a 4-dimensional theory of this kind leads to a tower of massless higher spin states, concretely implementing a Fronsdal's proposal [23] of introducing bosonic tensorial coordinates to describe massless higher spin theories (admitting helicity states greater than two). This is an active area of research, the main motivation being the investigation of the tensionless limit of superstring theory, corresponding to a tower of higher helicity massless particles (see e.g. [24]).

In a somehow "orthogonal" direction, a class of theories which can be investigated in the present framework is the class of supersymmetric extensions of Chern-Simon supergravities in higher dimensions, requiring as a basic ingredient a Lie superalgebra admitting a Casimir of appropriate order, see e.g. [25].

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