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DIRAC OSCILLATOR VIA R-DEFORMED HEISENBERG ALGEBRA

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Abstract

The complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg algebra is investigated.

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I. INTRODUCTION

The relativistic Dirac oscillator proposed by Moshinsky-Szczepaniac [1] is a spin $\frac{1}{2}$ object with the Hamiltonian which in the non-relativistic limit leads to that of a 3-dimensional isotropic oscillator shifted by a constant term plus a $\vec{L}\cdot\vec{S}$ coupling term for both signs of energy. There they construct a Dirac Hamiltonian, linear in the momentum $\vec{p}$ and position $\vec{r}$, whose square leads to the ordinary harmonic oscillator in the non-relativistic limit. The Dirac oscillator has been investigated in several context [2–8]

The R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique [9] was recently super-realized for the SUSY isotonic oscillator [17,18]. The R-Heisenberg algebra has also been investigated for the three-dimensional non-canonical oscillator to generate a representation of the orthosympletic Lie superalgebra $osp(3/2)$ [12].

The R-Heisenberg algebra has been found relevant in the context of integrable models [13], and the Calogero interaction [14,15]. Recently it has been employed for bosonization of supersymmetry in quantum mechanics [16], and the discrete space structure for the 3D Wigner quantum oscillator [20]. In this work, we obtain the complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg (RDH) algebra.

II. 3D WIGNER OSCILLATOR

In this Section, we provide a three dimensional representation of the Wigner system with its bosonic sector being the 3D isotropic oscillator (assumed to be of spin-$\frac{1}{2}$, to aid factorization).

The R-deformed Heisenberg (or Wigner-Heisenberg) algebra is given by following (anti-)commutation relations ($[A,B]_+\equiv AB+BA$ and $[A,B]_-\equiv AB-BA$) :

$$H=\frac{1}{2}[a^-,a^+]_+,\quad [H,a^\pm]_-=\pm a^\pm,\quad [a^-,a^+]_-=1+cR,\quad [R,a^\pm]_+=0,\quad R^2=1, \tag{1}$$

where c is a real constant associated to the Wigner parameter [17]. Note that when $c=0$ we have the standard Heisenberg algebra.

It is straightforward, following the analogy with the Ref. [17], to define the super-realizations for the ladder operators $a^\mp(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ for $H_W\equiv H(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ taking the explicit forms

$$a^\mp=a^\mp(\vec{\sigma}\cdot\vec{L}+\mathbf{1})=\frac{1}{\sqrt{2}}\left\{\mp\Sigma_1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\pm\frac{1}{r}(\vec{\sigma}\cdot\vec{L}+\mathbf{1})\Sigma_1\Sigma_3-\Sigma_1 r\right\} \tag{2}$$

which satisfy together with $H_W\equiv H(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ all the algebraic relations of the RDH algebra with the constant $\frac{c}{2}$ replaced by $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ and $R=\Sigma_3$. Note that $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ commutes with all the basic elements ($a^\mp$ and H_W) of the RDH algebra.

It may be observed that the RDH algebra that gets defined here is in fact three dimensional (one dimension for r and two for $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$) and is identically satisfied on any arbitrary three dimensional wave function.

On the eigenspaces of the operator $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$, the 3D Wigner algebra gets reduced to a 1D from with $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ replaced by its eigenvalue $\mp(\ell+1)$, $\ell=0,1,2,\cdots$, where ℓ is the orbital angular momentum quantum number. The eigenfuncitons of $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ for the eigenvalues $(\ell+1)$ and $-(\ell+1)$ are respectivaly given by the well known spin-spherical harmonics $y_{\mp}$.

Now, considering simultaneous eigenfuncitons of the mutually commuting H_W and $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ by

$$\psi_{W,+} = \begin{pmatrix}\tilde{R}_{1,+}(r)\\ \tilde{R}_{2,+}(r)\end{pmatrix} y_+, \quad (\vec{\sigma}\cdot\vec{L}+\mathbf{1})\psi_{W,+} = (\ell+1)\psi_{W,+}, \tag{3}$$

and

$$\psi_{W,-} = \begin{pmatrix}\tilde{R}_{1,-}(r)\\ \tilde{R}_{2,-}(r)\end{pmatrix} y_-, (\vec{\sigma}\cdot\vec{L}+\mathbf{1})\psi_{W,-} = (\ell+1)\psi_{W,-}, \tag{4}$$

(where the use of the subscript $+(-)$ indicates association with $[y_+(y_-)]$, we observe that the positive semi-definite form of H_W the ladder relations and the form of H_W dictat that the ground state energy $E_w^{(0)}(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) \geq 0$, where $E_W(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ indicates a function of $\vec{\sigma}\cdot\vec{L}+\mathbf{1}$, is determined by the annihilation condition which reads as two cases.

III. THE DIRAC OSCILLATOR MODEL

Adding an "anomalous momentum" in the form of a (nonlocal) linear and hermitian interaction,

$$\vec{\alpha}.\vec{\pi} \equiv -iM\omega\beta\vec{\alpha}.\vec{r} = (\vec{\alpha}.\vec{\pi})^{\dagger}, \tag{5}$$

in the (noncovariant) Dirac free particle equation with mass M and spin-$\frac{1}{2}$, in the natural sistem of units,

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}.\vec{p} + M\beta)\psi, \tag{6}$$

one obtains the equation for the Dirac oscillator [1]:

$$i\frac{\partial\psi}{\partial t} = \{\vec{\alpha}.(\vec{p}+\vec{\pi}) + M\beta\}\psi, \tag{7}$$

where M and ω are, respectively, the mass of the particle and the frequency of the oscillator, and the matrices $(\vec{\alpha}, \beta)$ satisfy the following properties:

$$[\alpha_i, \beta]_+ = 0, \quad [\alpha_i, \alpha_j]_+ = 2\delta_{ij}\mathbf{1}, \quad \beta^2 = \mathbf{1} = \alpha_i^2, \quad (i,j = 1,2,3). \tag{8}$$

Writing the Dirac spinor in terms of the upper and lower components, respectively, ψ_1 and ψ_2,

$$\Psi(\vec{r},t) = \exp(-\mathrm{iEt})\Psi(\vec{r}), \qquad \Psi(\vec{r}) = \begin{bmatrix}\psi_1(\vec{r})\\ \psi_2(\vec{r})\end{bmatrix} \tag{9}$$

the standard representation of the matrices $\vec{\alpha}$ and β.

IV. THE DIRAC OSCILLATOR VIA RDH ALGEBRA

In this section, we implement a new realization of the Dirac oscillator in terms of elements of the R-deformed Heisenberg algebra. To solve the Dirac equation, following the usual procedure, we consider the second order differential equation,

$$\tilde{H}_D\psi(\vec{r}) = E\psi(\vec{r}), \tag{10}$$

where $\tilde{H}_D$ is a second order Hamiltonian,

$$\tilde{H}_D = H_D^2 + M^2\mathbf{1}, \quad \tilde{E} = \frac{E^2 - M^2}{2M}. \tag{11}$$

In the spherical polar coordinate system, we obtain the non-relativistic form of the Hamiltonian of Ui [19], for an isotropic 3D SUSY harmonic oscillator with spin-$\frac{1}{2}$.

We consider a unitary operator in terms of the radial projection of the spin,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_r \end{bmatrix} = U^{-1} = U^{\dagger}, \tag{12}$$

to obtain the following relation between the transformed Dirac Hamiltonian, $\tilde{H}_D$, the 3D Wigner Hamiltonian, H_W, and the SUSY Hamiltonian, H_{SUSI} [19]:

$$H_{\mathrm{SUSI}} = U\tilde{H}_D U^{\dagger} = H_W - \frac{1}{2}\{1 + 2(\sigma\cdot\vec{L} + \mathbf{1})\Sigma_3\}\omega\Sigma_3. \tag{13}$$

A. The energy spectrum of the Dirac oscillator

The energy spectra of the operators $\tilde{H}_D$ and H_{SUSY} are identical, since these operators are related by a unitary transformation. However, the relation between the principal quantum number N and the angular momentum (ℓ) is different, in each case. Obviously, the energy spectrum associated with the two types of eigenspaces belonging to the eigenvalues $\pm(\ell+1)$:

Case(i) $\rightarrow \vec{\sigma}\cdot\vec{L} + \mathbf{1} \rightarrow \ell + 1 = j + \frac{1}{2}, \quad j = \ell + \frac{1}{2}$

$$\tilde{E}_{N\ell} = \frac{E^2 - M^2}{2M} = \begin{cases} 2m\omega = \tilde{E}^{+}_{N(\ell+1)}, \\ 2(m+1)\omega = \tilde{E}^{-}_{N\ell}, \end{cases} \tag{14}$$

where $m = 0, 1, 2, \ldots$.

Case(ii) $\rightarrow \vec{\sigma}\cdot\vec{L} + \mathbf{1} \rightarrow -(\ell+1) = -(j + \frac{1}{2}), \quad j = (\ell+1) - \frac{1}{2}$

$$\begin{aligned}\tilde{E}_{N\ell} &= \frac{E^2 - M^2}{2M} \\ &= \begin{cases} (N + j + 3/2)\omega = \tilde{E}^{+}_{N\ell}, \quad N = j - \frac{1}{2}, j + 3/2, j + 7/2, \ldots, \\ (N + j + 5/2)\omega = \tilde{E}^{-}_{N(\ell+1)}, \quad N = j + \frac{1}{2}, j + 5/2, \ldots \end{cases}.\end{aligned} \tag{15}$$

V. CONCLUSION

In this work we investigate an interesting quantum system, the so-called Dirac oscillator, first introduced by Moshinsky-Szczepaniak [1]; its spectral resolution has been investigated with the help of techniques of super-realization of the R-deformed Heisenberg algebra.

The Dirac oscillator with different interactions has been treated by Castaños *et al.* [7] and by Dixit *et al.* [8]. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $C\ell_7$. The seven elements of the Clifford algebra $C\ell_7$ generate the three linear momentum components, the three position coordinates components and the mass, and their squares are the 8x8-identity matrix $\mathbf{I}_{8\mathrm{X}8}$. Results of our analysis on Dirac oscillator via the Clifford algebra $C\ell_7$ are in preparation.

In a forthcoming paper we show that the Dirac oscillator equation can be resolved algebrically without having to transform it into a second order diferential equation. Therefore, the important connection for the Dirac 3D-isotropic oscillator with the linear ladder operators of the R-deformed Heisenberg algebra, satisfying the concomitant general oscillator quantum rule of Wigner, have explicited in this work.

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