CBPF-NF-031/04

Division Algebra, Generalized Supersymmetries and Octonionic M-Theory.*

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Abstract

This is the report of the talk given at the conference "Number, Time and Relativity", held at the Bauman University, Moscow, August 2004, concerning the recent research activity of the author and his collaborators about the inter-relation of the concepts of division algebras, representations of Clifford algebras, generalized supersymmetries with the introduction of an alternative description of the M-algebra in terms of the non-associative structure of the octonions.

*Talk given at the conference: Number, Time and Relativity 2004, Bauman Technical University, Moscow, August 2004. To appear in "Hypercomplex Numbers in Geometry and Physics".

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1 Introduction.

The unification program aiming at a unified description of the known interactions as well as a consistent quantum formulation for gravity, nowadays mostly points towards higher-dimensional supersymmetric theories. At present the most promising, however still conjectural, candidate should live in eleven dimensions and goes under the name of M-theory [1]. The theoretical (and phenomenological) consistency requirements put on any possible candidate for unification necessarily lead to a systematic investigation of the properties of Clifford algebras and spinors in space-times of arbitrary dimension and signature. The generalized supersymmetries going beyond the standard HŁS scheme [2] admit the presence of bosonic abelian tensorial central charges associated with the dynamics of extended objects (branes). It is widely known since the work of [3] that supersymmetries are related to division algebras. Indeed, even for the generalized supersymmetries, classification schemes based on the associative division algebras ($\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$) are now available. For what concerns the remaining division algebra, the octonions, much less is known due to the complications arising from their non-associativity. Octonionic structures were, nevertheless, investigated in [4, 5] in application to superstring theory.

Octonions are not just a curiosity. They are the maximal division algebra. This fact alone already justifies that they should receive the same kind of attention paid to, let's say, the maximal supergravity. However, their importance is more than that, they are at the very heart of many exceptional structures in mathematics and can be held responsible for their existence. Among these exceptional structures we can cite the 5 exceptional Lie algebras and the exceptional Jordan algebras. Indeed, the G_2 Lie algebra is the automorphism group of the octonions, while F_4 is the automorphism group of the 3×3 octonionic-valued hermitian matrices realizing the exceptional $J_3(\mathbf{O})$ Jordan algebra. F_4 and the remaining exceptional Lie algebras E_6, E_7, E_8 are recovered from the so-called "magic square Tits' construction" which associates a Lie algebra to any given pair of division algebras, if at least one of these algebras coincides with the octonionic algebra [6].

It has been pointed out several times, [7, 8] that the exceptional Lie algebras fit well into the grand-unification scenario. Moreover, the E_8 Lie algebra enters, through the $E_8 \times E_8$ tensor product, the anomaly-free heterotic string, while the G_2 holonomy of seven-dimensional manifolds is required, on phenomenological basis, to produce 4-dimensional $N = 1$ supersymmetric field theories by compactification of the eleven dimensions. This partial list of scattered pieces of evidence has brought to suggest, see e.g. [8], that for some deep reasons, Nature seems to prefer exceptional structures. In this context it deserves to be mentioned the special role of the exceptional Jordan algebra $J_3(\mathbf{O})$, not only associated to the unique consistent quantum mechanical system (in the Jordan framework, see [9]) based on a non-associative algebra, but also leading to a unique matrix Chern-Simon theory of Jordan type, see [10].

In this talk I will discuss the investigations presented in [11, 12] concerning the possibility of realizing general supersymmetries in terms of the non-associative division algebra of the octonions. In particular in [11] it was shown that the M algebra which supposedly underlines the M-theory comes in two (and only two, due to the absence of the complex and of the quaternionic structures) variants. Besides the standard realization of the M-algebra which involves real spinors and makes therefore use of the real structure, an alternative formulation, requiring the introduction of the octonionic structure, is also possible and can be exploited. This is made possible due to the existence of an octonionic description for the Clifford algebra defining the 11-dimensional Minkowskian spacetime and its related spinors. The features of this second variant, the octonionic M-superalgebra, are puzzling. While it is not at all surprising that it contains fewer bosonic generators, 52, w.r.t. the 528 of the standard M-algebra (this is after all expected, since the imposition of an extra structure always puts a constraint on a theory), what really came as an unexpected surprise is the fact that new conditions, not present in the standard M-theory, are now found. These conditions imply that the different brane-sectors are no longer independent. The octonionic 5-brane alone contains the whole set of degrees of freedom and is therefore equivalent to the octonionic $M1$ and $M2$ sectors. We can write this equivalence, symbolically, as $M5 \equiv M1 + M2$. This result is indeed very intriguing. It implies that quite non-trivial structures are found when investigating the octonionic construction of the M-theory. It is quite tempting to think that the exceptional structures that we mentioned before should be better understood from this octonionic variant of the M-algebra, rather than the standard real M-algebra.

The next passage consists in defining the closed algebraic structure which realizes the octonionic superconformal M-algebra. It turns out that the $OSp(1, 64)$ superconformal algebra of the real M-theory is replaced in the octonionic case by the $OSp(1, 8|\mathbf{O})$ superalgebra of supermatrices with octonionic-valued entries and total number of $7 + 232 = 239$ bosonic generators.

2 On Clifford algebras.

The classification of generalized supersymmetries requires the preliminary classification of Clifford algebras and spinors and of their association with division algebras.

To make this paper self-consistent, in this section we review the classification of the Clifford algebras associated to the $\mathbf{R}$,$\mathbf{C}$, $\mathbf{H}$ associative division algebras, following [13] and [14].

The most general irreducible *real* matrix representations of the Clifford algebra

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}, \qquad (1)$$

with $\eta^{\mu\nu}$ being a diagonal matrix of (p, q) signature (i.e. p positive, $+1$, and q negative, -1, diagonal entries)[1] can be classified according to the property of the most general S matrix commuting with all the Γ's ($[S, \Gamma^\mu] = 0$ for all μ). If the most general S is a multiple of the identity, we get the normal ($\mathbf{R}$) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, $\mathbf{C}$ case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the $\mathbf{H}$ case). According to [13] the *real* irreducible representations are of $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ type, according to the following table, whose entries represent the values $p - q$ *mod* 8

R	**C**	**H**
0, 2		4, 6
1	3, 7	5

(2)

The real irreducible representation is always unique unless $p - q$ *mod* $8 = 1, 5$. In these signatures two inequivalent real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma^\mu \mapsto -\Gamma^\mu$).

[1] Throughout this paper it will be understood that the positive eigenvalues are associated with space-like directions, the negative ones with time-like directions.

Let us denote as $C(p,q)$ the Clifford irreps corresponding to the (p,q) signatures. The normal (**R**), almost complex (**C**) and quaternionic (**H**) type of the corresponding Clifford irreps can also be understood as follows. While in the **R**-case the matrices realizing the irrep have necessarily real entries, in the **C**-case matrices with complex entries can be used, while in the **H**-case the matrices can be realized with quaternionic entries.

Let us discuss the simplest examples. The **C**-type $C(0,1)$ Clifford algebra can be expressed either through the 2×2 matrix with real-valued entries $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or through the imaginary number i.

The **H**-type Clifford algebra $C(0,3)$, on the other hand, can be realized as follows:
i) with three 4×4 matrices with real entries, given by the tensor products $\tau_A\otimes\tau_1$, $\tau_A\otimes\tau_2$ and $\mathbf{1}_2\otimes\tau_A$, where the matrices τ_A, τ_1 and τ_2 furnish a real irrep of $C(2,1)$
$(\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$,
ii) with three 2×2 complex-valued matrices given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$,
iii) with the three imaginary quaternions e_i (see for more details the section **3**).

The formulas at the items *i)* and *ii)* provide the real and complex representations, respectively, for the imaginary quaternions. They can be straightforwardly extended to provide real and complex representations for the **H**-type Clifford algebras by substituting the quaternionic entries with the corresponding representations (the quaternionic identity 1 being replaced in the complex representation by the 2×2 identity matrix $\mathbf{1}_2$ and by the 4×4 identity matrix $\mathbf{1}_4$ in the real representation).

It is worth noticing that in the given signatures $p-q \ mod\ 8 = 0,4,6,7$, without loss of generality, the Γ^μ matrices can be chosen block-antidiagonal (generalized Weyl-type matrices), i.e. of the form

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \qquad (3)$$

In these signatures it is therefore possible to introduce the Weyl-projected spinors, whose number of components is half of the size of the corresponding Γ-matrices[2].

A very convenient presentation of the irreducible representations of Clifford algebras with the help of an algorithm allowing to single out, in each arbitrary signature space-time, a representative (up to, at most, the sign flipping $\Gamma^\mu \leftrightarrow -\Gamma^\mu$) in each irreducible class of representations of Clifford's gamma matrices has been given in [14]. We recall and extend here the results presented in [14], making explicit the connection between the maximal-Clifford algebras in the table (6) below and their division-algebra property.

The construction goes as follows. At first one proves that starting from a given D spacetime-dimensional representation of Clifford's Gamma matrices, one can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D=p+q$ spacetime with (p,q) signature (namely, providing a representation for the

[2]This notion of Weyl spinors, which is convenient for our purposes, is different from the one usually adopted in connection with *complex*-valued Clifford algebras and has been introduced in [14].

$C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\begin{array}{rcl}\Gamma_j & \equiv & \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) & \mapsto & (p+1, q+1). \end{array} \tag{4}$$

or

$$\begin{array}{rcl}\Gamma_j & \equiv & \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) & \mapsto & (q+2, p). \end{array} \tag{5}$$

It is immediate to notice that the three matrices τ_A, τ_1, τ_2 introduced before and realizing the Clifford algebra $C(2,1)$ are obtained by applying either (4) or (5) to the number 1, i.e. the one-dimensional realization of $C(1,0)$.

All Clifford algebras of **R**-type are obtained by recursively applying the algorithms (4) and (5) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,7+8m)$ (with m non-negative integer), which must be previously known. Similarly, all Clifford algebras of **H**-type are obtained by recursively applying the algorithms to the Clifford algebras $C(0,3+8m)$, while the **C**-type Clifford algebras are obtained by recursively applying the algorithms to the Clifford algebras $C(0,1+8m)$ and $C(0,5+8m)$. This is in accordance with the scheme illustrated in the table below, taken from [14]. We get

Table with the maximal Clifford algebras (up to $d = 256$).

	1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
R	$\underline{(1,0)}$	⇒	(2,1)	⇒	(3,2)	⇒	(4,3)	⇒	(5,4)	⇒	(6,5)	⇒	(7,6)	⇒	(8,7)	⇒	(9,8)	⇒
				↗	(1,2)	→	(2,3)	→	(3,4)	→	(4,5)	→	(5,6)	→	(6,7)	→	(7,8)	→
C			$\underline{(0,1)}$															
				↘	(3,0)	→	(4,1)	→	(5,2)	→	(6,3)	→	(7,4)	→	(8,5)	→	(9,6)	→
						↗	(1,4)	→	(2,5)	→	(3,6)	→	(4,7)	→	(5,8)	→	(6,9)	→
H					$\underline{(0,3)}$													
						↘	(5,0)	→	(6,1)	→	(7,2)	→	(8,3)	→	(9,4)	→	(10,5)	→
								↗	(1,6)	→	(2,7)	→	(3,8)	→	(4,9)	→	(5,10)	→
C							$\underline{(0,5)}$											
								↘	(7,0)	→	(8,1)	→	(9,2)	→	(10,3)	→	(11,4)	→
								↗	(1,8)	→	(2,9)	→	(3,10)	→	(4,11)	→	(5,12)	→
R/O							$\underline{(0,7)}$											
								↘	(9,0)	→	(10,1)	→	(11,2)	→	(12,3)	→	(13,4)	→
												↗	(1,10)	→	(2,11)	→	(3,12)	→
C											$\underline{(0,9)}$							
												↘	(11,0)	→	(12,1)	→	(13,2)	→
														↗	(1,12)	→	(2,13)	→
H													$\underline{(0,11)}$					
														↘	(13,0)	→	(14,1)	→
																↗	(1,14)	→
C															$\underline{(0,13)}$			
																↘	(15,0)	→
																↗	(1,16)	→
R/O															$\underline{(0,15)}$			
																↘	(17,0)	→

(6)

Concerning the above table some remarks are in order. The columns are labeled by the matrix size d (in real components) of the maximal Clifford algebras. Their signature is denoted by the (p,q) pairs. Furthermore, the underlined Clifford algebras in the table can be named as "primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are the "maximal descendant Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (4) and (5). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices (this point has been fully explained in [14] and will not be further elaborated here).

The maximal Clifford algebras generated by the $C(0,7+8m)$ series are associated to both the real ($\mathbf{R}$) and octonionic ($\mathbf{O}$) division algebras, since (1), for the $(0,7+8m)$-signature, can be realized either associatively (in the normal, $\mathbf{R}$, case), or non-associatively through the octonions (see [14] and [16]).

The primitive maximal Clifford algebras $C(0,3)$ and $C(0,7)$ can be explicitly realized through, respectively, three 4×4 matrices (as already recalled) and seven 8×8 matrices given by

$$C(0,3) \equiv \begin{array}{c} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ \mathbf{1}_2\otimes\tau_A. \end{array} \tag{7}$$

and

$$C(0,7) \equiv \begin{array}{c} \tau_A\otimes\tau_1\otimes\mathbf{1}_2, \\ \tau_A\otimes\tau_2\otimes\mathbf{1}_2, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_1, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_2, \\ \tau_1\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_2\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_A\otimes\tau_A\otimes\tau_A. \end{array} \tag{8}$$

The complex primitive maximal Clifford algebras $C(0,1)$ and $C(0,5)$ can be obtained from $C(1,2)$ and $C(0,7)$, respectively, by deleting two gamma-matrices. From $C(0,7)$ we can, e.g., consider the last tensor-product column, eliminate the two terms containing τ_1 and τ_2 and replacing $\mathbf{1}_2\mapsto 1$, $\tau_A\mapsto i$, to get

$$C(0,5) \equiv \begin{array}{c} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ i\tau_1\otimes\mathbf{1}_2, \\ i\tau_2\otimes\mathbf{1}_2, \\ i\tau_A\otimes\tau_A. \end{array} \tag{9}$$

It is worth pointing out that the $C(0,1)$ and $C(0,5)$ series were correctly considered as "descendant" series in [14] due to the fact that they can be obtained from $C(1,2)$, $C(0,7)$ after erasing extra-Gamma matrices. We find however convenient here to explicitly insert them in table (6) and consider them as "primitive", since they admit a different division-algebra structure (they are almost complex, $\mathbf{C}$) w.r.t. the normal ($\mathbf{R}$)-type maximal Clifford algebras they are derived from.

The remaining primitive maximal Clifford algebras $C(0,x+8m)$, for positive integers $m=1,2,\ldots$ and $x=1,3,5,7$, can be recovered from the *mod* 8 properties of the Gamma-matrices. Let $\overline{\tau}_i$ be a realization of $C(0,x)$ for $x=1,3,5,7$. By applying the (4) algorithm to $C(0,7)$ we construct at first the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2=\mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j,

$j = 1, 2 \ldots, 8$, with ${\gamma_j}^2 = -\mathbf{1}$). We are now in the position [14] to explicitly construct the whole series of primitive maximal Clifford algebras $C(0, x+8n)$, through the formulas

$$C(0, x+8n) \quad \equiv \quad \begin{array}{lll} \overline{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \ldots \otimes \gamma_9, \\ \mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots & \ldots, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes \gamma_9 \otimes \gamma_j, \end{array} \tag{10}$$

Please notice that the tensor product of the 16-dimensional representation is taken n times.

3 On division algebras.

In the previous section we furnished a simple algorithm to explicitly construct any given Clifford irrep of specified division-algebra type. It is convenient to review here the basic features of division algebras which will be needed in the following.

The four division algebra of real ($\mathbf{R}$) and complex ($\mathbf{C}$) numbers, quaternions ($\mathbf{H}$) and octonions ($\mathbf{O}$) possess respectively 0, 1, 3 and 7 imaginary elements e_i satisfying the relations

$$e_i \cdot e_j \quad = \quad -\delta_{ij} + C_{ijk} e_k, \tag{11}$$

(i, j, k are restricted to take the value 1 in the complex case, $1, 2, 3$ in the quaternionic case and $1, 2, \ldots, 7$ in the octonionic case; furthermore, the sum over repeated indices is understood).

C_{ijk} are the totally antisymmetric division-algebra structure constants. The octonionic division algebra is the maximal, since quaternions, complex and real numbers can be obtained as its restriction. The totally antisymmetric octonionic structure constants can be expressed as

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{12}$$

(and vanishing otherwise).

The octonions are the only non-associative, however alternative (see [17]), division algebra.

Due to the antisymmetry of C_{ijk}, it is clear that we can realize (1) by associating the $(0, 3)$ and $(0, 7)$ signatures to, respectively, the imaginary quaternions and the imaginary octonions.

For our later purposes it is of particular importance the notion of division-algebra principal conjugation. Any element X in the given division algebra can be expressed through the sum

$$X \quad = \quad x_0 + x_i e_i, \tag{13}$$

where x_0 and x_i are real, the summation over repeated indices is understood and the positive integral i are restricted up to 1, 3 and 7 in the $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ cases respectively. The principal conjugate X^* of X is defined to be

$$X^* \quad = \quad x_0 - x_i e_i. \tag{14}$$

It allows introducing the division-algebra norm through the product $X^* X$. The normed-one restrictions $X^* X = 1$ select the three parallelizable spheres S^1, S^3 and S^7 in association with $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ respectively.

Further comments on the division algebras and their relations with Clifford algebras can be found in [14] and [17].

4 On fundamental spinors.

In section **2** we discussed the properties of the Clifford irreps, presenting a method to explicitly construct them and mentioning their division-algebra structure. It is worth reminding that the division-algebra character of fundamental spinors does not necessarily (depending on the given space-time) coincide with the division-algebra type of the corresponding Clifford irreps.

Fundamental spinors carry a representation of the generalized Lorentz group with a minimal number of real components in association with the maximal, compatible, allowed division-algebra structure.

The following table, taken from the results in [18] and [13], see also [14], presents the comparison between division-algebra properties of Clifford irreps ($\mathbf{\Gamma}$) and fundamental spinors ($\mathbf{\Psi}$), in different space-times parametrized by $\rho = s - t \quad mod \quad 8$. We have

ρ	$\mathbf{\Gamma}$	$\mathbf{\Psi}$
0	**R**	**R**
1	**R**	**R**
2	**R**	**C**
3	**C**	**H**
4	**H**	**H**
5	**H**	**H**
6	**H**	**C**
7	**C**	**R**

(15)

It is clear from the above table that, for $\rho = 2, 3$, the fundamental spinors can accommodate a larger division-algebra structure than the corresponding Clifford irreps. Conversely, for $\rho = 6, 7$, the Clifford irreps accommodate a larger division-algebra structure than the corresponding spinors. In several cases this mismatch of division-algebra structures plays an important role. For instance in [11] a method was introduced to construct superconformal algebras based on the minimal division algebra structure common to both Clifford irreps and fundamental spinors. This method can be straightforwardly modified to produce extended superconformal algebras based on the largest division-algebra structure. The price to be paid, in this case, would imply the introduction, for $\rho = 2, 3$, of reducible Clifford representations and, conversely, for $\rho = 6, 7$ of non-minimal spinors.

The reason behind the mismatch can be easily understood on the basis of the algorithmic construction of Section **2** and of table (6). Indeed, all the maximal, descendant Clifford algebras appearing in table (6) have all block-antidiagonal Gamma matrices with the exception of a single Gamma matrix given by $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$. Therefore, all non-maximal Clifford algebras which are produced by erasing this extra Gamma matrix (a detailed discussion can be found in [14]) are of block-antidiagonal form. We recall now that the fundamental spinors carry a representation of the generalized Lorentz group whose generators are given by the commutators among Gamma matrices, $[\Gamma_i, \Gamma_j]$. For the non-maximal Clifford algebras under considerations these commutators are all in 2×2 block-diagonal forms, allowing to introduce a (generalized, in the sense specified in [14]) Weyl projection for fundamental spinors, with non-vanishing upper or lower components.

It is convenient to explicitly discuss the simplest Minkowskian cases where the mismatch appears (the general procedure can be straightforwardly read from table (6)). In the ordinary $(3,1)$ space-time the $(\mathbf{R})$ Clifford irrep is obtained as the non-maximal Clifford algebra $(3,1) \subset (3,2)$, obtained from the maximal $(\mathbf{R})$ $(3,2)$ after erasing a time-like Gamma matrix. On the other hand, the fundamental complex spinors are obtained from the reducible Clifford representation $(3,1) \subset (4,1)$, obtained by erasing a space-like Gamma matrix from the $(\mathbf{C})$ Clifford irrep $(4,1)$.

In the other Minkowskian cases we get

i) $(4,1)$: $\mathbf{\Gamma}$ coincides with the maximal Clifford $(4,1)$ $(\mathbf{C})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(4,1) \subset (6,1)$ $(\mathbf{H})$,

ii) $(7,1)$: $\mathbf{\Gamma}$ coincides with the non-maximal Clifford $(7,1) \subset (7,2)$ $(\mathbf{H})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(7,1) \subset (8,1)$ $(\mathbf{C})$,

iii) $(8,1)$: $\mathbf{\Gamma}$ coincides with the maximal Clifford $(8,1)$ $(\mathbf{C})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(8,1) \subset (10,1)$ $(\mathbf{R})$.

5 Generalized supersymmetries: the M and F algebra examples

Three matrices, denoted as A, B, C, have to be introduced in association with the three conjugations (hermitian, complex and transposition) acting on Gamma matrices [3]. Since only two of the above matrices are independent we choose here, following [14], to work with A and C. A plays the role of the time-like Γ^0 matrix in the Minkowskian space-time and is used to introduce barred spinors. C, on the other hand, is the charge conjugation matrix. Up to an overall sign, in a generic (s,t) space-time, A and C are given by the products of all the time-like and, respectively, all the symmetric (or antisymmetric) Gamma-matrices[3]. The properties of A and C immediately follow from their explicit construction, see [3] and [14].

In a representation of the Clifford algebra realized by matrices with real entries, the conjugation acts as the identity, see (14). In this case the space-like gamma matrices are symmetric, while the time-like gamma matrices are antisymmmetric, so that A can be identified with the charge conjugation matrix C_A.

For our purposes the importance of A and the charge conjugation matrix C lies on the fact that, in a D-dimensional space-time ($D = s+t$) spanned by $d \times d$ Gamma matrices, they allow to construct a basis for $d \times d$ (anti)hermitian and (anti)symmetric matrices, respectively. It is indeed easily proven that, in the real and the complex cases (the quaternionic case is different), the $\left(\begin{array}{c} D \\ k \end{array}\right)$ antisymmetrized products of k Gamma matrices $A\Gamma^{[\mu_1 \ldots \mu_k]}$ are all hermitian or all antihermitian, depending on the value of $k \leq D$. Similarly, the antisymmetrized products $C\Gamma^{[\mu_1 \ldots \mu_k]}$ are all symmetric or all antisymmetric.

For what concerns the M-algebra, the 32-component real spinors of the $(10,1)$-spacetime admit anticommutators $\{Q_a, Q_b\}$ which are 32×32 symmetric real matrices with, at most, $32 + \frac{32\times 31}{2} = 528$ components. Expanding the r.h.s. in terms of the antisymmetrized product

[3]Depending on the given space-time (see [3] and [14]), there are at most two charge conjugations matrices, C_S, C_A, given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special space-time signatures they collapse into a single matrix C.

of Gamma matrices, we get that it can be saturated by the so-called M-algebra

$$\{Q_a, Q_b\} = (A\Gamma_\mu)_{ab} P^\mu + \left(A\Gamma_{[\mu\nu]}\right)_{ab} Z^{[\mu\nu]} + \left(A\Gamma_{[\mu_1\ldots\mu_5]}\right)_{ab} Z^{[\mu_1\ldots\mu_5]}. \tag{16}$$

Indeed, the $k = 1, 2, 5$ sectors of the r.h.s. furnish $11 + 55 + 462 = 528$ overall components. Besides the translations P^μ, in the r.h.s. the antisymmetric rank-2 and rank-5 abelian tensorial central charges, $Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ respectively, appear.

The (16) saturated M-algebra admits a finite number of subalgebras which are consistent with the Lorentz properties of the Minkowskian eleven dimensions. There are 6 such subalgebras which are recovered by setting either one or two among the three sets of tensorial central charges P^μ, $Z^{[\mu\nu]}$, $Z^{[\mu_1\ldots\mu_5]}$ identically equal to zero (a completely degenerate subalgebra is further obtained by setting the whole r.h.s. identically equal to zero).

The fact that the fundamental spinors in a $(10, 2)$-spacetime also admit 32 components is due to the existence of the Weyl projection. This implies that the saturated M-algebra admits a $(10, 2)$ space-time presentation, the so-called F-algebra, in terms of $(10, 2)$ Majorana-Weyl spinors $\tilde{Q}_{\tilde{a}}$, $\tilde{a} = 1, 2, \ldots, 32$.

In the case of Weyl projected spinors the r.h.s. has to be reconstructed with the help of a projection operator which selects the upper left block in a 2×2 block decomposition. Specifically, if $\mathcal{M}$ is a matrix decomposed in 2×2 blocks as $\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$, we can define

$$P(\mathcal{M}) \equiv \mathcal{M}_1. \tag{17}$$

The saturated M-algebra (16) can therefore be rewritten as

$$\left\{\tilde{Q}_{\tilde{a}}, \tilde{Q}_{\tilde{b}}\right\} = P\left(\tilde{A}\tilde{\Gamma}_{\tilde{\mu}\tilde{\nu}}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}\tilde{\nu}]} + P\left(\tilde{A}\tilde{\Gamma}_{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}, \tag{18}$$

where all tilde's are referred to the corresponding $(10, 2)$ quantities. The matrices in the r.h.s. are symmetric in the exchange $\tilde{a} \leftrightarrow \tilde{b}$. This time the rank-2 and selfdual rank-6 antisymmetric abelian tensorial central charges, $\tilde{Z}^{[\tilde{\mu}\tilde{\nu}]}$ and respectively $\tilde{Z}^{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}$, appear. Their total number of components is $66 + 462 = 528$, therefore proving the saturation of the r.h.s.. The saturated equation (18) is named the F-algebra.

6 Real, complex and quaternionic generalized supersymmetries.

For real n-component spinors Q_a, the most general supersymmetry algebra is represented by

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab}, \tag{19}$$

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n\times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space-time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of k-Gamma matrices, namely

$$\mathcal{Z}_{ab} = \sum_k (A\Gamma_{[\mu_1\ldots\mu_k]})_{ab} Z^{[\mu_1\ldots\mu_k]}, \tag{20}$$

where the values k entering the sum in the r.h.s. are restricted by the symmetry requirement for the $a \leftrightarrow b$ exchange and are specific for the given spacetime. The coefficients $Z^{[\mu_1 \dots \mu_k]}$ are the rank-k abelian tensorial central charges.

When the fundamental spinors are complex or quaternionic they can be organized in complex (for the **C** and **H** cases) and quaternionic (for the **H** case) multiplets, whose entries are respectively complex numbers or quaternions.

The real generalized supersymmetry algebra (19) can now be replaced by the most general complex or quaternionic supersymmetry algebras, given by the anticommutators among the fundamental spinors Q_a and their conjugate ${Q^*}_{\dot{a}}$ (where the conjugation refers to the principal conjugation in the given division algebra, see (14)). We have in this case

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab} \quad , \quad \{{Q^*}_{\dot{a}}, {Q^*}_{\dot{b}}\} = {\mathcal{Z}^*}_{\dot{a}\dot{b}}, \tag{21}$$

together with

$$\{Q_a, {Q^*}_{\dot{b}}\} = \mathcal{W}_{a\dot{b}}, \tag{22}$$

where the matrix $\mathcal{Z}_{ab}$ (${\mathcal{Z}^*}_{\dot{a}\dot{b}}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{W}_{a\dot{b}}$ is hermitian.

The maximal number of allowed components in the r.h.s. is given, for complex fundamental spinors with n complex components, by
ia) $n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ complex matrix $\mathcal{Z}_{ab}$ plus
iia) n^2 (real) bosonic components entering the hermitian $n \times n$ complex matrix $\mathcal{W}_{a\dot{b}}$.

Similarly, the maximal number of allowed components in the r.h.s. for quaternionic fundamental spinors with n quaternionic components is given by
ib) $2n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ quaternionic matrix $\mathcal{Z}_{ab}$ plus
iib) $2n^2 - n$ (real) bosonic components entering the hermitian $n \times n$ quaternionic matrix $\mathcal{W}_{a\dot{b}}$.

The previous numbers do not necessarily mean that the corresponding generalized supersymmetry is indeed saturated. This is in particular true in the quaternionic case, see [15].

Any real generalized supersymmetry admitting a complex structure can be re-expressed in a complex formalism with n-component complex spinors and total number of $n(2n+1)$ (real) bosonic components split into $n(n+1)$ components entering the symmetric matrix $\mathcal{Z}$ and n^2 components entering the hermitian matrix $\mathcal{W}$. The situation is different in the quaternionic case. The quaternionic structure requires a restriction on the total number of bosonic generators. n-component quaternionic spinors can be described as $4n$-component real spinors. However, the r.h.s. of a quaternionic (21) and (22) superalgebra admits at most $4n^2+n$ bosonic components, instead of $8n^2+2n$ of the most general supersymmetric real algebra. The Lorentz-covariance further restricts the number of bosonic generators in a quaternionic supersymmetry algebra.

We conclude this section mentioning the two big classes of subalgebras, respecting the Lorentz-covariance, that can be obtained from (21) and (22) in both the complex and quaternionic cases. They are obtained by setting identically equal to zero either $\mathcal{Z}$ or $\mathcal{W}$, namely

I) $\mathcal{Z}_{ab} \equiv {\mathcal{Z}^*}_{\dot{a}\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter the hermitian matrix $\mathcal{W}_{a\dot{b}}$ or, conversely,

II) $\mathcal{W}_{a\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter $\mathcal{Z}_{ab}$ and its conjugate matrix ${\mathcal{Z}^*}_{\dot{a}\dot{b}}$.

Accordingly, in the following we will refer to the (complex or quaternionic) generalized supersymmetries satisfying the *I)* constraint as "hermitian" (or "type I") generalized supersymmetries, while the (complex or quaternionic) generalized supersymmetries satisfying the *II)* constraint will be referred to as "holomorphic" (or "type II") generalized supersymmetries.

7 Generalized supersymmetries and the octonionic M-superalgebra

As already recalled, in the $D=11$ Minkowskian spacetime, where the M-theory should be found, the spinors are real and have 32 components. Since the most general symmetric 32×32 matrix admits 528 components, one can easily prove that the most general supersymmetry algebra in $D=11$ can be presented as

$$\{Q_a, Q_b\} = (C\Gamma_{\mu})_{ab}P^{\mu} + (C\Gamma_{[\mu\nu]})_{ab}Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{ab}Z^{[\mu_1\ldots\mu_5]} \tag{23}$$

(where C is the charge conjugation matrix), while $Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ are totally antisymmetric tensorial central charges, of rank 2 and 5 respectively, which correspond to extended objects [21, 22], the p-branes. Please notice that the total number of 528 is obtained in the r.h.s as the sum of the three distinct sectors, i.e.

$$528 \quad = \quad 11+66+462. \tag{24}$$

The algebra (16) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra, recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1\ldots\mu_5]} \equiv 0$.

The octonionic M-superalgebra is introduced by assuming an octonionic structure for the spinors which, in the $D=11$ Minkowskian spacetime, are octonionic-valued 4-component vectors. The algebra replacing (16) is given by

$$\{Q_a, Q_b\} = \{{Q^*}_a, {Q^*}_b\} = 0, \qquad \{Q_a, {Q^*}_b\} = Z_{ab}, \tag{25}$$

where $*$ denotes the principal conjugation in the octonionic division algebra and, as a result, the bosonic abelian algebra on the r.h.s. is constrained to be hermitian

$$Z_{ab} \quad = \quad {Z_{ba}}^*, \tag{26}$$

leaving only 52 independent components.

The Z_{ab} matrix can be represented either as the $11+41$ bosonic generators entering

$$Z_{ab} = P^{\mu}(C\Gamma_{\mu})_{ab} + Z_{\mathbf{O}}^{\mu\nu}(C\Gamma_{\mu\nu})_{ab}, \tag{27}$$

or as the 52 bosonic generators entering

$$Z_{ab} = Z_{\mathbf{O}}^{[\mu_1\ldots\mu_5]}(C\Gamma_{\mu_1\ldots\mu_5})_{ab}\,. \tag{28}$$

Due to the non-associativity of the octonions, unlike the real case, the sectors individuated by (27) and (28) are not independent. Furthermore, as we have already seen for $k=2$, in the antisymmetric products of k octonionic-valued matrices, a certain number of them are redundant (for $k = 2$, due to the G_2 automorphisms, 14 such products have to be erased). In the general case [14] a table can be produced expressing the number of independent components in D odd-dimensional spacetime octonionic realizations of Clifford algebras, by taking into account that out of the D Gamma matrices, 7 of them are octonionic-valued, while the remaining $D-7$ are purely real. We get the following table, with the columns labeled by k, the number of antisymmetrized Gamma matrices and the rows by D (up to $D = 13$)

$D \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
7	1	7	7	1	1	7	7	1						
9	1	9	22	22	10	10	22	22	9	1				
11	1	11	41	75	76	52	52	76	75	41	11	1		
13	1	13	64	168	267	279	232	232	279	267	168	64	13	1

(29)

For what concerns the octonionic equivalence of the different sectors, it can be symbolically expressed, in different odd space-time dimensions, according to the table

$D = 7$	$M0 \equiv M3$
$D = 9$	$M0 + M1 \equiv M4$
$D = 11$	$M1 + M2 \equiv M5$
$D = 13$	$M2 + M3 \equiv M6$
$D = 15$	$M3 + M4 \equiv M0 + M7$

(30)

In $D = 11$ dimensions the relation between $M1 + M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into the 4 real indices, labeled by $a, b, c, \ldots$ and the 7 octonionic indices labeled by $i, j, k, \ldots$. The 52 independent components are recovered from $52 = 4 + 2 \times 7 + 6 + 28$, according to

4	$M1_a$	$M5_{[aijkl]} \equiv M5_a$
7	$M1_i$, $M2_{[ij]} \equiv M2_i$	$M5_{[abcdi]} \equiv M5_i$,$M5_{[ijklm]} \equiv \widetilde{M5}_i$
6	$M2_{[ab]}$	$M5_{[abijk]} \equiv M5_{[ab]}$
$4 \times 7 = 28$	$M2_{[ai]}$	$M5_{[abcij]} \equiv M5_{[ai]}$

(31)

8 The octonionic superconformal M-algebra

The conformal algebra of the octonionic M-theory can be introduced [12] adapting to the eleven dimensions the procedure discussed in [5] for the 10 dimensional case. It requires the identification of the conformal algebra of the octonionic $D = 11$ M-algebra with the generalized Lorentz algebra in the $(11, 2)$-dimensional space-time. In such a space-time the octonionic Clifford's Gamma-matrices are 8-dimensional. The basis of the hermitian generators is given by the 64 antisymmetric two-tensors $C\Gamma_{[\mu_1\mu_2]}\mathcal{Z}^{\mu_1\mu_2}$ and the 168 antisymmetric three tensors

$C\Gamma_{[\mu_1\mu_2\mu_3]}\mathcal{Z}^{\mu_1\mu_2\mu_3}$ (or, equivalently, by the 232 antisymmetric six-tensors $C\Gamma_{[\mu_1...\mu_6]}\mathcal{Z}^{\mu_1..\mu_6}$). This is already an indication that the total number of generators in the conformal algebra is 232. We will show that this is the case.

According to [5] the conformal algebra can be introduced as the algebra of transformations leaving invariant the inner product of Dirac's spinors. In $(11,2)$ this is given by $\psi^\dagger C\eta$, where the matrix C, the analogous of the Γ^0, given by the product of the two space-like Clifford's Gamma matrices, is real-valued and totally antisymmetric. Therefore, the conformal transformations are realized by the octonionic-valued 8-dimensional matrices $\mathcal{M}$ leaving C invariant, i.e. satisfying

$$\mathcal{M}^\dagger C + C\mathcal{M} \;=\; 0. \tag{32}$$

This allows identifying the (quasi)-group of conformal transformations with the (quasi-)group of symplectic transformations. Indeed, under a simple change of variables, C can be recast in the form

$$\Omega \;=\; \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix}. \tag{33}$$

The most general octonionic-valued matrix leaving invariant Ω can be expressed through

$$\mathbf{M} \;=\; \begin{pmatrix} D & B \\ C & -D^\dagger \end{pmatrix}, \tag{34}$$

where the 4×4 octonionic matrices B, C are hermitian

$$B = B^\dagger, \qquad\qquad C = C^\dagger. \tag{35}$$

It is easily seen that the total number of independent components in (34) is precisely 232, as we expected from the previous considerations.

It is worth noticing that the set of matrices $\mathbf{M}$ of (34) type forms a closed algebraic structure under the usual matrix commutation. Indeed $[\mathbf{M},\mathbf{M}]\subset\mathbf{M}$ endows the structure of $Sp(8|\mathbf{O})$ to $\mathbf{M}$. For what concerns the supersymmetric extension of the superconformal algebra, we have to accommodate the 64 real components (or 8 octonionic) spinors of $(11,2)$ into a supermatrix enlarging $Sp(8|\mathbf{O})$. This can be achieved as follows. The two 4-column octonionic spinors α and β can be accommodated into a supermatrix of the form

$$\left(\begin{array}{c|cc} 0 & -\beta^\dagger & \alpha^\dagger \\ \hline \alpha & 0 & 0 \\ \beta & 0 & 0 \end{array}\right). \tag{36}$$

Under anticommutation, the lower bosonic diagonal block reduces to $Sp(8|\mathbf{O})$. On the other hand, extra seven generators, associated to the 1-dimensional antihermitian matrix A

$$A^\dagger \;=\; -A, \tag{37}$$

i.e. representing the seven imaginary octonions, are obtained in the upper bosonic diagonal block. Therefore, the generic bosonic element is of the form

$$\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D & B \\ 0 & C & -D^\dagger \end{array}\right), \tag{38}$$

with A, B and C satisfying (37) and (35).

The closed superalgebraic structure, with (36) as generic fermionic element and (38) as generic bosonic element, will be denoted as $OSp(1,8|\mathbf{O})$. It is the superconformal algebra of the M-theory and admits a total number of 239 bosonic generators.

9 Conclusions.

We have seen that, contrary to what is commonly believed, an alternative formulation for the M superalgebra and the M superconformal algebra can be consistently introduced in association with the non-associative maximal division algebra of the octonions. It presents peculiar features, like the non-independence of the different octonionic brane sectors, which is a reflection of the higher-rank antisymmetric octonionic tensorial identities discussed in section **5**. The existence of this second variant of the M algebra is puzzling. It could be ultimately related with the arising of exceptional structures (exceptional Lie and Jordan algebras) in the "Theory Of Everything" [19].

Since imaginary octonions admits a geometrical description in terms of the seven sphere S^7, it could be speculated that the higher-dimensional octonionic descriptions, e.g. of the eleven dimensions, corresponds to a particular compactification of the eleven-dimensional M theory down to $AdS_4 \times S^7$. This compactification corresponds to a natural solution for the 11 dimensional supergravity, see [20].

The octonionic superconformal algebra $OSp(1,8|\mathbf{O})$ has been explicitly derived. It corresponds to a supersymmetric extension of a bosonic conformal algebra which is mathematically interesting since it corresponds to a closed algebraic structure which goes beyond the standard notion of conformal algebra of a given Jordan algebra, see [12].

Besides this aspect, the notion of hermitian (complex and quaternionic) and holomorphic (complex and quaternionic) supersymmetries, as consistently division-algebra constrained generalized supersymmetries, has been presented.

Physical implications of these mathematical structures are quite obvious. The classification of generalized supersymmetries allow to understand the web of interrelated dualities of different classes of theories which can be either analitically continued (let's say, to the Euclidean) or recovered through dimensional reduction.

As an example, we can cite that the analytic continuation of the M algebra was proven in [23] to correspond to an eleven-dimensional complex holomorphic supersymmetry. It was further shown in [15] that the same algebra also admits a 12-dimensional Euclidean presentation in terms of Weyl-projected spinors. These two examples of Euclidean supersymmetries can find application in the functional integral formulation of higher-dimensional supersymmetric models.

There is an interesting class of models which nicely fits in the framework here described and is currently under intense investigation. It is the class of superparticle models, introduced at first in [24] and later studied in [25], whose bosonic coordinates correspond to tensorial central charges. It was shown in [26] that a 4-dimensional theory of this kind leads to a tower of massless higher spin states, concretely implementing a Fronsdal's proposal [27] of introducing bosonic tensorial coordinates to describe massless higher spin theories (admitting helicity states greater than two). This is an active area of investigation, the main motivation beingthe investigation the tensionless limit of superstring theory, corresponding to a tower of higher helicity massless particles (see e.g. [28]).

In a somehow "orthogonal" direction, a class of theories which can be investigated in the present framework is the class of supersymmetric extensions of Chern-Simon supergravities in higher dimensions, requiring as a basic ingredient a Lie superalgebra admitting a Casimir of appropriate order, see e.g. [29].

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