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THE q-DEFORMED WIGNER OSCILLATOR IN QUANTUM MECHANICS

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Abstract

Using a super-realization of the Wigner-Heisenberg algebra a new realization of the q-deformed Wigner oscillator is implemented.

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Dedicated to the memory of Prof. Jambunatha Jayaraman, 28 January 1945-19 June 2003.

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1 Introduction

In 1989, independently, Biedenharn and Macfarlame [1], introduced the q-deformed harmonic oscillator and constructed a realization of the $SU_q(2)$ algebra, using a q-analogue of the harmonic oscillator and the Jordan-Schwinger mapping. The q-deformation of $SU(2)$, denoted by $SU_q(2)$, is one of the simplest examples of a quantum group.

The deformation of the conventional quantum mechanical laws has been implemented via different definitions and studied by several authors in the literature [2, 3, 4, 5, 6, 7, 8, 9]. Also, recently Palev *et al.* have investigated the 3D Wigner oscillator [9].

The main purpose of this work is to set up a realization of the q-deformed Wigner oscillator [2].

2 The q-deformed usual harmonic oscillator

In this section, we consider the q-deformed ladder operators of the harmonic oscillator, a^- and its adjoint a^+, acting on the basis $\mid n>, \quad n = 0, 1, 2, \cdots$, as [1] $a_q^- \mid 0 >= 0, \quad \mid n >= \frac{(a_q^+)^n}{([n]!)^{\frac{1}{2}}} \mid 0 >$ where $[n]! = [n][n-1]\cdots[1]$. The classical limit $q \rightarrow 1$ yields to the conventional ladder boson operators $a^{\pm}$, which satisfies $[a^-, a^+] = 1, \quad a^-|n >= \sqrt{n}|n-1>, \quad a^+|n>= \sqrt{n+1}|n+1>$.

On the other hand, $su(1,1)$ algebra satisfies the following commutation relations $[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_0$ and the Casimir operator is given as $C = K_0(K_0 - 1) - K_+K_-$, where $K_0|0 >= k_0|0>$ and $K_-|0>= 0$. A usual representation for this algebra is given in terms of the ladder operators $a^- = (x + ip)/\sqrt{2}, \quad a^+ = (x - ip)/\sqrt{2}$. The $su(1,1)$ generators are given as $K_0 = \frac{1}{2}\left(N + \frac{1}{2}\right), \quad K_+ = \frac{1}{2}(a^+)^2$ and $K_- = \frac{1}{2}(a^-)^2$, where $N = a^+a$. Thus, the Casimir operator is given by $C = -\frac{3}{16}$. This system has two different representations whose k_0 is $\frac{1}{4}$ and $\frac{3}{4}$.

Its q-deformation, $su_{q^2}(1,1)$, is given [3] as

$$[\tilde{K}_0, \tilde{K}_{\pm}] = \pm\tilde{K}_{\pm} \quad [\tilde{K}_+, \tilde{K}_-] = -[2\tilde{K}_0]_{q^2}, \quad [x]_\mu \equiv (\mu^x - \mu^{-x})/(\mu - \mu^{-1}). \tag{1}$$

In Ref. [4] was found a realization of the $su_{q^2}(1,1)$ in terms of the generators of $su(1,1)$. The q-deformed ladder operators satisfy

$$a_q^- a_q^+ = [N+1], \quad a_q^+ a_q^- = [N], \tag{2}$$

where N is the number operator which is positive semi-definite. The q-analogue operators can be found in terms of the usual ladder boson operators a^- and a^+.

Note that we can write $\mid n >= \frac{a_q^+}{\sqrt{[n]!}} \frac{(a_q^+)^{n-1}}{([n-1]!)^{\frac{1}{2}}} \mid 0 >= \frac{a_q^+}{\sqrt{[n]!}} \mid n-1>$ so that we obtain

$$a_q^+ \mid n-1 >= [n]^{\frac{1}{2}} \mid n > \Rightarrow a_q^+ \mid n >= [n+1]^{\frac{1}{2}} \mid n+1 > . \tag{3}$$

Also, from (2) and $a_q^- a_q^+ \mid n >= [n+1]^{\frac{1}{2}} a_q^- \mid n+1 >$ we get

$$a_q^- \mid n >= [n]^{\frac{1}{2}} \mid n-1 > . \tag{4}$$

It's easy to verify that $[N, a_q^+] = a_q^+, \quad [N, a_q^-] = -a_q^-, \quad [N, q^N] = [a_q^- a_q^+, q^N] = 0, \quad a_q^- a_q^+ - q a_q^+ a_q^- = q^{-N}$. We will show that a structure of this type exists for the Wigner oscillator.

3 The q-deformed Wigner Oscillator

The one-dimensional Wigner super-oscillator Hamiltonian in terms of the Pauli's matrices (σ_i, i=1,2,3) is given by

$$H(\lambda+1) = \begin{pmatrix} H_-(\lambda) & 0 \\ 0 & H_+(\lambda) \end{pmatrix}, H_-(\lambda) = \frac{1}{2}\left\{-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\lambda(\lambda+1)\right\}, \tag{5}$$

where $H_+(\lambda) = H_-(\lambda+1)$. The even sector $H_-(\lambda)$ is the Hamiltonian of the oscillator with barrier or isotonic oscillator or Calogero interaction.

Thus, from the super-realized Wigner oscillator, its first order ladder operators given by [2] $a^\pm(\lambda+1) = \frac{1}{\sqrt{2}}\left\{\pm\frac{d}{dx} \pm \frac{(\lambda+1)}{x}\sigma_3 - x\right\}\sigma_1$, the Wigner Hamiltonian and the Wigner-Heisenberg(WH) algebra ladder relations are readily obtained as

$$H(\lambda+1) = \frac{1}{2}\Big[a^+(\lambda+1), a^-(\lambda+1)\Big]_+, \Big[H(\lambda+1), a^\pm(\lambda+1)\Big]_- = \pm a^\pm(\lambda+1). \tag{6}$$

Equations (6) and the commutation relation

$$\Big[a^-(\lambda+1), a^+(\lambda+1)\Big]_- = 1 + 2(\lambda+1)\sigma_3 \tag{7}$$

constitutes the WH algebra [2] or deformed Heisenberg algebra [5, 7].

Let us consider an extension of the q-deformed harmonic oscillator commutation relation,

$$a_W^- a_W^+ - q a_W^+ a_W^- = q^{-N}(1 + c\sigma_3), \quad c = 2(\lambda+1) \tag{8}$$

as a q-deformation of the Wigner oscillator commutation realization. These operators may be written in terms of the Wigner oscillator ladder operators, viz.,

$$a_W^- = \beta(N)a^-(\lambda+1), \qquad a_W^+ = a^+(\lambda+1)\beta(N), \quad N = a^+(\lambda+1)a^-(\lambda+1). \quad (9)$$

Acting the ladder operators of the WH algebra in the Fock space, spanned by the vectors

$$a^-(\lambda+1)|2m>_c = \sqrt{2m}|2m-1>_c,$$

$$a^-(\lambda+1)|2m+1>_c = \sqrt{2m+c+1}|2m>_c,$$

$$a^+(\lambda+1)|2m>_c = \sqrt{2m+c+1}|2m+1>_c,$$

$$a^+(\lambda+1)|2m+1>_c = \sqrt{2(m+1)}|2m+2>_c,$$

we obtain a recursion relation given by

$$(2m+2)\beta^2(2m+1) - q(2m+1+c)\beta^2(2m) = q^{-(2m+1)}(1-c). \quad (10)$$

Also

$$(2m+1+c)\beta^2(2m) - 2mq\beta^2(2m-1) = q^{-2m}(1+c). \quad (11)$$

The q-deformed Wigner Hamiltonian and the commutator $[a_W^-, a_W^+]$, for $c=0$ become the q-deformed harmonic oscillator

$$H_W = \frac{1}{2}[a_W^-, a_W^+]_+ = H_b = \frac{1}{2}([N+1]+[N]), \quad [a_W^-, a_W^+] = [N+1]-[N]. \quad (12)$$

Ten equation (10) and (11) have the following solution $\beta(N) = \sqrt{\frac{[N+1]}{N+1}}$. The general case is under investigation and the results will be published elsewhere.

4 Conclusion

In this work, we firstly presented a brief review on the q-deformation of the conventional quantum mechanical laws for the unidimensional harmonic oscillator. We have also implemented a new approach for the WH algebra. Indeed, the q-deformations of WH algebra are investigated via the super-realization introduced by Jayaraman-Rodrigues [2].

Also, we do not assume the relations of operators $a_W^+ a_W^-$ and $a_W^- a_W^+$. They are derived from our defining set of relations $a_W^- = a_q^- = \sqrt{\frac{[N+1]}{N+1}} a^-$ and $a_W^+ = a_q^+ = a^+ \sqrt{\frac{[N+1]}{N+1}}$, for vanish Wigner parameter ($c = 0$) given by Eq. (2). The case with $c \neq 0$ will be presented in a forthcoming paper.

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