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Fermions on Lattice by Means of Mandelstam-Wilson Phase Factors

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ABSTRACT

We propose a Mandelstam-Wilson phase factor approach to the solve problem of handling correctly fermions fields on lattice. We apply this approach to fermionize exactly Q.C.D ($U(\infty)$) at the leading limit of the strong coupling limit.

Key-words: Lattice; Phase factors; Expansão $1/N_c$; Effective meson actions.

One of the long-standing unsolved problem in the lattice approach to Q.C.D. is how to handle discretized fermionic fields ([1]). In this rapid communication we propose a solution for the above mentioned problem by considering as the Q.C.D natural field variable to discretize in lattice the Mandelstam-Wilson phase factor defined by the color singlet quark currents. Additionaly, we show the usefulness of this propose by obtaining in a unambiguously way the associated Q.C.D Nambu-Jona-Lasinio fermionic model, which, upon being bosonized, leads to low energy theory of meson and baryon of Q.C.D.

Let us start our study by considering the Euclidean Q.C.D. ($U(N_c)$) generating functional for the color singlet scalar and vectorial quark currents.

$$
\begin{aligned}
Z[\sigma(x), J_\mu(x)] &= \int D^F[A_\mu(x)]\, exp\left(-\frac{1}{4}\int d^4x\; Tr(F_{\mu\nu}(A))^2(x)\right) \\
&\left\{\int D^F[\psi_{(x)}, \bar{\psi}_{(x)}]\, exp\, \left(-\int d^4x(\bar{\psi}[i\gamma_\mu\partial_\mu + ig\gamma_\mu A_\mu]\psi)(x)\right)\right. \\
&\left. exp\, \left(-\int d^4x[(\bar{\psi}\psi)\,\sigma + J_\mu(\bar{\psi}\gamma^\mu\psi) + (\bar{\psi}\gamma^5\psi)\beta + (\bar{\psi}\gamma^\mu\gamma^5\psi).k_\mu](x)\right)\right\} \qquad (1)
\end{aligned}
$$

Where $\psi_{(x)}, \bar{\psi}_{(x)}$ are the independent Euclidean quark fields, $\sigma(x), \beta(x)$ and $J_\mu(x), k_\mu(x)$ are the external sources for the scalar, scalar-axial, and axial-vectorial Q.C.D. quark currents. $A_\mu(x)$ denotes the $U(N_c)$ gluon field.

The framework to obtain effective quark field theories in Q.C.D. consists in trying to integrate out the gluon degrees of freedom in the above written functional integral ([2]), namely

$$
\begin{aligned}
I[\psi, \psi] \;&=\; \int D^F[A_\mu(x)]\, exp\left(-\frac{1}{4}\int d^4x\; Tr(F_{\mu\nu}(A))^2(x)\right) \\
&\quad exp\left(ig\int d^4x(\bar{\psi}\gamma^\mu\psi)(x)A_\mu(x)\right) \qquad (2)
\end{aligned}
$$

Our idea to evaluate equation (2) is, firstly, introduce a Lattice Space-Time and writing the associated (gauge-invariant) lattice path integral. At this point we put forward our idea to treat unambiguosly the fermionic fields on Lattice. Since it is impossible to have well-defined procedure to define massless Fermion Fields on the usual Lattice $\{x_\mu = [n_\mu], n_\mu \in Z\}$ (with spacing a) ([1]) we propose to consider directly the bosonic quark Fermion current on the lattice by means of its associated Mandelstam-Wilson phase factor

$$\Phi_\alpha([n_\mu]) = exp\left(ia(\bar{\psi}(\gamma^\alpha)\psi)([n_\mu])\right)^{(i,j)} \qquad (3)$$

Note that the above written Phase Factor has index (i, j) on the group $U(N)$ and a index α related to the Lorentz Group as it should be.

The associated gluon $U(N)$ group-valued Mandelstam-Wilson phase factor is still given by

$$U_\mu\left([n_\alpha]\right) = exp\left(iaA_\mu([n_\alpha])\right) \qquad (4)$$

At this point of our study, we remark that the Quark-Gluon coupling in Lattice may be written as a product of the Mandelstam-Wilson Phase Factor Eq. (3) and Eq. (4)

since we have the usual lattice result

$$\lim_{a\to 0} a^2 \left[\sum_{\{[n_\alpha]\}} Tr_{color} \{ (U_\mu([n_\alpha]) - \mathbf{1}\ )(\phi_\mu([n_\alpha]) - \mathbf{1}\ )\} \right]$$
$$= ig \int d^4x\, A_\mu(x).(\bar{\psi}\gamma^\mu\psi)(x) \tag{5}$$

The gauge-invariant Lattice version of the Gluon Functional Integral Eq. (2) is, thus, given by

$$\hat{I}[\psi,\bar{\psi}] = \int D^H[U_\mu([n_\alpha])]\, exp\left(-\frac{1}{4g^2} \sum_{\{[n_\alpha]\}} Tr_{color}(U(\square) + h.c) \right)$$
$$\times\, exp\left(-a^2 \sum_{\{[n_\alpha]\}} Tr_{color}\, [h.c + (U_\mu([n_\alpha]) - \mathbf{1}\ )(\Phi_\mu([n_\alpha] - \mathbf{1}\ )] \right) \tag{6}$$

The advantage of this Lattice Phase Factor approach to analyze Eq. (2) is its allowance to exactly integration of the Lattice Gluons Phase Factors in both perturbative and non-perturbative regime. Let us show its usefulness by evaluating in closed form Eq. (6) in the leading limit of the number of colors and in the leading limit of strong coupling as in Ref. ([3]) - Eq. (3.17)).

$$\hat{I}[\psi,\bar{\psi}, g^2 \to \infty, N_c \to \infty] = \int D^H[U_\mu[n_\mu]] exp\{-a^2 \sum_{\{[n_\alpha]\}} Tr_{color}(U_\mu([n_\alpha]).\Phi_\mu([n_\mu]) + h.c)\}$$

$$= exp\left\{ \frac{(\Lambda_{Q.C.D.}(a) \times a^4)}{N_c} \sum_{\{[n_\alpha]\}} Tr_{color}\, [(\phi_\mu([n_\alpha]).\phi_\mu([n_\alpha]))^+] \right\} \tag{7}$$

where $\Lambda(a)$ is the Q.C.D. strong coupling phenomenological scale with dimension of inverse of area (the gluon non perturbative condensate) which by its turn is Lattice spacing dependent. The formal continuum limit $a \to 0$ of the result Eq. (7), after a Fierz transformation, leads to the following quartic fermionic action in the continuum

$$I[\psi,\bar{\psi}, g^2 \to \infty, N_c \to \infty] = exp\left\{ \frac{g_F^2}{N_c} \int d^4x [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2 \right.$$
$$\left. -\frac{1}{2}(\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma^\mu\gamma^5\psi)^2](x) \right\} \tag{8}$$

Here the fermionic effective coupling constant g_F^2 is defined in the continuum by the formal limit $g_F^2 = \lim_{a\to 0} \Lambda_{Q.C.D.}(a).a^2$ and signaling the usual Q.C.D. dimensional transmutation phenomena.

After substituting Eq. (8) into Eq. (1) we get our propose fermionization for quantum chromodynamics in the very low energy region with the gluon field $U(N_c)$ integrated out for large N_c in the sense of ref.[3].

We remark that by introducing the Hubbard-Stratonovich Ansatz to linearize the quartic fermion interactions we obtain the $U(1)$-chiral scalar and vectorial Q.C.D. ($U(\infty)$) meson theory which differs from that considered in reference [4] by using phenomenological guessing arguments.

$$
\begin{aligned}
&Z[\sigma+\gamma_5\beta, J_\mu+\gamma_5 k_\mu] = \int D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{k}_\mu] \\
&exp\left(-\frac{N_c}{g_F^2}\int d^4x\left[\left(\frac{1}{2}\hat{\sigma}^2+\frac{1}{2}\hat{\beta}^2\right)+\left(\frac{1}{2}\hat{J}_\mu^2+\frac{1}{2}\hat{k}_\mu^2\right)\right]\right)(x) \\
&det^{N_c}\left[i\gamma\partial+(\sigma+\hat{\sigma})+\gamma_5(\beta+\hat{\beta})+(J_\mu+\hat{J}_\mu)+\gamma^5(k_\mu+\hat{k}_\mu)\right]
\end{aligned}
\qquad (9)
$$

Note that in Eq. (9) $(\hat{\sigma}+\gamma_5\hat{\beta})$ and $(\hat{J}_\mu+\gamma_5\hat{k}_\mu)$ should be identified with the $U(1)$-chiral scalar and vetorial low energy physical meson fields.

Let us comment that dynamics for the physically identified meson fields above comes from the evaluation of the quark functional determinant. In the limit of the heavy scalar meson mass $\langle\hat{\sigma}\rangle \to \infty$ one can easily implement the technique of refs. [5] to get the full effective hadronic action in terms of $1/\langle\hat{\sigma}\rangle$ power series.

In the case of Baryon-like field excitations of the form $\beta(x) = \varepsilon_{ijk}\psi_j(x)\psi_j(x)\bar{\psi}_k(x)$ it is still possible analyze them in our proposed framework. For this task we consider a Hubbard-Stratonovich Ansatz to write the generating functional for the Baryon-like excitation $B(x)$, namely

$$
\begin{aligned}
&Z[k] = \int D^F[\Delta]D^F[\lambda]D^F[A_\mu]D^F[\psi]D^F[\bar{\psi}] \\
&exp\left(-\int d^4x[\bar{\psi}(i\gamma_\mu\partial_\mu)\psi + i\psi_p\bar{\psi}_q\lambda_{qp} + iq\bar{\psi}\gamma_\mu A_\mu\psi](x)\right) \\
&exp\left(-\int d^4x[k(x)\varepsilon_{ijk}\psi_i(x)\Delta_{jk}(x)]\right) \\
&exp\left(-i\int d^4x[\lambda_{pq}(x).\Delta_{qp}(x)]\right)
\end{aligned}
\qquad (10)
$$

where (p,q) are $U(N)$ indexes and the auxiliary fields (Δ,λ) belong to the adjoint $U(N)$-representation.

After integrating out the gluon field $A_\mu(x)$ as in Eq. (7) and the quark field as in Eq. (9) we get our proposed effective Q.C.D.-Baryon field theory

$$
\begin{aligned}
&Z[k] = \int D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{k}_\mu]D^F[\Delta]D^F[\lambda] \\
&exp\left(-\frac{N_c}{g_F^2}\int d^4x\left[\frac{1}{2}\hat{\sigma}^2+\frac{1}{2}\hat{\beta}^2\right](x)+\left[\frac{1}{2}\hat{J}_\mu^2+\frac{1}{2}\hat{k}_\mu^2\right](x)\right)exp\left(-i\int d^4x Tr_{color}(\lambda\Delta)(x)\right) \\
&det\left\{[i\gamma\partial+(\hat{\sigma}+\gamma_5\hat{\beta})+(\hat{J}_\mu+\gamma_5\hat{k}_\mu)]\delta_{rj}-i\lambda_{rj}\right\} \\
&exp\left\{-\int d^4x\, d^4y\, k(x)(\varepsilon_{ijk}\Delta_{jk}(x)[(i\gamma\partial+\hat{\sigma}+\gamma_5\hat{\beta}+\hat{J}_\mu+\gamma_5\hat{k}_\mu- \right. \\
&\left. -\lambda]^{-1}_{ij}(x,y)\varepsilon^{i'j'k'}\Delta_{j'k'}(y))k(y)\right\}
\end{aligned}
\qquad (11)
$$

It is instructive remark that Eq. (11) indicates the impossibility to consider baryons excitations without interaction with the meson excitations in a effective Q.C.D. field theory.

It is worth point out that strong coupling corrections from the neglected gluon field kinetic action in Eq. (7) are straightforwardly implemented in lattice by using the usual Q.F.T. perturbation theory with the external lattice gluon source coupling $J_\mu([n_\mu].U_\mu([n_\mu])$.

$$\hat{I}[\psi,\bar{\psi},N_c\to\infty]=\lim_{J_\mu[n_\alpha]\to 0}\left(exp\left\{-\frac{1}{4g^2}Tr_{color}\sum\left(\frac{\delta}{\delta J}(\Box)+h.c.\right)\right\}\right.$$

$$\left.\left\{\int D^H[U_\mu[n_\mu]]exp\left(\frac{\Lambda_{Q.C.D.}(a)\times a^4}{N_c}\sum_{\{[n_\alpha]\}}Tr_{color}[(\Phi_\mu[n_\alpha]+J_\mu[n_\alpha])(\Phi_\mu[n_\alpha]+J_\mu[n_\mu])^+]\right)\right\}\right) \tag{12}$$

with

$$\sum Tr_{collor}\left(\frac{\delta}{\delta J}(\Box)\right)=\sum_{\{[n_\mu]\}}Tr_{color}\left(\frac{\delta}{\delta J_{\mu_1}[n_\alpha]}\frac{\delta}{\delta J_{\mu_2}[n_\alpha+a_{\mu_2}]}\frac{\delta}{\delta J_{-\mu_2}[n_\alpha]}\frac{\delta}{\delta J_{\mu_2-\mu_1}[n_\alpha]}+h.c.\right) \tag{13}$$

The associated $1/g^2$ corrected fermionized Q.C.D. ($U(\infty)$ effective theory will be given by non-local current-current quark correlation functions averaged with the fermionized strong coupling $g^2\to\infty$ Q.C.D. ($U(\infty)$) theory Eq. (8).

Finally, one can take in principle corrections to the large N_c limit in the lattice result Eq. (7) by using Eq. (3.12) of reference [3].

Work on these above mentioned corrections and its implication for Q.C.D. low energy nuclear physics as in ref [6] will be reported elsewhere.

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