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QUANTIZED WILSON LOOPS WITH DIMENSIONAL REGULARIZATION

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CENTRO BRASILEIRO DE PESQUISAS FÍSICAS - CBPF/CNPq
Av. Wenceslau Braz, 71, fundos
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ABSTRACT

Calculation of Wilson Loops (WL) to second order, are performed for different shapes of the path, using dimensional regularization techniques. Some useful formulae are developped. In particular a discussion is given on the influence of points of contact, cusps and intersections, on the residue of the resultant pole. For smooth curves WLs are finite.

1. INTRODUCTION

It is now clear that the Wilson Loop Functional (WL) is a good candidate to act as a gauge invariant dynamical variable, containing all information about the gauge fields. Attempts are being made to find its equations of motion[1]. However, the appearances of infinities in the actual computations somehow obscure its meaning and interpretation[2],[3],[4],[5],[6]. The dimensional regularization method [7] thus appear as an appropriate tool for treating those infinities and clarify the meaning of the singular equations that are presently being considered[1]. The quantized WL is defined as

$$W(C) = \langle Tr P e^{\oint A_\mu dx^\mu} \rangle \tag{1}$$

In second order, we have

$$W^{(2)}(C) = -\frac{1}{2} \oint dx^\alpha \oint dy^\beta D_{\alpha\beta}(x-y) \tag{2}$$

For simplicity we shall limit the discussion to planar curves in ν dimensional spaces. We shall also condider formulae of the type of eq.(2) in which C is not necessarily closed. For open curves, we shall use the notation V(C).

The propagator is:

$$D_{\alpha\beta}(x-y) = \int \frac{d^\nu k}{(2\pi)^\nu} \left\{ \frac{g_{\alpha\beta}}{k^2} - A \frac{k_\alpha k_\beta}{k^4} \right\} e^{ik.x} \tag{3}$$

where A is a gauge fixing constant.

In configuration space (3) leads to [8]

$$D_{\alpha\beta}(x-y) = \frac{\Gamma(\frac{\nu-2}{2})}{4\pi^{\nu/2}} \{g_{\alpha\beta}|x-y|^{2-\nu} - \frac{A}{2(\nu-4)} \partial^x_\alpha \partial^y_\beta |x-y|^{4-\nu}\} \tag{4}$$

From which we get:

$$V(C) = -\frac{\Gamma(\frac{\nu-2}{2})}{8\pi^{\nu/2}} \{\iint_C dx^\alpha dy^\alpha |x-y|^{2-\nu} + \frac{A}{\nu-4}|a-b|^{4-\nu}\} \tag{5}$$

a,b being the end points of C. Obviously W(C)(a=b) is gauge independent. For reasons which will become clear later (see below, form. (11), (12)) we shall choose

$$A = \frac{2}{3-\nu} \quad \text{(dimensional gauge)} \tag{6}$$

Sometimes it is convenient to compute V(C) in momentum space. By defining the linear functional over C

$$f^\alpha_C(k) = \int_C e^{ik.x} dx^\alpha \tag{7}$$

We have

$$V(C) = -\frac{1}{2(2\pi)^\nu} \int d^\nu k \left[\frac{g_{\alpha\beta}}{k^2} - \frac{2k_\alpha k_\beta}{(3-\nu)k^4}\right] f^\alpha_C(k) f^{\beta *}_C(k) \tag{8}$$

Suppose now that the curve C is contained in some subspace with n dimensions. We can decompose then, the vector k as:

$k_\mu = k^\perp_\mu + \bar{k}_\mu$ where $\bar{k}_\mu$ is the projection of k_μ over the subspace containing C, and $k^{\perp}$ is the orthogonal component to $\hat{k}$. Obviously

$$k^2 = k^{\perp 2} + \hat{k}^2$$

One can readily perform the integration over $k^{\perp}$ in form (8), with

$$\Omega(\nu-n) \int_0^{\infty} \frac{d|k^{\perp}|(k_{\perp}^2)^{\frac{\nu-n-1}{2}}}{(k^{\perp 2}+\hat{k}^2)^{\alpha}} = \pi^{\frac{\nu-n}{2}} \frac{(\hat{k}^2)^{\frac{\nu-n}{2}-\alpha}\Gamma(\alpha-\frac{\nu-n}{2})}{\Gamma(\alpha)} \tag{10}$$

where $\Omega(\nu-n)$ comes from the $(\nu-n)$ angular integration. Using (10) in (8) and since $f_c(k)$ only depends on $\hat{k}$, we get

$$V(C) = -\frac{\Gamma(1-\frac{\nu-n}{2})}{2^{\nu+1}\pi^{\frac{\nu+n}{2}}} \int d^n\hat{k}(\hat{k}^2)^{\frac{\nu-n}{2}-1} \left[g_{\alpha\beta} - \frac{2-\nu+n}{3-\nu} \frac{\hat{k}_\alpha \hat{k}_\beta}{\hat{k}^2}\right] f_c^{\alpha}(\hat{k}) f_c^{\beta *}(\hat{k}) \tag{11}$$

We see that for n=1, the choice (6) for the gauge constant makes the integrand identically zero.i.e. for any segment

$$V(C) \equiv 0 \tag{12}$$

From now on we drop the hat over k and shall consider curves only with n=2.

Note that

$$k_\alpha f^\alpha = \frac{1}{i} \int_a^b dx^\alpha \partial_\alpha e^{ikx} = \frac{1}{i}(e^{ik.b} - e^{ik.a}) \tag{13}$$

$$|k_\alpha f^\alpha|^2 = 4\mathrm{sen}^2 \frac{k.(b-a)}{2} \tag{14}$$

For a=b:

$$k.f = 0 \qquad \text{(for any closed curve)} \tag{15}$$

Using (14) and (11) (for n=2) and

$$\int_{-\infty}^{+\infty} dk_2 (k_1^2+k_2^2)^{\frac{\nu-6}{2}} = (k_1^2)^{\frac{\nu-5}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5-\nu}{2})}{\Gamma(\frac{6-\nu}{2})} \tag{16}$$

$$\int_{-\infty}^{+\infty} dk_1 (k_1^2)^{\frac{\nu-5}{2}} \operatorname{sen}^2 \frac{k_1(a-b)}{2} = - \frac{\Gamma(\nu-4)\cos(\frac{\nu-4}{2})\pi}{(b-a)^{\nu-4}} \tag{17}$$

the following result is obtained

$$V^{(2)}(C) = - \frac{2\Gamma(\frac{4-\nu}{2})}{(4\pi)^{\frac{\nu+2}{2}}} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 (k_1^2+k_2^2)^{\frac{\nu-4}{2}} |f|^2 + \frac{\Gamma(\nu-4)|a-b|^{4-\nu}}{(4\pi)^{\frac{\nu-1}{2}}\Gamma(\frac{\nu-1}{2})} \tag{18}$$

For closed planar curves for which (15) is valid $f^{\alpha}(k)$ will be of the form:

$$f^{\alpha} = \varepsilon^{\alpha\beta} k_{\beta} f \tag{19}$$

Now we proceed as follows

$$\varepsilon^{\alpha\beta} k_{\beta} f^{\alpha} = k^2 f = \oint dx^{\alpha} \varepsilon^{\alpha\beta} k_{\beta} e^{ik.x}$$

and using Gauss theorem

$$k^2 f = \iint_{\Sigma} d\sigma\, \partial_{\alpha}(k_{\alpha} e^{ik.x}) = ik^2 \iint_{\Sigma} d\sigma e^{ik.x} \tag{20}$$

i.e. (21) $f = i \iint_{\Sigma} d\sigma e^{ik.x}$

where Σ is the surface enclosed by C.

2. APPLICATIONS

a) We shall first consider a circle. Using (19) and (21) we find

$$f= i\int_0^R rdr \int_0^{2\pi} d\phi \; e^{ikr\cos\phi} = \frac{i2\pi R}{k} \quad J_1(kR) \tag{22}$$

$$f^\alpha f^*_\alpha = 4\pi^2 R^2 J_1^2(kR) \tag{23}$$

Inserting (23) in (18), with a=b

$$W(\text{circle}) = - \frac{\Gamma(\frac{4-\nu}{2})}{(4\pi)^{\frac{\nu+2}{2}}} \, 16\pi^3 R^2 \int_0^\infty dk \; (k^2)^{\frac{\nu-3}{2}} J_1^2(kR) \tag{24}$$

and, from [9] p.692

$$W(\text{circle}) = - \frac{R^{4-\nu} \; \Gamma(\frac{3-\nu}{2}) \; \Gamma(\frac{\nu}{2})}{2^{\nu-1} \; \pi^{\frac{\nu-3}{2}} \; \Gamma(\frac{6-\nu}{2})} \tag{25}$$

which, for $\nu=4$ gives:

$$W^{(2)}(\text{circle}) = \frac{1}{4} \tag{26}$$

(25) shows that $W^{(2)}$ for a circle is zero for even $\nu>4$ and has a pole for odd ν. The pole for $\nu=3$ is a "physical" one, as $W^{(2)}$ represents then the self inductance of an infinitesimally thin circular ring.

For $\nu=2$ (27) $W^{(2)}(\text{circle}) = - \frac{1}{2} \pi R^2$ (area law)

b) Ellipse with semiaxis, a, b.

If we change variables in (7)

$ax_1' = x_1 \quad bx_1' = x_2$ we obtain,

$$f^1_{ell}(k_1;\, k_2) = a\ f^1_{cir}\ (ak_1;bk_2)$$

$$f^2_{ell}(k_1;\, k_2) = b\ f^2_{cir}\ (ak_1;bk_2)$$

substituting in (18) and with a new change of variables

$k_1' = ak_1 \quad k_2' = bk_2$ one is led to

$$W^{(2)}_{ell} = W^{(2)}_{Cir}(R=1)\circ(a\circ b)^{\frac{4-\nu}{2}}\, P_{\frac{\nu-2}{2}}\left(\frac{a^2+b^2}{2ab}\right) \tag{28}$$

where P_α is a Lengendre function ([9],p.384)

In particular,

$$\nu = 4,\ W^{(2)}_{el} = \frac{a^2+b^2}{8ab} \qquad \nu=2 \qquad W^{(2)}_{ell} = -\frac{\pi ab}{2}$$

As we Know [5] that an infinitely elongated WL is associated with the force law in the transverse direction, we shall write $b = \frac{R}{2}$, $a = \frac{T}{2}$, $T\to\infty$. The asymptotic expression obtained is

$$W^{(2)}_{el} = -\frac{\Gamma(\frac{3-\nu}{2})\Gamma(\nu-1)}{2^{\nu+1}\pi^{\frac{\nu-3}{2}}\ \Gamma(\frac{6-\nu}{2})\ \Gamma(\frac{\nu}{2})}\ \frac{T}{R^{\nu-3}} \tag{29}$$

For $\nu=4$

$$W^{(2)}_{el} \to \frac{1}{8}\ \frac{T}{R} \tag{30}$$

One should note that (29) is the Coulomb law in $(\nu-1)$

dimensions. The proportionality constant depends on the shape of the contour.

c) Rectangle

In the dimensional gauge, the self interaction of each segment is zero. For the mutual interaction we first calculate from (7) the f^{α} corresponding to each of the segments. The total f^{α} will be the sum of the four.

Replacing it in (18) and performing the integration, the result is

$$W^{(2)}(C) = \frac{\Gamma(\frac{\nu-4}{2})}{4\pi^{\frac{\nu}{2}}} \{L_2^{4-\nu} F(-\frac{1}{2};\frac{\nu-4}{2};\frac{1}{2};-\frac{L_1^2}{L_2^2}) + +L_1^{4-\nu} F(-\frac{1}{2};\frac{\nu-4}{2};\frac{1}{2};-\frac{L_2^2}{L_1^2}) - \frac{\nu-2}{\nu-3}(L_2^{4-\nu}+L_1^{4-\nu})\} \qquad (31)$$

If we take the limit $L_1 \to \infty$, we get as dominant term:

$$W^{(2)}(C) \to \frac{\Gamma(\frac{\nu-3}{2})}{4\pi^{\frac{\nu-1}{2}}} \frac{L_1}{L_2^{\nu-3}} \xrightarrow{\nu=4} \frac{1}{4\pi}\frac{T}{R}$$

We see that form (32) reproduces,up to a constant, the results of form (29) and (30).

3. SINGULAR POINTS

We are interested in the computation of the effect of intersections. To simplify the discussion we first consider the case of an infinite straight line with a circle. From (7) we have for the line

$$f_L^1 = 2\pi\delta(k_1) \qquad f_L^2 = 0 \tag{33}$$

and for a circle whose center is shifted a distance D from the line

$$f_C^\alpha = \frac{2\pi i R}{|k|}\,\varepsilon^{\alpha\beta} k_\beta e^{ik_2 D} J_1(kR) \tag{34}$$

Adding (33) and (34) we obtain for the interaction

$$W_{int} = \frac{2R\Gamma(\frac{4-\nu}{2})}{(4\pi)^{\frac{\nu-2}{2}}} \int_0^\infty dk_2 |k_2|^{\nu-4} \,\text{sen}\, k_2 D.\, J_1(|k_2|R) \tag{35}$$

There are three cases to be considered. (See [9] p. 747)

a) D>R The circle does not intersect the line.

$$W_{int} = \frac{\Gamma(\frac{4-\nu}{2})\; R^2 D^{2-\nu}}{(4\pi)^{\frac{\nu-2}{2}}}\,\Gamma(\nu-2)\,\text{sen}\frac{\pi}{2}(\nu-2)\,F(\frac{\nu-1}{2},\frac{\nu-2}{2};2;\frac{R^2}{D^2}) \tag{36}$$

b) D=R The circle just touches the line.

$$W_i = \frac{1}{2}\,\frac{R^{4-\nu}}{\pi^{\frac{\nu-3}{2}}}\,\frac{\Gamma(\frac{7-2\nu}{2})\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{5-\nu}{2})\;\Gamma(\frac{6-\nu}{2})} \tag{37}$$

c) D<R The circle intersects the line

$$W_i = 2\,\frac{\Gamma(\frac{4-\nu}{2})2^{\nu-3}R^{3-\nu}D}{(4\pi)^{\frac{\nu-2}{2}}\;\Gamma(\frac{5-\nu}{2})}\,\Gamma(\frac{\nu-1}{2})\;F(\frac{\nu-1}{2},\;\frac{\nu-3}{2};\frac{3}{2};\frac{D^2}{R^2}) \tag{38}$$

For $\nu=4$ (36) and (37) have no pole, while (38) has one with residue

$$\frac{1}{\pi}\;\text{cotg}\alpha \tag{39}$$

Analogously, for a segment and a straight line we have the result (Ref [9] p.422)

$$V_i = -\mathrm{cotg}\alpha \frac{\Gamma(\frac{4-\nu}{2})\Gamma(\frac{\nu-3}{2})}{8\pi^{\frac{\nu-1}{2}}\Gamma(\frac{6-\nu}{2})}\{(\frac{L\mathrm{sen}\alpha}{2} + D)^{4-\nu} +$$

$$+ \left|\frac{L\mathrm{sen}\alpha}{2} - D\right|^{4-\nu}\mathrm{sgn}(\frac{L\mathrm{sen}\alpha}{2} - D)\} \qquad (40)$$

where L is the length of the segment and D is the distance from the line to the middle of the segment. Again for $\nu=4$, we have three cases

a) $\frac{L\mathrm{sen}\alpha}{2} < D$. There is no pole.

b) $\frac{L\mathrm{sen}\alpha}{2} = D$. The segment just touches the straight line.There is a pole with residue:

$$R = -\frac{1}{4\pi}\ \mathrm{cotg}\alpha \qquad (41)$$

c) $\frac{L\mathrm{sen}\alpha}{2} > D$. The segment crosses the line and there is a pole with residue

$$R = -\frac{1}{2\pi}\ \mathrm{cotg}\alpha \qquad (42)$$

Compare (42) with (39) and note that there, there are two intersting points.

From the previous discussion we have seen that a crossing generates a pole. We shall now see that the same happens when the path has a cusp (or angle). To that end, we shall first consider an angle formed by two segments.

The interaction between them, according with the general formalism, is

$$V_i = -2\,\frac{\Gamma(\frac{\nu-2}{2})}{8\pi^{\frac{\nu}{2}}} \int_1 dx^\beta \int_2 dy^\beta \,|(x_1-y_1)^2 + x_2^2|^{\frac{2-\nu}{2}} -$$

$$- \frac{\Gamma(\frac{\nu-2}{2})(L_1^{4-\nu}+L_2^{4-\nu} - |\vec{L}_1-\vec{L}_2|^{4-\nu})}{4\pi^{\frac{\nu}{2}}\,(3-\nu)(4-\nu)} \tag{43}$$

Performing now the changes of variables:

$$x_1 = \xi\cos\alpha \qquad dx^\beta dy^\beta = \cos\alpha\, d\xi\, dy$$

$$x_2 = \xi\sin\alpha$$

$$\eta = (y-\chi) = \lambda x_1$$

Integrating over ξ we obtain a pole at $\nu=4$ whose residue is given by:

$$R = \frac{1}{4\pi^2}\,\frac{\cos^2\alpha}{\operatorname{sen}^2\alpha} \int_{-1}^{\operatorname{sgn}(\cos\alpha)\infty} d\lambda\,(\lambda^2 \operatorname{cte}^2\alpha + 1)^{-1} + \frac{1}{4\pi^2} \tag{44}$$

with the result: (Cf. |4|)

$$R = \frac{(\pi-\alpha)\operatorname{ctg}\alpha + 1}{4\pi^2} \tag{45}$$

Note that this result is valid when one of the arrows enter and the other leaves the vertex. In other cases, the overall sign must be changed.

By joining smoothly the two segments we get a loop with a cusp. The residue at the pole of this loop is given by (45). (See [2]). Of course, if $\alpha\to\pi$ (smooth curve) $R\to$zero. For a rectangle, we must multiply (45) (for $\alpha = \frac{\pi}{2}$) by four, as it has four cusps, with the result: $R=\frac{1}{\pi^2}$ which coincides with the residue of (31) at $\nu=4$. Proceeding in analogous way it is easy to prove that for one segment just touching another (not at the end points), we have

$$R = \frac{\cot g\alpha}{4\pi} \tag{46}$$

α being the angle between the segments with the arrows following one another ("conserved current")(46) is then the contribution of a contac point to the residue of the pole.

Again, when the two segments cross each other we have.

$$R = \frac{\cot g\alpha}{2\pi} \tag{47}$$

which is twice the contribution of (46). (47) is the contribution of an intersection (Cf. (42) and (39)). Now, in more complicated loops where these cusps and intersections are structural elements, their contributions to the residue are simply to be summed to obtain the total residue.

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