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Prototype for Memory Effects: Analytical Results

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ABSTRACT

We present analytical results for a one-dimensional cellular automaton which is a good prototype for memory effects on damage. The dynamics of the system is studied through the analysis of the associated Hamming distance as a function of time. The characteristics of its fluctuations indicate the existence of some type of memory.

Key-words: Cellular automaton; Hamming distance; Memory effects; Noise.

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INTRODUCTION

Memory effects are present in a huge number of complex dynamical systems (magnetic systems, neural networks, social groups models, etc.). Therefore, the study of those effects is relevant for enlightening our understanding of critical phenomena. One of the ways of recognizing the presence of memory effects is based on the study of the spreading of a damage introduced in the system. This may be achieved through the analysis of the time evolution of two different replicas of the system, between which a small difference is introduced, for instance, in the initial conditions or in the random sequence which generates the evolution of the system. In order to characterize the damage, the quantity whose time evolution is followed typically is the Hamming distance[1]. This quantity behaves as a function of time, in a more or less noise-like manner, as well as the quantities which characterize a large number of complex systems comprising from the discrete sandpile model[2] to models describing the dynamics of social groups[3].

Recently, it was introduced by Tsallis et al.[4] a simple prototype whose associated Hamming distance as a function of time mimics many complex dynamical systems. From the study, by means of computational simulations, of the time evolution of the associated Hamming distance, the prototype was shown to exhibit effects of memory in the evolution of damages as a large number of systems and theoretical models do.

For different values of the external parameters the behaviour of the prototype varies from a noise-like to a plateaux-like one. In the present work, we find an analytical expression which describes its behaviour and we compare our results with those obtained previously[4] through numerical simulations. Moreover, as suggested in ref. [4], we also describe analytically the distribution of probabilities ($P(\tau)$) of finding plateaux of size τ and the momenta of $P(\tau)$.

THE PROTOTYPE

As described in ref. [4], we assume a semi-infinite linear chain of sites ($i = 0, 1, 2, ...$) occupied by binary random variables $\{S_i\}$ ($S_i = 0, 1,\ \forall i$). The prototype is defined through the following construction rules:

- $S_0 = 1$
- $S_{i+1} = S_i$ with probability p (thus $S_{i+1} \neq S_i$ with probability $1-p$), for all $i = 0, 1, 2, ...$

We consider two equivalent replicas of the system $\{S_i^A\}$ and $\{S_i^B\}$, constructed with the same value of p but corresponding to different random sequences.

In order to study the time dynamics of the damage evolution in the system, we define the following Hamming distance:

$$H(t) = \frac{1}{L} \sum_{i=i_0}^{i_0+L-1} \mid S_i^A - S_i^B \mid, \tag{1}$$

where L is the length of the window we will analyze, $i_0 \equiv Jt$, J being a fixed positive integer number and t the discrete time ($t = 0, 1, 2, \ldots$).

As already discussed in ref. [4], the Hamming distance $H(t)$ as a function of time fluctuates, exhibiting, for different values of the external parameters, behaviours which vary from a noise-like to a plateaux-like one. It is worth recalling that these behaviours are present in a large number of complex dynamical systems that plausibly belong to the same "class of universality" that the prototype under consideration. It is specially interesting to analyze the behaviour of the distribution of plateaux present in the fluctuations of the Hamming distance. We say that there is a plateau at time t if $H(t) = H(t+1)$ and the plateau is of length τ if $H(t-1) \neq H(t) = H(t+1) = ... = H(\tau - 1) = H(\tau) \neq H(\tau+1)$. For fixed values of the external variables (p, J, L), $H(t)$ yields a distribution of plateaux $P(\tau)$ $(\sum_{\tau=0}^{\infty} P(\tau) = 1))$ and the probability of having finite-size plateaux is:

$$M(p, J, L) \equiv 1 - P(0). \tag{2}$$

RESULTS

First, we analyze the mean value of the Hamming distance $< H(t) >$. It follows directly that:

$$< H(t) > = \frac{1}{2} - \frac{(2p-1)^{2Jt}}{8\, L\, p\, (1-p)} (1 - (2p-1)^{2L}). \tag{3}$$

So that, in the limit $L \to \infty$, for $0 < p < 1$, we have:

$$\lim_{L \to \infty} \;\; < H(t) > = \frac{1}{2}, \tag{4}$$

hence, there are no long-range memory effects for any value of J and $0 < p < 1$.

For a more detailed study, we want to know the probability of having finite size plateaux, M(p,J,L), which is the probability that $H(t) = H(t+1)$ for an arbitrary time t. From the analysis of all possible configurations producing $H(t) = H(t+1)$ (see APPENDIX), we obtain an analytical expression for $M(p, J, L)$:

$$\begin{aligned} M(p, J, L) &= M(R, a, b) = \\ &= \frac{1}{2} R^{2(a-1)} \left(1 + W_1(R, a) + (2R-1)^{b-a+1} \left[1 + W_2(R, a)\right]\right), \end{aligned} \tag{5}$$

where $a \equiv min\{J, L\}$, $b \equiv max\{J, L\}$ and $R \equiv R(p) = p^2 + (1-p)^2$,

$$W_1(R, a) = \frac{1}{2} \sum_{l=1}^{a-1} \left(\sigma_3(R, a, l) + 2\sigma_2(R, a, l) + \sigma_1(R, a, l)\right)^2 \tag{6}$$

and

$$W_2(R, a) = \frac{1}{2} \sum_{l=1}^{a-1} \left(\sigma_3(R, a, l) - \sigma_1(R, a, l)\right)^2, \tag{7}$$

with

$$\sigma_1(R, a, l) = \sum_{k=1}^{l-1} \binom{l-1}{k} \binom{a-l-1}{k-1} \left(\frac{1-R}{R}\right)^{2k}, \tag{8}$$

$$\sigma_2(R, a, l) = \sum_{k=0}^{l-1} \binom{l-1}{k} \binom{a-l-1}{k} \left(\frac{1-R}{R}\right)^{2k+1}, \tag{9}$$

$$\sigma_3(R, a, l) = \sum_{k=1}^{l} \binom{l-1}{k-1} \binom{a-l-1}{k-1} \left(\frac{1-R}{R}\right)^{2k}. \tag{10}$$

The comparison of these results with the experimental ones as obtained in ref. [4] shows an excellent agreement within the standard deviation of the values from simulations. For a wide range of values of the external parameters, the maximal percent difference between theoretical and experimental values of M was of the order of 1%.

In the particular case $p = 1/2$ (when $\frac{1-R}{R} = 1$), by means of the property: $\sum_{i=0}^{q} \binom{m}{i} \binom{n}{q-i} = \binom{m+n}{q}$, we obtain:

$$M(R = 1/2, a, b) = \frac{1}{4^a} \binom{2a}{a}, \tag{11}$$

which corresponds to the result obtained, through another treatment, in ref. [4], in the discussion of the completely randomic case ($p = 1/2$).

If $p = 0, 1$, then $M = 1$, $\forall J, L$.

For $J = 1$ (Fig. 1(a)), M has a local minimum at $p_{min} = 1/2$, being $M_{min} = M(p_{min}) = 1/2$, $\forall L \geq 1$. For $0 < p < 1$ and $p \neq 1/2$, M decreases for increasing L, i.e. the concavity of the curve increases, such that as $L \to \infty$, $M \to 1/2$, $\forall 0 < p < 1$.

For $1 < J \leq L$ (illustrated in Fig. 1(b)), M has a local maximum at $p = 1/2$, and two absolute minima at p_{min} and $1 - p_{min}$, such that, for a given value of J, M_{max} is independent on L while p_{min} and $M_{min} = M(p_{min}) = M(1 - p_{min})$ decrease as L increases down to a constant value. This value decreases for increasing values of J. In the limit $J \to \infty$ (hence $L \to \infty$), $M_{max} \to \dfrac{1}{\sqrt{\pi J}} \to 0$, $p_{min} \to 0$, $M_{min} \to 0$. Thus, when p approaches either zero or unity, memory persists for increasing values of J, while, memory effects disappear ($M \to 0$) for $J \to \infty$ ($L \to \infty$).

Other cases are reduced to the previous ones recalling that M is a symmetrical function of J and L.

Finally, we want to determine the probability distribution $P(\tau)$, for $0 < p < 1$. From the precedent discussion we note that the probability of having a plateau at time t, that is $M = \text{pr.}(H(t) = H(t+1))$, is independent of t for any value of the external parameters. Recalling also that there is a plateau of length τ if $H(t-1) \neq H(t) = H(t+1) = ... = H(\tau - 1) = H(\tau) \neq H(\tau + 1)$, then it follows that:

$$P(\tau) = \tau \, (1 - M)^2 \, M^{\tau}, \quad \text{for} \quad \tau \geq 1, \tag{12}$$

which verifies $\sum_{\tau=1}^{\infty} P(\tau) = M$.

The mean size of the plateaux is $< \tau > = \sum_{\tau=0}^{\infty} \tau P(\tau)$, then, we have:

$$< \tau > = \frac{M(1+M)}{1-M} \, . \tag{13}$$

Straightforwardly, the n^{th} momenta of $P(\tau)$ ($< \tau^n >$) may be obtained from the following recurrence equation:

$$< \tau^n > = M \; \frac{\partial < \tau^{n-1} >}{\partial M} + \frac{M}{1-M} < \tau^{n-1} > . \tag{14}$$

By calculating the square deviation $\sigma^2 =< \tau^2 > - < \tau >^2$, in particular, we get:

$$\frac{< \tau >}{\sigma_\tau} = \frac{1+M}{\sqrt{2+(1-M)(1+M)^2/M}} \,. \tag{15}$$

The distribution of probabilities of the plateaux size, $P(\tau)$, presents a maximum at $\tau_{max} \sim \frac{-1}{\ln M}$, then τ_{max} increases with increasing M and so does the mean size of the plateaux $< \tau >$. Above τ_{max}, $P(\tau)$ decreases following a power law with exponent τ. Its standard deviation is such that $0 \leq \frac{< \tau >}{\sigma_\tau} \leq \sqrt{2}$, $\frac{< \tau >}{\sigma_\tau}$ increasing for increasing M and being unitary for $M \sim 0.75$, then, the mean size of the plateux is not well defined.

DISCUSSION

The Hamming distance associated to the present prototype behaves as a function of time in a more or less noise-like manner, gradually varying its behaviour from a noise-like to a plateaux-like one, the presence of jumps between plateaux indicating the existence of some type of memory. Such behaviour is found in many complex systems and theoretical models. Thus, the present prototype mimics many complex dynamical systems and could belong to the same "universality class" as some of them.

Recently, connections between spread of damage and relevant thermal equilibrium quantities of discrete statistical models have been established[5-7]. In particular, for the Ising model, relations between the Hamming distance and some of their thermal quantities may be found[5]. The relevance of these correspondences stays in the fact that they provide a new approach in order to calculate thermostatistical quantities. Therefore, the study of the present prototype may lead to a better understanding of the effects of memory in statistical systems and their consequences in relation to critical phenomena.

APPENDIX: DETERMINATION OF $M(p, J, L)$

In order to calculate the probability that $H(t) = H(t+1)$ for an arbitrary time t, let us rewrite Eq. (1) as:

$$H(t) = \frac{1}{L} \sum_{i=Jt}^{Jt+L-1} h_i, \tag{A1}$$

where $h_i =\mid S_i^A - S_i^B \mid$. For comparing $H(t)$ vs. $H(t+1)$, we only need to consider their non common terms. For a systematic analysis, let us first study the special case $J = 1$ and later the wider case $J > 1$.

Case $J = 1$

In this case, the non common terms are h_t and h_{t+L}, $\forall\ L$. Therefore:

$$M(p, 1, L) = \text{pr.}(h_t = h_{t+L}). \tag{A2}$$

We consider arrays $a_i = \begin{bmatrix} S_i^A \\ S_i^B \end{bmatrix}$ which represent the values of the random variable S_i on the two replicas A and B. The possible arrays are: $a^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $a^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $a^4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, from Eq. (A2), we have:

$$M(p, 1, L) = \sum_{\alpha,\beta=1}^{2} \text{pr.}(a_t = a^\alpha \wedge a_{t+L} = a^\beta) \ + \sum_{\alpha,\beta=3}^{4} \text{pr.}(a_t = a^\alpha \wedge a_{t+L} = a^\beta). \tag{A3}$$

By considering the prototype construction rules, it follows:

$$\begin{aligned} \text{pr.}(S_{i+k} = S_i) &= (1 + (2p-1)^k)/2, \quad \forall i, k \geq 0 \quad \text{hence,} \\ \text{pr.}(S_{i+k} \neq S_i) &= (1 - (2p-1)^k)/2. \end{aligned} \tag{A4}$$

From Eq. (A4) and taking into account that the values of S_{t+L} and S_t are not independent, we have:

$$p^{++} = \left(\frac{1 + (2p-1)^L}{2}\right)^2, \tag{A5}$$

$$p^{--} = \left(\frac{1 - (2p-1)^L}{2}\right)^2, \tag{A6}$$

$$p^{+-} = p^{-+} = \frac{1 - (2p-1)^{2L}}{4}, \tag{A7}$$

where p^{++}, p^{--} and p^{+-} (p^{-+}) are, respectively, the probabilities that either both, none or one of the elements of the arrays a_t and a_{t+L} be equal.

Therefore, by substituting Eqs. (A5) and (A6) in Eq. (A3), we get:

$$M(p,1,L) = (1+(2p-1)^{2L})/2. \qquad \text{(A8)}$$

Case $J > 1$

For the case $J > 1$, in order to compare $H(t)$ with $H(t+1)$, we consider 2 cases: (a) $L \geq J$ and (b) $1 \leq L \leq J$.

Subcase (a): $L \geq J$

Now, we must consider arrays of $2 \times J$ elements, e.g. $\begin{bmatrix} \overbrace{0\ 1\ \cdots\ 1}^{J} \\ 1\ 1\ \cdots\ 0 \end{bmatrix}$. In this case $H(t)$ and $H(t+1)$ share $L-J$ terms. Thus, we must compare J terms: $(h_{Jt}, \cdots, h_{Jt+J-1})$ vs. $(h_{Jt+L}, \cdots, h_{Jt+L+J-1})$. In order to perform this comparison, we consider pairs of $2 \times J$-arrays corresponding to $[a_{Jt}, \cdots, a_{Jt+J-1}]$ vs. $[a_{Jt+L}, \cdots, a_{Jt+L+J-1}]$.

For a given array, let us call l $(0 \leq l \leq J)$ the number of columns formed by equal elements (i.e. either $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$). Since we are interested in the configurations giving $H(t)$=$H(t+1)$, we have to consider only pairs of arrays of $2 \times J$ elements with the same value of l which will have the same Hamming distance $J-l$. So that, we calculate the contributions M_l corresponding to each value of l to the probability M. We have:

$$\begin{aligned} M_l &= \text{pr.}(h_{Jt} + \cdots + h_{Jt+J-1} \ = \ h_{Jt+L} + \cdots + h_{Jt+L+J-1} \ = \ J-l) \\ &= \sum_{all\ arrays(J,l)} \text{pr.}([a_{Jt},\cdots,a_{Jt+J-1}]_{J,l}) \ \text{pr.}([a_{Jt+L},\cdots,a_{Jt+L+J-1}]_{J,l}/[a_{Jt},\cdots,a_{Jt+J-1}]_{J,l}), \end{aligned} \qquad \text{(A9)}$$

subindices J and l in the arrays indicate the total number of columns and the number of columns with equal elements respectively. If incompatible with the form of the array, then pr.(array$_{J,l}$)=0, e.g. pr.$\left(\begin{bmatrix} 1 \cdots 0 \\ 0 \cdots 0 \end{bmatrix}_{J,J}\right) = 0$, since in this case $(l = J)$ all columns should have identical elements.

Besides, we have

$$\text{pr.}([a_{Jt},\cdots,a_{Jt+J-1}]_{J,l}) \ \text{pr.}([a_{Jt+L},\cdots,a_{Jt+L+J-1}]_{J,l}/[a_{Jt},\cdots,a_{Jt+J-1}]_{J,l}) =$$

$$\text{pr.}(a_{Jt}) \ \text{pr.}([a_{Jt},\cdots,a_{Jt+J-1}]_{J,l}) \ \text{pr.}([a_{Jt+L},\cdots,a_{Jt+L+J-1}]_{J,l}) \ \text{pr.}(a_{Jt+L}/a_{Jt+J-1}), \qquad \text{(A10)}$$

where the probabilities will be defined in the following paragraphs:

- pr.(a_{Jt}) in Eq. (A10) is the probability that the first column of the first array (i.e. 2×1-array at position Jt) be equal to a given a^α, $1 \leq \alpha \leq 4$.

- The probability pr.($\mathrm{array}_{J,l}$) in Eq. (A10), is the probability of having two given sequences of length J corresponding to rows A and B of the array. This probability is not that of finding the sequences at a given position of the samples but that of just having a given sequence without caring for the position. Thus, if we interchange all 1's in the array by 0's and vice versa, as well as if we interchange rows, or reverse the order of the columns, the probability is the same.

- In order to calculate the conditional probability pr.(a_{Jt+L}/a_{Jt+J-1}) in Eq. (A10), we must now consider the last column of the first array and the first column of the second array. Here, the "distance" k between the arrays, that is, the difference between the first site of the second array ($Jt + L$) and the last site of the first array ($Jt + J - 1$) is $k = L - J + 1$, so that, we just have to replace L in Eqs. (A5)-(A7) by $L - J + 1$. Thus, let us define the following probabilities: P_J^+ = probability that the last column of the first array and the first column of the second have two or none equal elements and P_J^- = probability that those columns have only one equal element:

  $P_J^+ = p_J^{++} + p_J^{--} = (1 + (2p-1)^{2(L-J+1)})/2,$

  $P_J^- = p_J^{+-} + p_J^{-+} = (1 - (2p-1)^{2(L-J+1)})/2,$

  where p_J^{++}, p_J^{--}, p_J^{+-} and p_J^{-+} were calculated in the same way as the probabilities in Eqs. (A5)-(A7), corresponding to $J = 1$, but substituting L by $L - J + 1$.

Considering that the arrays with $l = \lambda$ and $l = J - \lambda$ (with, $0 \leq \lambda \leq J$) are obtained one from the other by interchanging 0's and 1's in one of the rows and also considering that $\sum_{\alpha=1}^{4} \mathrm{pr.}(a_{Jt} = a^\alpha) = 1$, then, from Eqs. (A9) and (A10), the contribution $M_{l,J-l} = M_l + M_{J-l}$ is:

$$M_{l,J-l} = 2\, P_J^- \,\mathrm{pr.}\left(\begin{bmatrix} 0 \cdots \\ 0 \cdots \end{bmatrix}_{J,l}\right) \mathrm{pr.}\left(\begin{bmatrix} 0 \cdots \\ 1 \cdots \end{bmatrix}_{J,l}\right)$$

$$+\ P_J^+\left(\left(\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,l})\right)^2+\left(\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 1 & \cdots\end{matrix}\right]_{J,l})\right)^2\right), \tag{A11}$$

where, there is an additional 1/2 factor if $l = J - l$.

Since $\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 1 & \cdots\end{matrix}\right]_{J,l}) = \mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,J-l})$, then, we just need to know $\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,l})$.

⋆ For $l = J, 0$, from Eq. (A11), we obtain $M_{l=J,0} = P_J^+ \times \left(\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,J})\right)^2$. It is easy to see that $\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,J}) = (p^2 + (1-p)^2)^{2(J-1)}$, then, we have:

$M_{l=J,0} = (1 + (2p-1)^{2(L-J+1)})/2\ \ (p^2 + (1-p)^2)^{2(J-1)}$.

⋆ Now, let us calculate $\mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,l})$ when, $l \neq J, 0$. Let us define

$$V_{J,l} = \mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 0 & \cdots\end{matrix}\right]_{J,l}), \tag{A12}$$

$$W_{J,l} = \mathrm{pr.}(\left[\begin{matrix}0 & \cdots\\ 1 & \cdots\end{matrix}\right]_{J,l}) \tag{A13}$$

and

$$R = R(p) = p^2 + (1-p)^2. \tag{A14}$$

It is easy to see that:

$$V_{J,l} = R\,V_{J-1,l-1} + (1-R)\,W_{J-1,l-1}, \tag{A15}$$

$$W_{J,l} = (1-R)\,V_{J-1,l} + R\,W_{J-1,l}. \tag{A16}$$

Then, we get the following recurrence relation:

$$V_{J,l} = R\,V_{J-1,l-1} + (1-R)\,V_{J-1,J-l}, \tag{A17}$$

which solution is:

$$V_{J,l} = R^{J-1}(\sigma_1(R,J,l) + \sigma_2(R,J,l)), \tag{A18}$$

where

$$\sigma_1(R,J,l) = \sum_{k=1}^{l-1}\binom{l-1}{k}\binom{J-l-1}{k-1}\left(\frac{1-R}{R}\right)^{2k}, \tag{A19}$$

$$\sigma_2(R,J,l) = \sum_{k=0}^{l-1}\binom{l-1}{k}\binom{J-l-1}{k}\left(\frac{1-R}{R}\right)^{2k+1}, \tag{A20}$$

thus,

$$W_{J,l} \;=\; V_{J,J-l} \;=\; R^{J-1}(\sigma_3(R,J,l)+\sigma_2(R,J,l)), \tag{A21}$$

where

$$\sigma_3(R,J,l)=\sigma_1(R,J,J-l)=\sum_{k=1}^{l}\binom{l-1}{k-1}\binom{J-l-1}{k}\left(\frac{1-R}{R}\right)^{2k}. \tag{A22}$$

Note also that $\sigma_2(R,J,J-l)=\sigma_2(R,J,l)$.

By substitution in Eq. (A11) we obtain:

$$M_{l,J-l}=\frac{1}{2}\,(V_{J,l}+W_{J,l})^2+\frac{1}{2}\,(2p-1)^{2(L-J+1)}(V_{J,l}+W_{J,l})^2. \tag{A23}$$

Then:

$$M=M_{J,0}+\sum_{l=1}^{\left[\frac{J}{2}\right]-1}M_{l,J-l}+\frac{1}{2}\,M_{\left[\frac{J}{2}\right],\left[\frac{J}{2}\right]}=M_{J,0}+\frac{1}{2}\sum_{l=1}^{J-1}M_{l,J-l}. \tag{A24}$$

Subcase (b): $1\le L\le J$

In this case we have two arrays of $2\times L$ elements and $k=J-L+1$. Since the calculation of M will only depend on the size of the arrays and on the "distance" k, then case (b) is reduced to case (a) just by inverting the roles of J and L, being $M(p,J,L)=M(p,L,J)$.

CAPTION FOR FIGURE

Figure 1: $M(p, J, L)$ vs p as obtained from Eq. (5). (a) $J = 1$ and different values of L; (b) $J = 3$ and different values of $L \geq J$. Values of L are indicated on the figure. Scales are the same in both graphs for comparison.

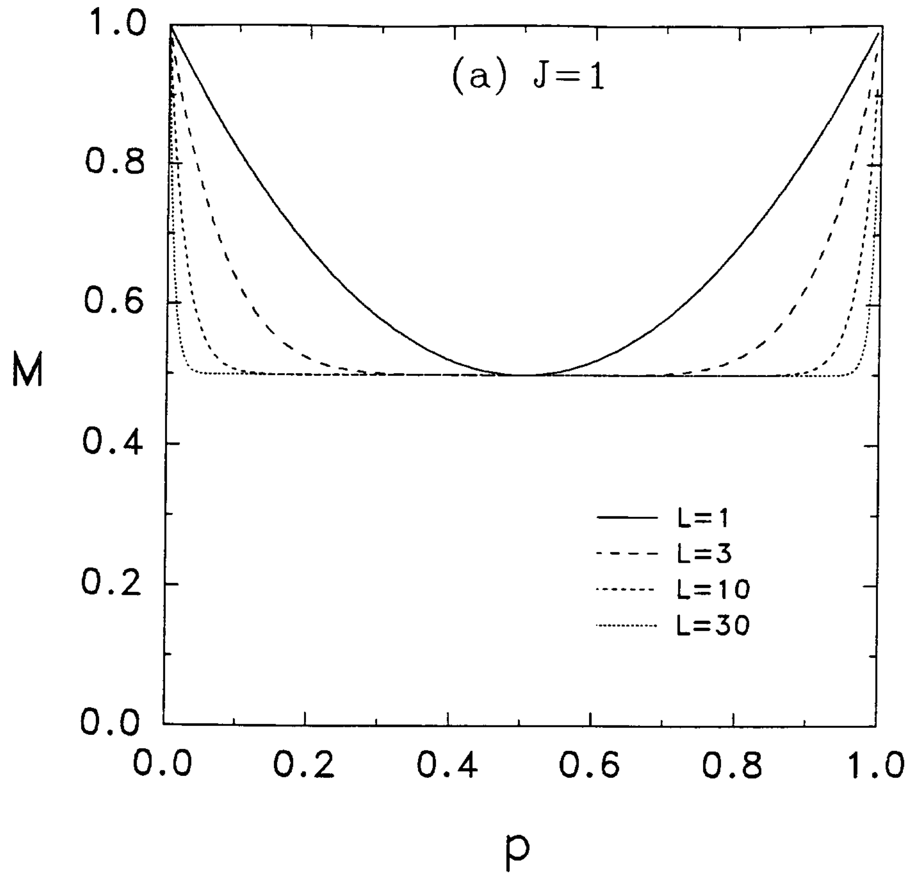

Figure 1.a

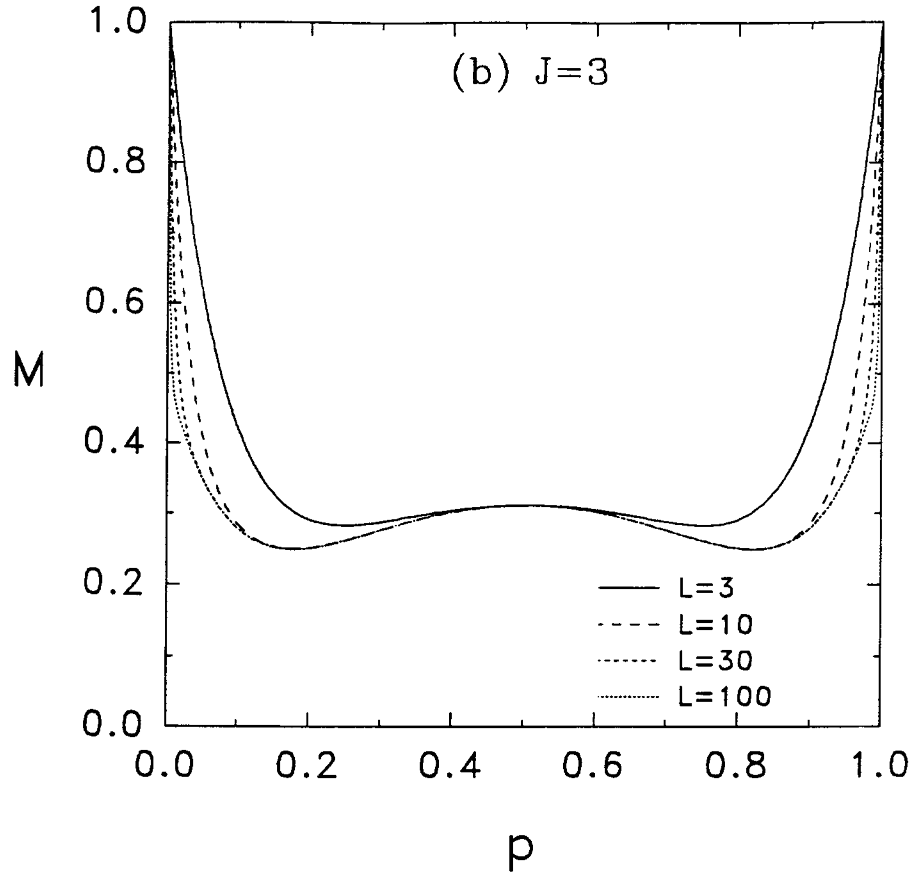

Figure 1.b

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