ISSN 0029 - 3865

CBPF-NF-029/84

POTTS FERROMAGNET CORRELATION LENGTH IN HYPERCUBIC LATTICES: RENORMALISATION-GROUP APPROACH

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ABSTRACT

Through a real space renormalisation group approach, the q-state Potts ferromagnet correlation length on hierarchical lattices is calculated. These hierarchical lattices are build in order to simulate hypercubic lattices. The high-and-low temperature correlation length asymptotic behaviours tend (in the Ising case) to the Bravais lattices correlation length ones when the size of the hierarchical lattices cells tends to infinity. We conjecture that the asymptotic behaviours for several values of q and d (dimensionality) so obtained are correct. Numerical results are obtained for the full temperature range of the correlation length.

Key-words: Potts model; Correlation Length; Hierarchical lattices.

1 INTRODUCTION

The study of Bravais lattices through the use of hierarchical ones (which are not translationally invariant) has received a growing attention in the last few years, especially in the area of phase transitions (Reynolds et al. 1977, Yeomans and Stinchcombe 1979, Bleher and Zalys 1979, Berker and Ostlund 1979, de Magalhães et al. 1980, Levy et al. 1980, Curado et al. 1981, Griffiths and Kaufman 1982, Kaufman and Griffiths 1984).

The question whether the limit of functions calculated on families of hierarchical lattices, with basic cells (Melrose 1983a) of increasing size b (see, for example, fig. 1), converges to the respective functions on Bravais lattices is a point that is not clear today. An argument that favours the convergence is obtained if we adopt the Melrose (1983a) definition of dimension (D) and connectivity (Q) of hierarchical lattices (on whose values depend the critical exponents). For example, in fig. 1, the limit $b \to \infty$ leads to $D = 2$ and $Q = 1$ which coincides with the values of the square lattice. Other arguments are given in several works which exhibit an apparent convergence towards the corresponding results in Bravais lattices (Curado et al. 1981, Martin and Tsallis 1981, Oliveira 1982 Kaufman and Mon 1984, Hauser and Tsallis 1984, Curado et al. 1984). However, Tsallis and Levy 1981, Tsallis 1984, have provided results for the correlation length initial exponent ($b = 2,3,4$ and 5) which suggest that the convergence, if it exists, is not so clear.

In this work we investigate this problem through another point of view. We calculate, within the framework of real space renormalisation group (RG), the Potts model correlation length for hierarchical lattices and investigate how both high and low temperature asymptotic behaviours converge to the true hyper cubic lattice correlation length as $b \to \infty$.

2 POTTS MODEL ON HIERARCHICAL LATTICES AND RG

The families of cells that we choose for simulate hypercubic lattices are shown in fig. 1 for d = 2 (square lattice), in fig. 2a for d = 3 (simple cubic), fig. 2b for d = 4 and so on. The reasons for the word "simulate" are the following:

i) the intrinsic dimension D and the connectivity Q of these families of cells lead to $D = d$ and $Q = d - 1$ as $b \to \infty$ (d is the dimension of the Euclidean space where the cells are embedded, see Melrose 1983a). This leads to $D = 1 + Q$ in that limit, which is typical for Bravais lattices.

ii) the number of sites and bonds, of a basic cell with size b and intrinsic dimension D (embedded in a Euclidean space of dimension d) are

$$N_{sites} = b^Q (b-1) + 2 = b^{d-1}(b-1) + 2, \tag{1}$$

$$N_{bonds} = b^D = b^d + (d-1)b^{d-2}(b-1)^2 \tag{2}$$

which yields

$$\lim_{b\to\infty} \frac{N_{bonds}}{N_{sites}} = d \tag{3}$$

as for a d-dimensional hypercubic lattice.

In order to build a model on these cells we associate with each of their sites a Potts variable σ_i (=0,1...,q-1) and to each bond of the cells a coupling constant $J > 0$. The interac tions between the variables are given by (Potts model)

$$\mathcal{H} = - \sum_{<i,j>} qJ\delta_{\sigma_i\sigma_j} \tag{4}$$

where $\delta_{\sigma_i\sigma_j}$ is the Kroenecker's delta and <i,j> means nearest neighbours sites.

Performing the partial trace over the "internal" sites (full circles, see figures 1 and 2) of the chöosen cell we renormali ze it into a smaller cell (b') with a coupling constant K'. This can be done in a simple way adopting the break-collapse method (Tsallis and Levy, 1981). Using the transmissivity

$$t = \frac{1 - e^{-qK}}{1 + (q-1)e^{-qK}} \qquad K = \frac{J}{k_B T} \tag{5}$$

we obtain recursion relations in the form

$$R^d_{b'}(t') = R^d_b(t) . \tag{6}$$

where $R^d_b(t) = N^d_b(t)/D^d_b(t)$ being $N^d_b(t)$ and $D^d_b(t)$ polynomials functions of t.

For example to b = 2, b' = 1 and d = 2 (fig. 1) we have

$$t' = R^2_2(t) = \frac{2t^2 + 2t^3 + 5(q-2)t^4 + (q-2)(q-3)t^5}{1 + 2(q-1)t^3 + (q-1)t^4 + (q-1)(q-2)t^5} \tag{7}$$

It is important to observe in the two dimensional case that the cells are self-dual (see Melrose 1983b), like the square lattice, and yield the correct critical temperature of the square lattice Potts model for any value of b.

The break-collapse method enables us to calculate the asymptotic behaviour (Curado et al. 1981, Hauser and Tsallis 1984, Curado et al. 1984) of these functions $R_b^d(t)$ for $t \to 1 \left(\frac{k_B T}{J} \to 0\right)$ and $t \to 0 (k_B T/J \to \infty)$. The results are

$$R_b^2(t) \sim \begin{cases} bt^b + 2(b-1)^2 t^{b+1} + \ldots \quad (t \to 0) & (8a) \\ 1 - qb\left(\frac{1-t}{q}\right)^b - q\left[b^2(q-1) + 2(b-1)^2\right]\left(\frac{1-t}{q}\right)^{b+1} \ldots \quad (t \to 1) & (8b) \end{cases}$$

$$\begin{matrix} R_b^d(t) \sim \\ (d > 2) \end{matrix} \begin{cases} b^{d-1}t^b + 2(d-1)b^{d-2}(b-1)^2 t^{b+1} + \ldots \quad (t \to 0) & (9a) \\ 1 - qb\left(\frac{1-t}{q}\right)^{b^{d-1}} - \ldots - q^{2d-1}b^{d-1}(b-1)\left(\frac{1-t}{q}\right)^{b^{d-1}+2(d-1)} \ldots \quad (t \to 1) & (9b) \end{cases}$$

where in eq. (9b) we have indicated only the first dominant terms which contribute to the asymptotic correlation length behaviour in the $b \to \infty$ limit. We remark that the two-dimensional case is different from all the $d > 2$ cases, because of its peculiar topological features. For example if we break <u>one</u> "horizontal"(*) bond (any one) of figure 1b($b = 3$, $d = 2$) and collapse the rest of the horizontal ones, the resulting graph is <u>different</u> from that obtained if we collapse all the horizontal bonds. However if we break <u>one</u> horizontal bond of figure 2($b = 2$ or 3, $d = 3$) and collapse the re-

(*) We define a "horizontal" plane as that determined by all horizontal bonds located at a same "distance" to an arbitrary vertex (terminal), where "distance" means the minimum number of bonds that connect the bonds of the plane to this vertex. In a cell of size b there are $b - 1$ horizontal planes, see figures 1 and 2.

maining horizontal ones we obtain the <u>same</u> graph that is obtained if <u>all</u> the horizontal bonds are collapsed. This result is true for all $d > 2$.

In order to obtain the longitudinal correlation length (measured along one axis of a hypercubic lattice) as a function of temperature, we must have another equation besides of (6). This new equation arises from the well-known length scaling under the RG operations. So, we obtain

$$\xi(T') = \frac{\xi(T)}{b/b'} \quad . \tag{10}$$

We observe that eq. (10) remains invariant when it is multiplied by any factor $c(q;d)$ independent of T but that may depend of q and d. This factor cannot be obtained within the present formalism.

3 TWO DIMENSIONS: ASYMPTOTIC BEHAVIOUR AND NUMERICAL RESULTS OF ξ

The asymptotic behaviours of the correlation length $\xi_{b,b'}$ are obtained through the use of equations (6) with $d = 2$, (8) and (10) (Curado et al. 1981). We obtained the following expressions:

$$(\xi_{b,b'})^{-1} \sim - \frac{\ln(b/b')}{(b-b')} + \ln\left(\frac{k_BT}{J}\right) - \left[\frac{q-2}{2} + 2\left(\frac{b-1}{b}\right)^2\right]\frac{J}{k_BT} + \ldots \left(\text{if } \frac{k_BT}{J} \to \infty\right) \tag{11}$$

$$(\xi_{b,b'})^{-1} \sim \frac{c(q)J}{k_BT}\left[1 - \frac{\ln(b/b')}{b-b'}\frac{k_BT}{qJ} - 2\left(\frac{b-1}{b}\right)^2\left(\frac{k_BT}{qJ}\right)\exp(-qJ/k_BT)\ldots\right]\left(\text{if } \frac{k_BT}{J} \to 0\right) \tag{12}$$

It is interesting to note that if in eq. (12) we adopt for the invariant factor the value $c(q) = q$ we obtain qJ/k_BT as a "natural" variable of this equation suggesting that this can be the correct value for this constant. The same is not possible to do in eq. (11) where there is no factor that leads to a "natural" variable. The exact longitudinal correlation length of the square lattice Ising model (q=2) is known (Onsager 1944, Fisher and Burford 1967, Baxter 1982) and is the following

$$\xi_{EX}^{-1} = \ln \coth \frac{J}{k_BT} - \frac{2J}{k_BT} \qquad T > T_c \qquad (13a)$$

$$\sim \ln \frac{k_BT}{J} - \frac{2J}{k_BT} \qquad \frac{k_BT}{J} \to \infty \qquad (13b)$$

$$\xi_{EX}^{-1} = \frac{2J}{k_BT} - \ln \coth \frac{J}{k_BT} \qquad T < T_c \qquad (14a)$$

$$\sim \frac{2J}{k_BT}\left[1 - \frac{k_BT}{J} \exp(-2J/k_BT) \ldots\right] \qquad \frac{k_BT}{J} \to 0 \qquad (14b)$$

Therefore, the asymptotic behaviours of eqs. (11) and (12) [this with $c(q) = q = 2$] in the limit $b \to \infty$ reproduce the <u>exact</u> asymptotic behaviours given by expressions (13b) and (14b) respectively. So, we conjecture that the asymptotic expressions given by eqs. (11) and (12) are the correct ones for the square lattice, in the limit $b \to \infty$, for <u>all</u> q.

It is interesting to note that, as $\xi^{-1} = \gamma/k_BT$ (γ = surface tension), the term $\ln b/(b-1)$ of the eq. (12) (where we put $b' = 1$ because in this case our treatment corresponds to uniform hierarchical lattices) is associate with the zero temperature surface entropy $(\partial\gamma/\partial T)$ of the hierarchical lattice (of size b) as we can see from eq. (15) in the work of Curado et al. 1981. Also, following the comments and nomenclature of Kaufman and Griffiths 1984 we can see that, for the hierarchical lattices of this section, the ground state degeneracy D with antiperiodic boundary conditions and order $N = 1$ (basic cell) are given by $D = b$ leading the following expression for the degeneracy at Nth order D_N

$$D_N = b^{(b^N - 1)/(b-1)}.$$

This degeneracy arises when we associate a state α to one terminal (surface site for Griffiths and Kaufman 1982) and a state β (one of the remaining $q-1$ ones) $\neq \alpha$ to the other one.

The surface entropy per unit area at 0^0K provides the same term of the equation (12)

$$\lim_{N\to\infty} \frac{1}{b^N} \ln D_N = \frac{\ln b}{b-1}.$$

We also remark that $\lim_{b\to\infty} \frac{\ln b}{b-1} \to 0$ leads to the correct surface entropy of the Potts model on the square lattice with antiperiodic boundary conditions.

This same term $(\ln b/(b-1))$ appears in equation (11) and can be associated with the zero temperature surface entropy of the dual hierarchical lattice. As the hierarchical lattices presented in this section are self-duals (for any b), their surface tension are the same as those of their dual lattices. So, we can associate the dual lattice surface tension to the inverse of the original lattice correlation length (for $T > T_c$).

Along the lines of Curado et al. (1981) we use equations (6), (10), (11) and (12) to obtain the numerical results for the correlation length as a function of the temperature. They are shown in figures (3) and (4). In figure (3) we plot the $\xi_{b,b'}$ corresponding to $b = 2,3$ and 4 $(b' = 1)$ for the case $q = 2$ (where the exact answer is known) and we note that the RG results tend to approach the exact answer for increasing b. In figure (4) we plot the RG results for typical values of q. We note that for a fixed $T/T_c(q)$ the correlation length decreases as q increases. This can be intuitively understood as the higher probability that two variables become uncorrelated if there are more states to choose.

To analyse the behaviour of the $\xi_{b,b'}$ near T_c,

$$\xi_{b,b'} \sim A_{b,b'}^{(q)} \left(\frac{T - T_c(q)}{T_c(q)}\right)^{-\nu_{b,b'}(q)} , \qquad (15)$$

we studied the amplitude $A_{b,1}(q)$ $(T \gtrsim T_c(q))$ for several values of q with $b = 2,3,4$ (the critical exponent $\nu_{b,b'}(q)$ has been studied in previous works, see for example Tsallis and Levy 1981). Some values of them are shown in Table I. The results for $T \lesssim T_c(q)$ are obtained dividing those of Table I by $c(q)$.

4 d. DIMENSIONS: ASYMPTOTIC BEHAVIOUR OF ξ $(d > 2)$

In a similar way, with the use of equations (6), (9) and (10) we obtain the following expressions for ξ^{-1} (in the limit $b \to \infty$) for dimensions $d > 2$:

$$\lim_{\substack{b\to\infty \\ b'<b}} \xi_{b,b'}^{-1} \sim \ell n\left(\frac{k_B T}{J}\right) - \left[\frac{q-2}{2} + 2(d-1)\right]\frac{J}{k_B T} - \ldots$$

$$\forall\ d > 2, \forall q \left(\frac{k_B T}{J} \to \infty\right) \qquad (16)$$

$$\lim_{\substack{b\to\infty \\ b'<b}} \xi_{b,b'}^{-1} \sim c(q;d)\left(\frac{J}{k_B T}\right)^{1/(d-1)}\left\{1 - \frac{q^{2(d-1)}}{d-1}\left(\frac{k_B T}{qJ}\right) \exp\left[-\frac{2q(d-1)J}{k_B T}\right] - \right.$$

$$\left. - \ldots\right\} \quad \forall q, \forall d > 2 \quad \left(\frac{k_B T}{J} \to 0\right) \qquad (17)$$

The two terms of expression (16) reproduce (with $q = 2$ and $d = 3$) the first two terms of the high-temperature correlation length expansions (the only possible comparison, in the best of our knowledge) carried out by Fisher and Burford (1967). It is worthwhile to note that in three dimensions the hierarchical lattices we have used are not self-dual, which exhibits that this property is not a necessary condition to obtain the correct results as might be thought if we look only for the two-dimensional case. For the low-temperature case, eq. (17), we do not know of any result to compare with. Then we conjecture that the asymptotic behaviours of the correlation length of the adopted hierarchical lattices, given by equations (16) and (17), reproduce the exact asymptotic behaviours of the corresponding hypercubic lattices.

5 CONCLUSION

We calculated, with RG techniques on the family of hierarchical lattices shown in figures (1) and (2), the asymptotic behaviour of the Potts model longitudinal correlation length. In the limit $b \to \infty$ this correlation length reproduces (in the few known cases where we can compare) the exact asymptotic behaviour of the longitudinal correlation length in the corresponding hypercubic lattice. For the cases where there are no exact results to compare, we believe that the asymptotic behaviours obtained by us are the correct ones in the $b \to \infty$ limit. It is interesting to note that in several works, with different functions (see, for example, the specific heat in the work of Martin and Tsallis 1981, the surface tension in the works of Curado et al. 1981, 1984, and Hauser and Tsallis 1984), the convergence of the asymptotic behaviours of these functions (constructed on hierarchical lattices) to the corresponding functions on hipercubic lattices as $b \to \infty$ was shown. So, we believe that the asymptotic behaviours (for a large class of functions) obtained for these families of hierarchical lattices converge to the exact ones on the corresponding hypercubic lattices as $b \to \infty$.

Therefore, these results strongly suggest that if any discrepancy exist between the $b \to \infty$ limit of these families of hierarchical lattices, and the corresponding hypercubic lattices then it must be localized in the neighbourhood of T_c. To test this suggestion, full analytic expressions for arbitrary b would be necessary. The numerical calculations of the amplitude of ξ near T_c seem to converge to the exact values but are not conclusive. Finally we remark that these RG's, on hierarchical lattices, are a good alternative way to high and low temperature series, at least for the Potts model.

We want to thank especially Professor C. Tsallis for several discussions on this problem. We are indebted to A.C.N. de Magalhães, A.M. Mariz, L.R. da Silva and U.M.S. Costa for many valuable discussions. Also we acknowledge I. Roditi.

CAPTIONS (Figures)

Fig. 1. - Family of hierarchical lattices adopted to simulate the square lattice. They are self-dual ($\forall b$). The full circles are the "internal" sites and the open are the "terminals".

Fig. 2. - Family of hierarchical lattices adopted to simulate ⓐ the simple cubic lattice and ⓑ the hypercubic lattice for $d = 4$.

Fig. 3. - Comparison among the RG - $\xi_{b,1}$ results for several sizes of cells (full lines) and the exact one (dashed lines) for the cases $q = 2$ (Ising), $d = 2$.

Fig. 4. - RG - $\xi_{b,1}$ results for several values of q and $b = 4$.

CAPTION (Table I)

Table I. - The RG - ξ two dimensional amplitude $A_{b,1}(q)$ for several values of b and q ($T \gtrsim T_c(q)$). The only exact value known is the $q = 2$ case where the amplitude value is $[2 \ell n(1 + \sqrt{2}]^{-1} \simeq 0.5673$.

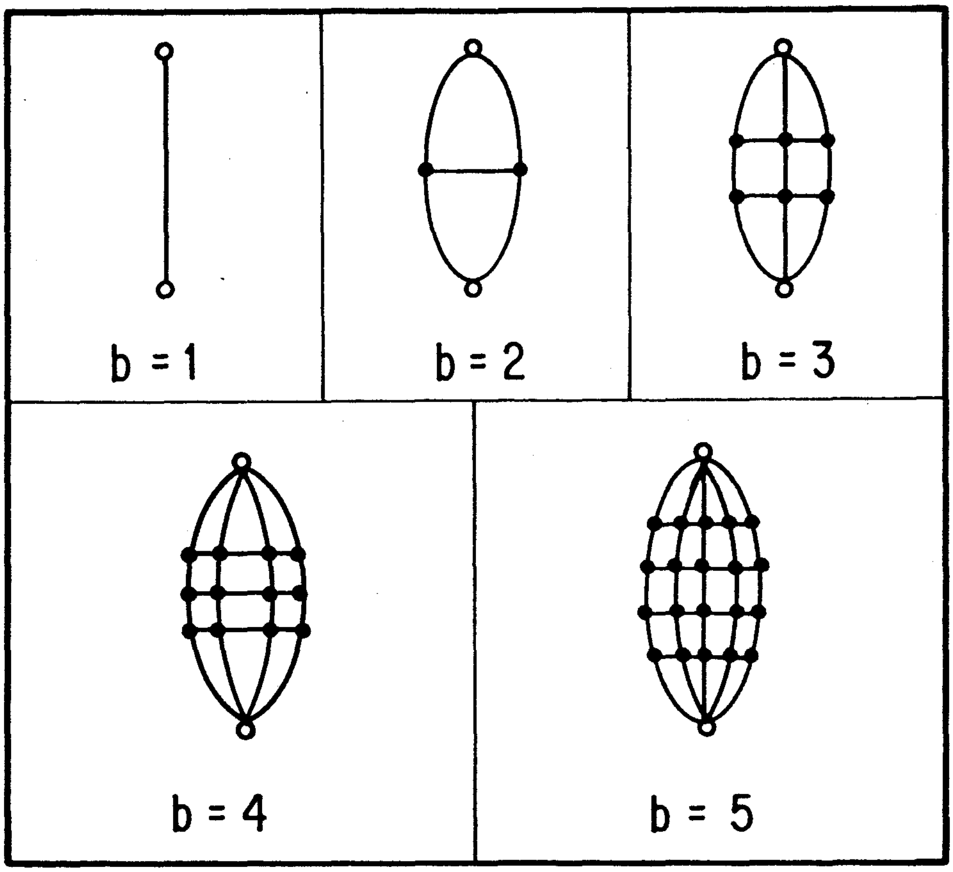

FIG. 1

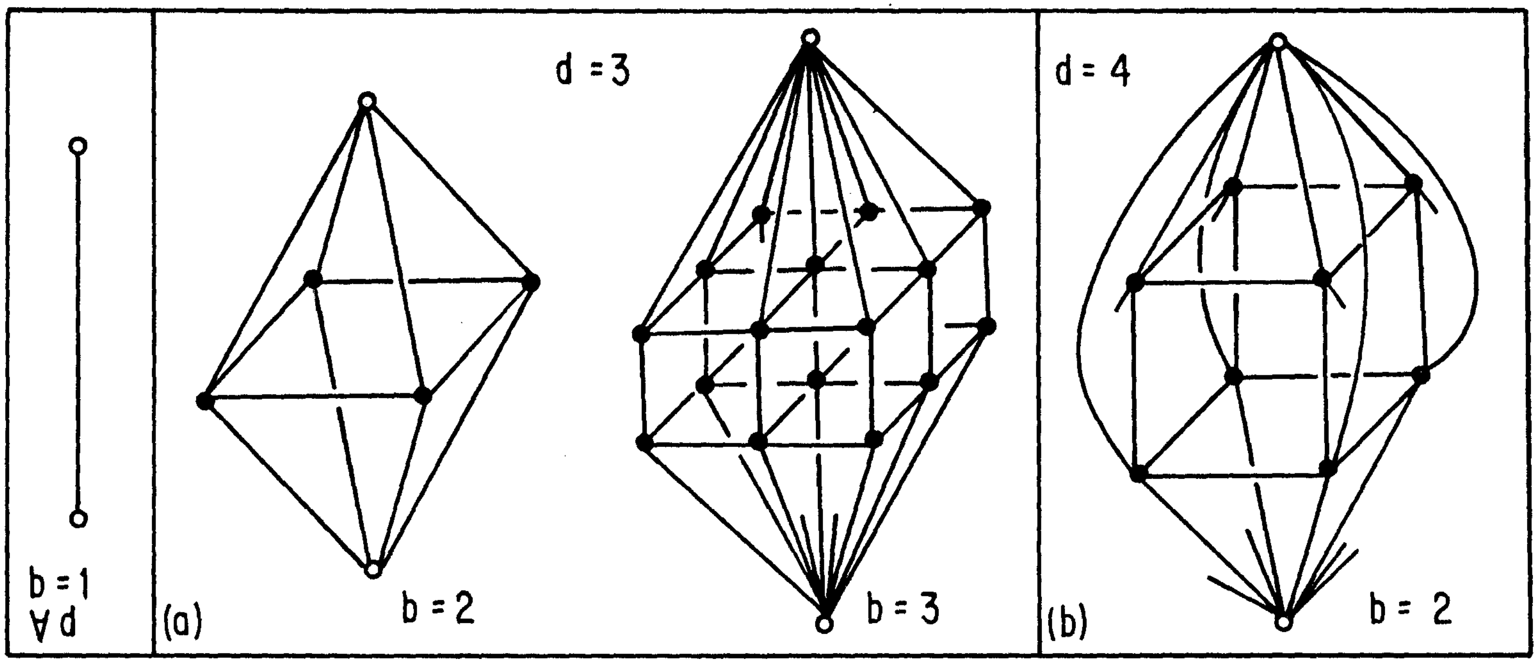

FIG. 2

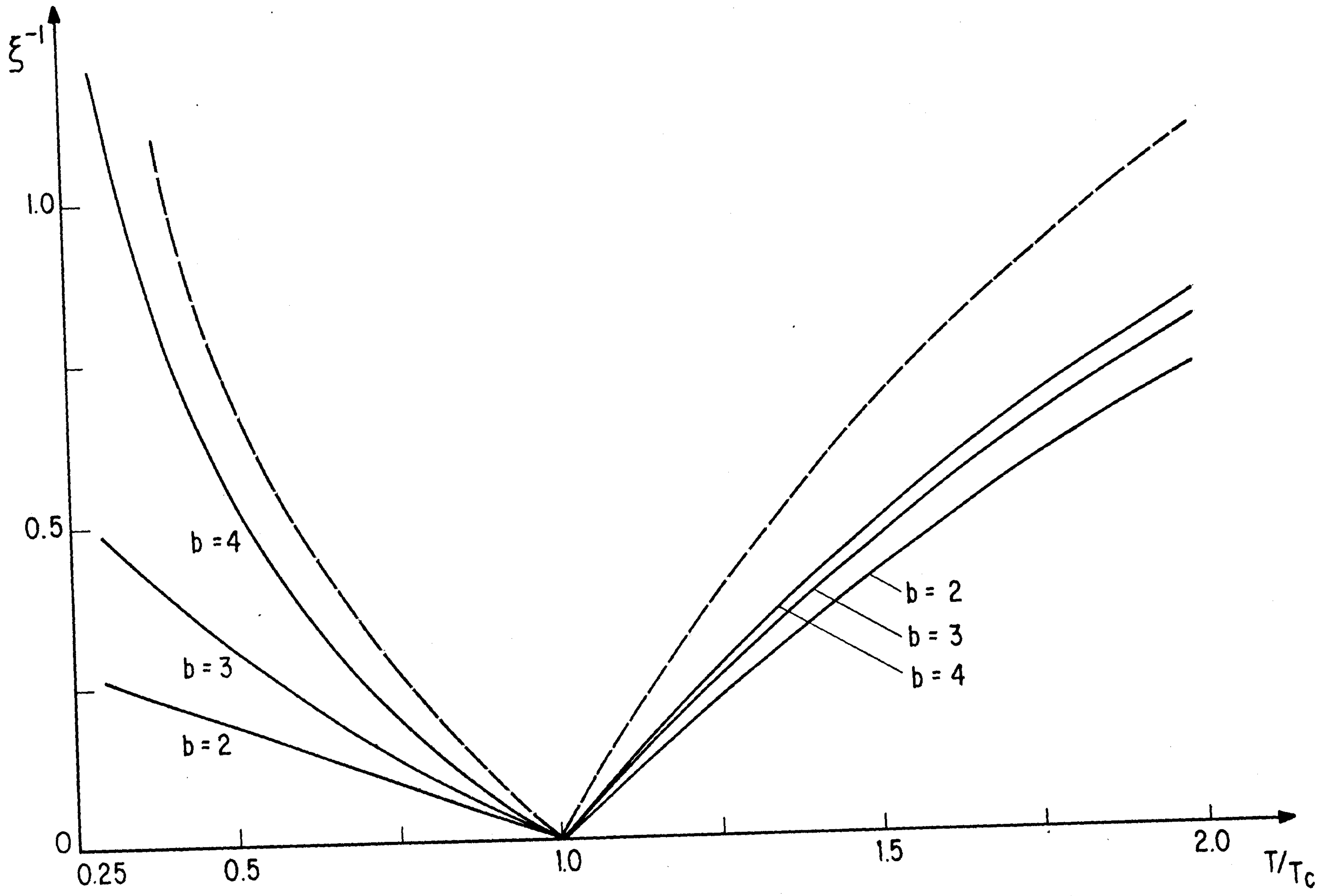

FIG.3

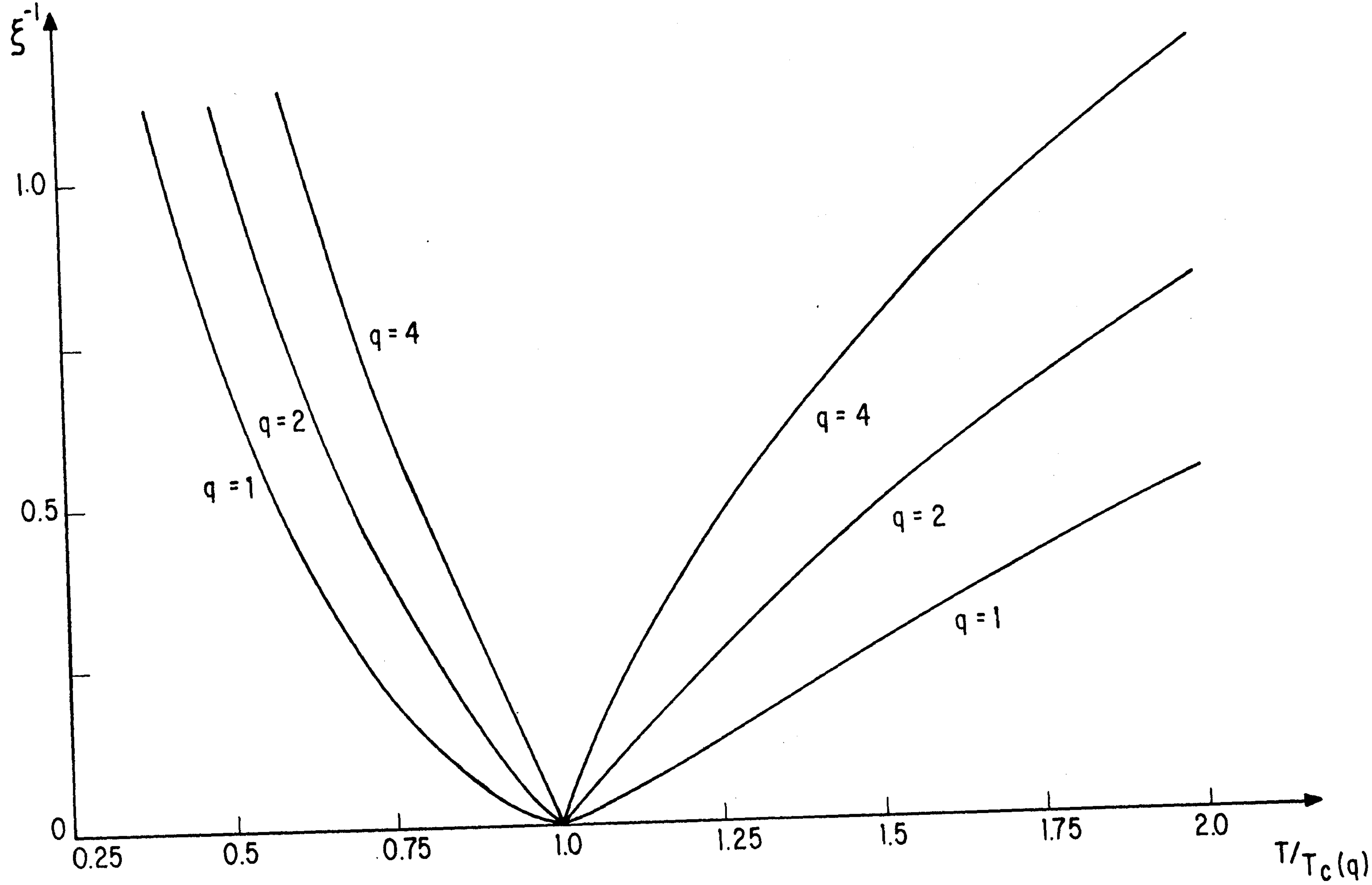

FIG.4

Table I

b \ q	0.1	0.5	1.0	2.0	3.0	4.0
2	8.130	1.765	1.144	0.805	0.675	0.602
3	6.998	1.596	1.049	0.746	0.629	0.563
4			1.00	0.715	0.604	0.541
Extrap. linear	4.734	1.258	0.857	0.626	0.534	0.481

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