CBPF-NF-029/80

CONJECTURE ON THE CRITICAL FRONTIER OF THE POTTS FERROMAGNET ON PLANAR LATTICES

by

Constantino Tsallis and
Aglaē C.N. de Magalhẽs

Centro Brasileiro de Pesquisas Fĩsicas/CNPq
Av. Wenceslau Braz, 71, fundos
22290 - Rio de Janeiro - R.J. - Brazil

ABSTRACT

A conjecture is proposed for the approximate critical frontiers of the q-state Potts ferromagnets on planar lattices. This conjecture is verified, within a satisfactory degree of accuracy, for a variety of planar lattices (as well as for the first and second neighbour square one), and enables the prediction of a considerable number of new results (29 independent critical points and a few critical lines).

Reduced title: Potts Ferromagnet on Planar Lattices.

Titre: Conjecture sur la frontière critique du modèle de Potts ferromagnétique sur des réseaux plans.

Résumé:

On propose une conjecture sur les frontières critiques approchées du modèle ferromagnétique de Potts à q états sur des réseaux plans. Cette conjecture est verifée avec une precision satisfaisante pour une varieté de réseaux plans (et aussi pour le réseau carré avec premiers et seconds voisins), et permet de prédire un nombre considerable de résultats nouveaux (29 points critiques independants et quelques lignes critiques).

It is well known that the ferromagnetic q-state Potts model presents a second-order (first-order) phase transition for all dimensionalities $d>1$ and number of states $q\leq q_c(d)$ $(q>q_c(d))$; in particular $q_c(2)=4$. We remind that $q=2$ corresponds to the 1/2-spin Ising model, and the $q\to 1$ and $q\to 0$ limits correspond respectively to the standard and tree-like bond percolations[1,2]. The critical frontiers associa ted to the anisotropic square, triangular and honeycomb lattices are presently, as far as we know, the only ones to be exactly known[3] for all q. And of course several other planar lattices have been solved for $q=2$[4]. In the present work we make a conjecture which essentially states that the knowledge, for a *particular* value of $q(1\leq q\leq 4)$, of the simple[5] paramagnetic-ferromagnetic critical frontier (CF) associa-ted to any given *planar* lattice (isotropic or not, homo-geneous or not) enables one to calculate, at least approxi-mately, the critical frontier associated to the same lattice for the *other* values of $q(1\leq q\leq 4)$.

Let us consider a single Potts bond (coupling constant J); it is convenient to introduce a variable $t^{(q)}$ (referred hereafter as *thermal transmissivity*, see Refs. (6-8) and references therein) defined as follows

$$t^{(q)} \equiv \frac{1-e^{-qJ/k_BT}}{1+(q-1)e^{-qJ/k_BT}} \qquad (J>0) \qquad (1)$$

It is interesting to remark that the $t^{(q)}$ -variable precisely coincides, for all q, with the p-variable Stephen[2] founds interesting to work with. Let us stress that $t^{(1)}=1-e^{-J/k_BT}$ is precisely the variable isomorphic[1] to the occupation probability of the standard bond percolation problem. It is straightforward to verify[8] that the equivalent transmissivity $t_s^{(q)}$ of a *series* array of two bonds with transmissivities $t_1^{(q)}$ and $t_2^{(q)}$ is given by

$$t_s^{(q)} = t_1^{(q)}\ t_2^{(q)} \tag{2}$$

whereas for a *parallel* array it is

$$t_p^{(q)D} = t_1^{(q)D}\ t_2^{(q)D} \tag{3}$$

where we have introduced[8] the *dual*[9] transmissivity

$$t_i^{(q)D} \equiv \frac{1-t^{(q)}}{1+(q-1)t^{(q)}} \qquad \forall i \tag{4}$$

Let us now introduce[7] a new variable $s^{(q)}$ through the relation

$$s^{(q)} \equiv \frac{\ln\left[1+(q-1)t^{(q)}\right]}{\ln q} \tag{5}$$

We verify a remarkable property, namely,

$$s^{(q)}\left\{t^{(q)D}\right\} = 1-s^{(q)}\left\{t^{(q)}\right\} \tag{6}$$

i.e. the s-variable *transforms, under duality, like a probability*. We verify as well that, in the limit $q\to 1$, $s^{(1)}=t^{(1)}$.

We are now prepared to state our conjecture. We shall assume known, for a certain *planar* lattice, the CF for a fixed value q_0 $(1 \leq q_0 \leq q_c(2))$, namely, the equation

$$\phi\left\{s_1^{(q_0)}, s_2^{(q_0)}, \ldots, s_n^{(q_0)}\right\} = 0 \tag{7}$$

where $s_1^{(q_0)}, s_2^{(q_0)}, \ldots, s_n^{(q_0)}$ are associated to the independent coupling constants $J_1, J_2, \ldots, J_n$ of the system under consideration. Our conjectural statement will be that the equation

$$\phi\left\{s_1^{(q)}, s_2^{(q)}, \ldots, s_n^{(q)}\right\} = 0 \tag{8}$$

represents also, either exactly or within great accuracy, the CF of the corresponding q-state Potts model for the *other* values of $q(1 \leq q \leq q_c(2))$.

Anisotropic Square Lattice - The bond percolation CF for this lattice is given by[10]

$$p_1 + p_2 = 1 \tag{9}$$

which, within the present framework, will be generalized into

$$s_1^{(q)} + s_2^{(q)} = 1 \tag{10}$$

which reproduces the *exact*[3] critical temperature for *any* value of q. We see, therefore, that for this system the present conjecture is *rigorously true* $(0 \leq q \leq 4)$. Moreover, this might happen only for this lattice, as a consequence of its self-duality.

Anisotropic Triangular and Honeycomb Lattices - The bond percolation CF frontier for the anisotropic triangular lattice is given by[10]

$$1 - p_1 - p_2 - p_3 + p_1 p_2 p_3 = 0 \tag{11}$$

(for the honeycomb lattice do $p_i \to 1-p_i$ $\forall i$). Therefore, within the present framework, the CF for any value of $q (1 \leqslant q \leqslant 4)$ is approached by

$$1-s_1^{(q)}-s_2^{(q)}-s_3^{(q)}+s_1^{(q)} s_2^{(q)} s_3^{(q)} = 0 \qquad (11')$$

Let us now compare this eq. with the exact[3] one. We immediately verify that they coincide whenever one of the three coupling constants vanishes (anisotropic square lattice limit). Next we perform the comparison for the maximal error case, namely, the isotropic limit. Our conjecture leads to $s^{(q)}=2 \sin \frac{\pi}{18}$ for this lattice, whereas the exact answer is given by

$$\frac{t^{(q)}}{1-t^{(q)}} = \frac{2}{q} \cos \left[\frac{1}{3} \text{arc cos} \left(\frac{q}{2} - 1 \right) \right] - \frac{1}{q} \qquad (q \leqslant 4) \qquad (12)$$

The results are presented in the Fig. 1. We remark that the error is smaller than 2.4% for $1 \leqslant q \leqslant 4$. It is straightforward to verify[11] that, in the $q \to 0$ limit, $t^{(q)} \sim 1 - L\sqrt{q}$, hence $s^{(q)} \sim \frac{1}{2} + \frac{\ln L}{\ln q}$, where L is a lattice-dependent pure number ($L = 1$, $\sqrt{3}$, $1/\sqrt{3}$ for the square, triangular and honeycomb lattices respectively; as a matter of fact for any pair of dual lattices it holds $L^D=1/L$). The non-vanishing error comes from the tendency of $s^{(q)}$ towards 1/2, and its smallness from the infinite slope at $q=0$.

<u>"Inhomogeneous" 4-8 Lattice and its Dual</u> - The exact CF's associated to the Ising model in the "inhomogeneous" 4-8 lattice (figure 10 of Ref.(4)) and its dual (square lattice with non-crossing diagonal bonds) are known[4]. Since there is a straightforward relation between the CF's of any pair of dual lattices, we shall restrict our discussion here to the 4-8

lattice (we note J_1 and J_2 the coupling constants respectively associated to the different bonds which are in a 2:1 ratio). The exact CF in the $s_1^{(2)}-s_2^{(2)}$ space is represented in Fig. 2. We may verify that $s_1^{(2)}=s_2^{(2)} \simeq 0.6792$, $\left.\frac{ds_1^{(2)}}{ds_2^{(2)}}\right|_{s_2^{(2)}=1} \simeq -0.414$ and

$$\left(s_2^{(2)}-\frac{1}{2}\right) \sim A \left(s_1^{(2)}-1\right)^2 \qquad (A\simeq 1.39)$$

In the present framework, this CF should, within satisfactory accuracy, be the same for $q\neq 2$; let us compare it with a recent conjecture[12] (completely unrelated to the present one) for bond percolation in the same system, namely

$$3\left(s_2^{(1)}-1/2\right) -4\left[\left(1-s_1^{(1)}\right)^2+\left(1-s_1^{(1)}\right)^3\right] = 0 \qquad (13)$$

From this equation it comes that $s_1^{(1)}=s_2^{(1)}\simeq 0.6801$ (which compares well with 0.675 ± 0.027[13] and 0.684[14], is included in the conjectural interval 0.645-0.707[15], and whose discrepancy with the value 0.6792 is only 0.13%), $\left.ds_1^{(1)}/ds_2^{(1)}\right|_{s_2^{(1)}=1} = -3/7 \simeq -0.429$ (whose discrepancy with the value -0.414 is 3.5%), and the same assymptotic behaviour mentioned previously for q=2 is satisfied with A=4/3 (which differs by 4.2% from 1.39). As we see, the CF's associated to q=1 and q=2 are satisfactorily coincident, therefore we conjecture that eq.(13) or the one corresponding to Fig. 2 can be used as well for *all* values of q ($1\leq q\leq 4$).

Some Other Planar Lattices - The exact critical points for the isotropic Kagomé, Diced, 3-12 and Asanoha (Hemp-Leaf) lattices (see figs.14, 15 and 19 of Ref. (4)) are known[4] only for q=2. Thus by imposing the exact critical value $s^{(2)}$ to be equal to $s^{(q)}$ we have obtained the critical points indicated in the Table which compares well with previous q=1 results[14,15]. For the Archimedean[14] lattices we used Neal[14] estimates' for $p_c = s^{(1)}$ to predict the critical points for $q \neq 1$ shown in the Table.

Some First- and Higher-Neighbour Lattices - The present conjecture, mainly supported by the probability-like transformation of the s-variable under duality, is not expected to *necessarily* lead to satisfactory approximate CF's for non-planar lattices. For example, for the sc lattice (fcc, bcc) $t^{(2)} \simeq 0.2181$[20] (0.1017[20], 0.1561[20]) hence $s^{(2)} \simeq 0.2846$ (0.1398, 0.2093), whereas $t^{(1)} = s^{(1)} \simeq 0.2526 \pm 0.0013$ [21] (0.119 ± 0.001[22], 0.1785 ± 0.0020[22]), therefore there is a discrepancy of about 13% (17%, 17%). Although the first and second as well as the first, second and third neighbour square and triangular lattices are not strictly planar (in the sense that they cannot be embedded in the plane; remark however that their three-dimensional extension is a *finite* one), we have also considered them in this work since our conjecture is well verified ($s^{(1)}$ equals $s^{(2)}$ within a 0.4% error) for the first and second neighbour square lattice (see the Table). Therefore, we have used the Ising critical temperature[17,23] to predict the $q \neq 2$ critical points for the three last lattices of the Table.

In the Table we present the critical points for several isotropic and homogeneous lattices and values of q. The estimated error bars simultaneously take into account the values available in the literature as well as the error (exactly known for the triangular and honeycomb lattices) introduced by the present conjecture. Cross-checking, by other procedures, of the critical points appearing in this Table, as well as of the critical lines associated to the 4-8 lattice would be very wellcome.

We are indebted to G.Schwachheim, S.V.F.Levy and E.M.F.Curado for useful remarks.

REFERENCES

1. P.W.Kasteleyn and C.M.Fortuin, J.Phys. Soc. Japan (suppl.) 26, 11 (1969).
2. M.J.Stephen, Phys. Lett. 56A, 149 (1976).
3. T.W.Burkhardt and B.W.Southern, J.Phys. A 11, L247 (1978); R.J.Baxter, H.N.V.Temperley and S.E.Ashley, Proc. R. Soc. A 358, 535 (1978); and references therein.
4. I.Syozi, "Phase Transitions and Critical Phenomena", vol.1 pg. 270 (1972; edited by C.Domb and M.S.Green) and references therein.
5. By "simple" we mean that *all* the coupling constants are of the ferromagnetic type.
6. C.Tsallis and S.V.F.Levy, J.Phys. C 13, 465 (1980); E.M.F. Curado, C.Tsallis, S.V.F.Levy and M.J.de Oliveira, Phys. Rev. B 23 (1981).
7. S.V.F.Levy, C.Tsallis and E.M.F.Curado, Phys. Rev. B 21, 2991 (1980); C.Tsallis, J. Phys. C 14, L85 (1981).
8. C.Tsallis and S.V.F.Levy (1980), to be published.
9. "Dual" stands for the fact that this transformation unifies the series and parallel algorithms (2) and (3); see also Ref.(4).
10. M.F.Sykes and J.W.Essam, Phys. Rev. Lett. 10, 3 (1963).
11. Though we cannot prove it, we believe that $t^{(q)} \sim 1-L\sqrt{q}$ holds for *all* planar lattices, i.e. they all become, for $q \to 0$, self-dual in the sense that $s^{(q)} \to 1/2$.
12. A.C.N. de Magalhães, C.Tsallis and G.Schwachheim, J.Phys. C14, 1393 (1981).
13. P.Dean, Proc. Camb. Phil. Soc. 59, 397 (1963).
14. D.G.Neal, Proc. Camb. Phil. Soc., 71, 97 (1972); the present Archimedean lattices are described by L.F.Tóth, "Regular Figures" (Pergamon Press, 1964).
15. H.Ottavi, J.Physique 40, 233 (1979).

16. C.Domb and R.B.Potts, Proc. Roy. Soc. A210, 125 (1951).

17. N.W.Dalton and D.W.Wood, J.Math. Phys. 10, 1271 (1969); M.Nauenberg and B.Nienhuis, Phys. Rev. Lett. 33, 944 (1974).

18. C.Fan and F.Y.Wu, Phys. Rev. 179, 560 (1969).

19. R.W.Gibberd, J.Math. Phys. 10, 1026 (1969).

20. J.Zinn-Justin, J. de Physique 40, 969 (1979).

21. A.C.N. de Magalhães, C.Tsallis and G.Schwachheim, J.Phys. C 13, 321 (1980) and references therein.

22. M.F.Sykes, D.S.Gaunt and M.Glen, J. Phys. A: Math. Gen. 9, 1705 (1976).

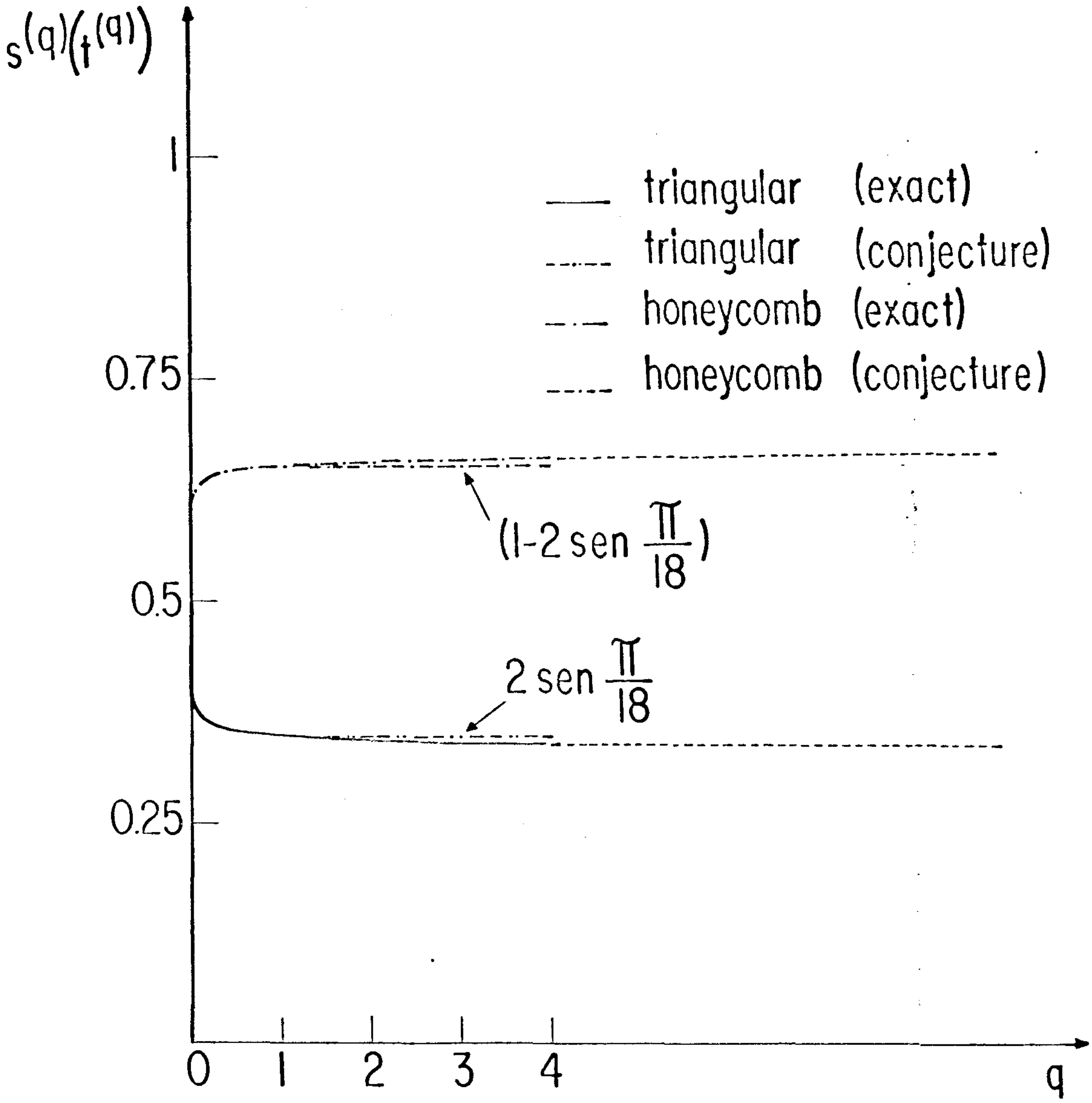
23. C.Domb and N.W.Dalton, Proc. Phys. Soc. 89, 859 (1966).

CAPTIONS FOR FIGURES AND TABLE

FIG. 1 Comparison between the exact and conjectural critical points for the isotropic triangular and honeycomb lattices (the analytic extension for $q>4$ has been represented as well).

FIG. 2 The exact para(P) - ferro(F) - magnetic critical frontier of the 4-8 lattice Ising model (as a matter of fact the proposed[12] $q=1$ critical line is indistinguishable from the $q=2$ one, within the present scale).

TABLE Critical points for isotropic and homogeneous q-state Potts ferromagnets in a set of lattices. $t^{(q)}$ and $s^{(q)}$ are related to $k_B T_c/J$ through eqs. (1) and (5); (...) denotes an exact value. For the triangular and honeycomb lattices the exact, rather than the conjectural, values are indicated. The region delimited by a heavy line contains results that, as far as we know, have not yet been checked by any other procedure.† This central value has been adopted after consideration of the Ising critical lines associated to the case where the first and second coupling constants are not necessarily equal.



s(q)(t(q))
1
0.75
0.5
0.25
0
1
2
3
4
q
triangular (exact)
triangular (conjecture)
honeycomb (exact)
honeycomb (conjecture)
(1-2 sen π/18)
2 sen π/18

FIG.1

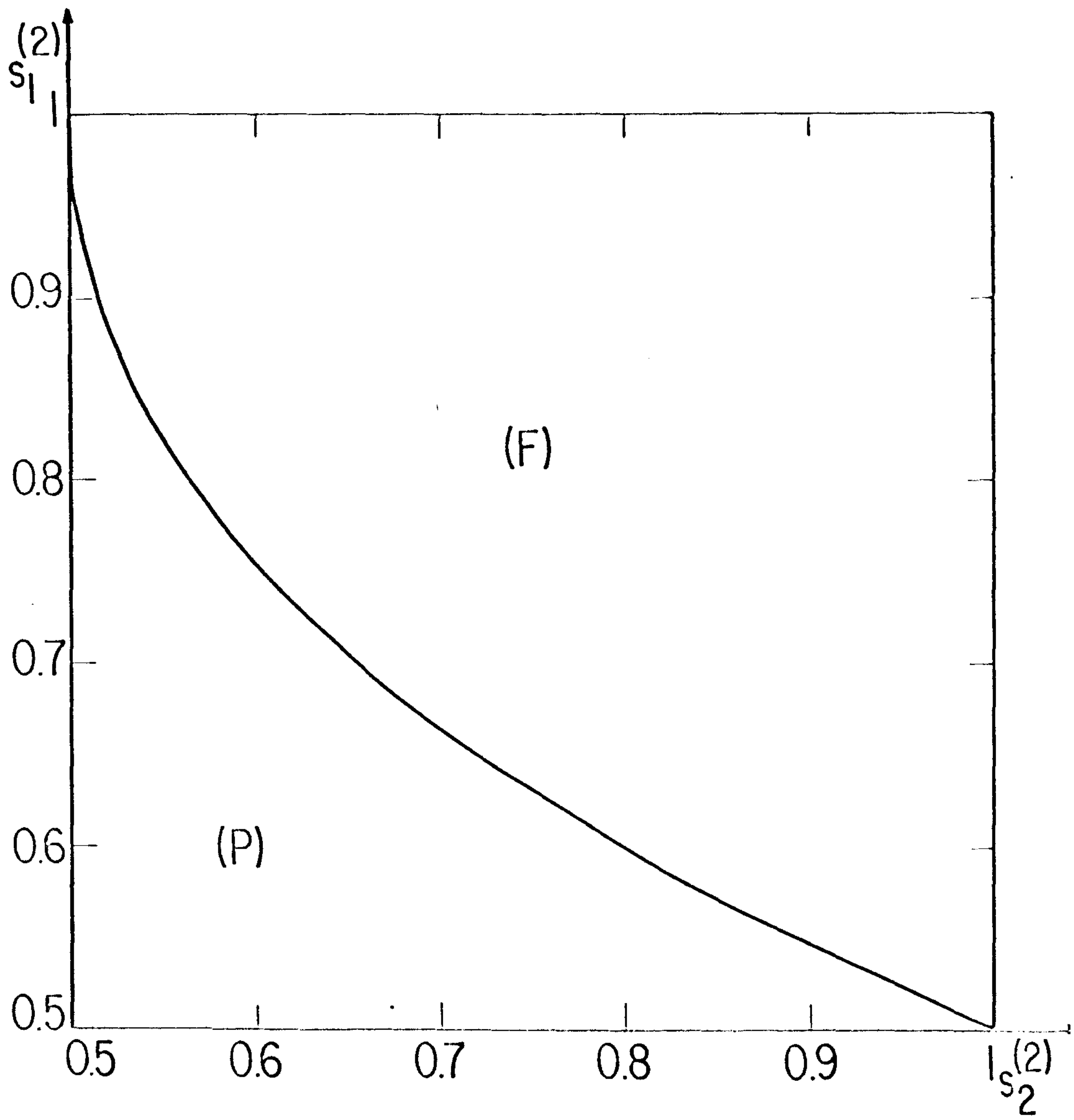


FIG. 2

TABLE

Lattices	q=1	q=2		q=3		q=4	
	$s^{(1)}=t^{(1)}$	$s^{(2)}$	$t^{(2)}$	$s^{(3)}$	$t^{(3)}$	$s^{(4)}$	$t^{(4)}$
Square	1/2 [10]	1/2	0.414... [4]	1/2	0.366... [3]	1/2	1/3 [3]
Triangular	0.347... [10]	0.342...	0.268... [4]	0.340...	0.227... [3]	0.339...	1/5 [3]
Honeycomb	0.653... [10]	0.658...	0.577... [4]	0.660...	0.532... [3]	0.661...	1/2 [3]
4-8	0.679±0.006 [12] 0.680±0.005 [13] 0.675±0.027 0.684 [14] [0.645,0.707] [15]	0.679...	0.601... [4]	0.679±0.003	0.554±0.003	0.679±0.005	0.521±0.006
Non-Crossing Diagonal Square-Lattice	0.321±0.006	0.321...	0.249... [4]	0.321±0.003	0.211±0.002	0.321±0.005	0.187±0.004
Kagomé	0.521±0.006 0.526 [14] [0.522,0.529] [15]	0.521...	0.435... [4]	0.521±0.003	0.387±0.003	0.521±0.005	0.354±0.005
Diced	0.479±0.006	0.479...	0.393... [4]	0.479±0.003	0.346±0.003	0.479±0.005	0.314±0.004
3-12	0.740±0.011 0.751 [14]	0.740...	0.671... [4]	0.740±0.005	0.628±0.006	0.740±0.008	0.597±0.010
Asanoha	0.260±0.011	0.260...	0.197... [4]	0.260±0.005	0.165±0.004	0.260±0.008	0.144±0.005
(4,6,12)	0.693 [14]	0.693±0.012	0.617±0.013	0.693±0.017	0.571±0.020	0.693±0.021	0.538±0.025
(3,4,6,4)	0.525 [14]	0.525±0.012	0.439±0.012	0.525±0.017	0.390±0.017	0.525±0.021	0.357±0.020
(3,3,3,3,6)	0.439 [14]	0.439±0.012	0.356±0.011	0.439±0.017	0.310±0.015	0.439±0.021	0.279±0.018
(3,3,3,4,4) and (3,3,4,3,4)	0.422 [14]	0.422±0.012	0.340±0.011	0.422±0.017	0.295±0.015	0.422±0.021	0.265±0.017
1st and 2nd Neighbour Square Lattice	0.249±0.011 0.250±0.003 [12]	0.249±0.007	0.188±0.006 † 0.189 [16] 0.188 [17] 0.184 [18] 0.199 [19]	0.249±0.008	0.157±0.006	0.249±0.009	0.137±0.006
1st and 2nd Neighbour Triangular Lattice	0.155±0.004	0.155±0.001	0.1135±0.0010 0.1135 [17] 0.1131 [23]	0.155±0.002	0.093±0.001	0.155±0.003	0.080±0.002
1st, 2nd and 3rd Neighbour Square Lattice	0.154±0.004	0.154±0.001	0.1130±0.0010 0.1130 [23]	0.154±0.002	0.092±0.001	0.154±0.003	0.080±0.002
1st, 2nd and 3rd Neighbour Triangular Lattice	0.099±0.003	0.099±0.001	0.0711±0.0010 0.0711 [23]	0.099±0.002	0.057±0.001	0.099±0.002	0.049±0.001