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SUSY QM VIA 2x2 MATRIX SUPERPOTENTIAL

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Abstract

The $N = 2$ supersymmetry in quantum mechanics involving two-component eigenfunction is investigated.

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1 Introduction

The algebraic technique of the supersymmetry in quantum mechanics (SUSY QM) formulated by Witten [1], in which the essential idea is based on the Darboux procedure on second-order differential equations, has been extended in order to find the 2x2 matrix superpotencial [2, 3, 4, 5].

In this work we show that the superpotential for the SUSY QM with two-component wave functions is a Hermitian matrix, and we consider the application to a planar physical system, a neutron interacting with the magnetic field [6].

2 Supersymmetry for two-component eigenfunction

In this section we consider a non-relativistic Hamiltonian ($\mathbf{H}_1$) for a two-component wave function in the following bilinear forms

$$\begin{aligned}\mathbf{H}_1 \quad &= \mathbf{A}_1^+\mathbf{A}_1^- + E_1^{(0)} \\ &= -\mathbf{I}\frac{d^2}{dx^2} + \left(\frac{d}{dx}\mathbf{W}_1(x)\right) + \mathbf{W}_1(x)\frac{d}{dx} - \mathbf{W}_1^\dagger(x)\frac{d}{dx} + \mathbf{W}_1^\dagger\mathbf{W}_1(x), \end{aligned} \tag{1}$$

$$\begin{aligned}\mathbf{H}_2 \quad &= \mathbf{A}_1^-\mathbf{A}_1^+ + E_1^{(0)} \\ &= -\mathbf{I}\frac{d^2}{dx^2} - \left(\frac{d}{dx}\mathbf{W}_1^\dagger(x)\right) - \mathbf{W}_1^\dagger(x)\frac{d}{dx} + \mathbf{W}_1(x)\frac{d}{dx} + \mathbf{W}_1(x)\mathbf{W}_1^\dagger, \end{aligned} \tag{2}$$

where

$$\mathbf{A}_1^- = -\mathbf{I}\frac{d}{dx} + \mathbf{W}_1(x), \quad \mathbf{A}_1^+ = \left(\mathbf{A}_1^-\right)^\dagger. \tag{3}$$

So far $\mathbf{W}_1(x)$ can be a two by two non-Hermitian matrix, but we will now show that $\mathbf{H}_1$ and $\mathbf{H}_2$ are exactly the Hamiltonians of the bosonic and fermionic sectors of a SUSY Hamiltonian if and only if the matrix superpotential is a Hermitian one. Indeed (comparing the pair SUSY Hamiltonians $\mathbf{H}_\pm$ with the Hamiltonians $\mathbf{H}_1$ and $\mathbf{H}_2$) we see that only when the hermiticity condition of the $\mathbf{W}_1$ is readily satisfied, i.e., $\mathbf{W}_1^\dagger = \mathbf{W}_1$, we may put $\mathbf{H}_1$ in a bosonic sector Hamiltonian. In this case $\mathbf{H}_1$ ($\mathbf{H}_2$) becomes exactly $\mathbf{H}_-$ ($\mathbf{H}_+$) of a SUSY Hamiltonian model, analogous to the Witten model, viz.,

$$\mathbf{H}_{SUSY} = -\frac{1}{2}\mathbf{I}\frac{d^2}{dx^2} + \frac{1}{2}\left\{\mathbf{W}^2(x) + \mathbf{W}'(x)\sigma_3\right\} = \begin{pmatrix}\mathbf{H}_- & 0 \\ 0 & \mathbf{H}_+\end{pmatrix}, \tag{4}$$

where σ_3 is the Pauli matrix. Only under the hermiticity condition one can to call $\mathbf{W}_1 = \mathbf{W}(x)$ of a matrix superpotential.

Let $\mathbf{H}_\pm$ be the bosonic (-) and fermionic (+) sector Hamiltonians for a two-component eigenstate Ψ_-, given by

$$\mathbf{H}_\pm = -\mathbf{I}\frac{d^2}{dx^2} + \mathbf{V}_\pm(x), \quad \Psi_-(x) = \left(\begin{array}{c} \psi_{-,1}(x) \\ \psi_{-,2}(x) \end{array}\right), \quad E_-^{(0)} = 0, \tag{5}$$

where $\mathbf{I}$ denotes the 2x2 unit matrix and the pair of SUSY potential $\mathbf{V}_-(x)$, is a 2x2 matrix potential which may be written in terms of a 2x2 matrix superpotential $\mathbf{W}$(x), viz.,

$$\mathbf{V}_\pm(x) = \mathbf{W}^2(x) \mp \mathbf{W}'(x). \tag{6}$$

Let us consider the eigenvalue equations for the bosonic and fermionic sector Hamiltonians, viz.,

$$\mathbf{H}_\pm \Psi_\pm^{(n)} = E_\pm^{(n)} \Psi_\pm^{(n)}, n = 0, 1, 2, \cdots. \tag{7}$$

These systems can exhibit bound and continuous eigenstates under the annihilation conditions

$$\mathbf{A}^- \Psi_-^{(0)} = 0, \quad \Psi_-^{(0)}(x) = \left(\begin{array}{c} \psi_{-,1}^{(0)}(x) \\ \psi_{-,2}^{(0)}(x) \end{array}\right) \tag{8}$$

or

$$\mathbf{A}^+ \Psi_+^{(0)} = 0, \quad \Psi_+^{(0)}(x) = \left(\begin{array}{c} \psi_{+,1}^{(0)}(x) \\ \psi_{+,2}^{(0)}(x) \end{array}\right). \tag{9}$$

In this case we see that one cannot put $\Psi_+^{(0)}(x)$ in terms of $\Psi_-^{(0)}(x)$ and vice-versa in a similar manner to the case of one-component eigenfunction system. However, if $\Psi_-^{(0)}(x)$ is normalizable we have

$$\int_{-\infty}^{+\infty} \left(|\psi_{-,1}^{(0)}|^2 + |\psi_{-,2}^{(0)}|^2\right) dx = 1. \tag{10}$$

Note that in Eq. (5) of ref. [4] the author has taken a particular Hermitian matrix for his superpotential in such a way that the validity of his development is ensured.

Let us now consider the interesting application of the above development for a bidimensional physical system in coordinate space associated to a Neutron with magnetic momentum $\vec{\mu} = \mu(\sigma_1, \sigma_2, \sigma_3)$ in a static magnetic field [6]. In this case, $x = \rho > 0$, the Ricatti equation in matrix form is given by

$$\mathbf{V}_-(\rho) = \mathbf{W}'(\rho) + \mathbf{W}^2(\rho) = \left(\begin{array}{cc} \frac{m^2-\frac{1}{4}}{\rho^2} & \frac{-2F}{\rho} \\ \frac{-2F}{\rho} & \frac{(m+1)^2-\frac{1}{4}}{\rho^2} \end{array}\right) - \mathbf{I}\tilde{E}_1^{(0)}, \tag{11}$$

which has the following particular solution for the 2x2 matrix superpotential given by $\mathbf{W}_m = \left(\begin{array}{cc} \frac{m+\frac{1}{2}}{\rho} & -\frac{F}{m+1} \\ -\frac{F}{m+1} & \frac{m+\frac{3}{2}}{\rho} \end{array}\right)$, where the energy eigenvalue of the ground state is $\tilde{E}_1^{(0)} = -\frac{F^2}{2(m+1)^2}$, $F \propto -\mu I$, $m = 0, \pm 1, \pm 2, \cdots$, and ρ is the usual cylindrical coordinate. We are considering the current I located along the z-axis, and we have used units with $\hbar = 1 = mass$. The current I generate a static magnetic field. Also, note that $\mathbf{V}_-(\rho)$ has zero ground state energy, $E_-^{(0)} = 0$, thus SUSY is said to be unbroken.

The algebra of SUSY in quantum mechanics is characterized by one anti-commutation and two commutation relations given below

$$H_{SUSY} = [Q_-, Q_+]_+, \quad [H_{SUSY}, Q_\pm]_- = 0 = (Q_-)^2 = (Q_+)^2. \tag{12}$$

One representation of the $N = 2$ SUSY superalgebra is the following

$$H_{SUSY} = [Q_-, Q_+]_+ = \left(\begin{array}{cc} \mathbf{A}^+\mathbf{A}^- & 0 \\ 0 & \mathbf{A}^-\mathbf{A}^+ \end{array}\right)_{4X4} = \left(\begin{array}{cc} \mathbf{H}_- & 0 \\ 0 & \mathbf{H}_+ \end{array}\right). \tag{13}$$

The supercharges $Q_\pm$ are differential operators of first order and can be given by $Q_- = \left(\begin{array}{cc} 0 & 0 \\ \mathbf{A}^- & 0 \end{array}\right)_{4X4}$, $Q_+ = \left(\begin{array}{cc} 0 & \mathbf{A}^+ \\ 0 & 0 \end{array}\right)_{4X4}$, where $\mathbf{A}^\pm$ are 2x2 non-Hermitian matrices given by Eq. (3).

3 Conclusion

In this work we investigate an extension of the supersymmetry in non-relativistic quantum mechanics for two-component wave functions. This leads to 4x4 supercharges and supersymmetric Hamiltonians whose bosonic sectors are privileged with two-component eigenstates.

We have considered the application for a Neutron in interaction with a static magnetic field of a straight current carrying wire [6].

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