THE PROPAGATOR FOR AN OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

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ABSTRACT - The propagator for a one-dimensional oscillator with time-dependent frequency is calculated by the path-integral method.

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* Partially supported by CNPq (Brazil)

** Partially supported by CNPq/FINEP (Brazil)

I. INTRODUCTION

The propagator between the states $|x',t'>$ and $|x'',t''>$ of a one-dimensional quantum system characterized by a Hamitonian

$$H = \frac{p^2}{2m} + V(x,t) \tag{1}$$

is given by[1]

$$K(x'',t'';x',t') = \lim_{n\to\infty} \int \prod_{i=1}^{n} dx_i \prod_{j=1}^{n+1} \left(\frac{dp_j}{2\hbar}\right) \exp\left\{\frac{i}{\hbar}\left[\sum_{k=1}^{n+1} p_k(x_k - x_{k-1}) - \varepsilon H_k\right]\right\} \tag{2}$$

where

$$x_i = x(t_i),\ t_0 = t',\ x_0 = x',\ x_{n+1} = x'',\ \varepsilon = \frac{t''-t'}{n+1},\ t_{i+1} = t_i + \varepsilon \tag{3}$$

and $$H_k = \frac{p^2{}_k}{2m} - V\left[x(t_{k,k-1}), t_{k,k-1}\right] \tag{4}$$

where $t_{k,k-1}$ is any value of t belonging to the interval $\left[t_{k-1}, t_k\right]$(*).

The explicit calculation using the form (2) is limited to some special cases due to technical problems in the evaluation of the integrals involved. Hence it is of interest to make the evaluation in as many cases as possible.

(*) Usually $x(t_{k,k-1})$ is taken as the mean-value between x_k and x_{k-1}. But to order ε, one may take for $t_{k,k-1}$ any value of the interval $\left[t_{k-1}, t_k\right]$ without changing the integral, what amounts to take any value for x in the interval $\left[x_{k-1}, x_k\right]$.

In this note we consider the calculation of the propagator for a one-dimensional harmonic oscillator with time-dependent frequency i.e. for the cases when

$$V(x,t) = \frac{m}{2}\,\omega^2(t)x^2 \tag{5}$$

A calculation of this case has already appeared in literature[2]. The method used involves a complicated series of manipulations of finite-difference equations. Also certain results stated in the appendix cannot be obtained using the conditions stated. In contrast our calculation involves very simple techniques. We present our calculation in next section.

II. CALCULATION OF THE PROPAGATOR

We now consider the evaluation of the integral in eq.(2) for the special case of the potential of eq.(6). Choosing $t_{k,k-1} = \frac{t_k + t_{k-1}}{2}$, eq.(4) becomes

$$H_k = \frac{p_k^2}{2m} - \frac{m}{2}\,\omega^2_{k,k-1}\left(\frac{x_k + x_{k-1}}{2}\right)^2 \tag{6}$$

where

$$\omega_{k,k-1} = \omega(t_{k,k-1}) \tag{7}$$

By the repeated use of the Fresnel integral

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp \left| i\varepsilon \left(p\dot{x} - \frac{p^2}{2} \right) \right| = (2\pi\, i\varepsilon)^{-1/2} \exp \left(\frac{i\varepsilon\, \dot{x}^2}{2} \right) \tag{8}$$

we get

$$K(x'', t''; x', t') = \left(\frac{m}{2\ \ i\hbar} \right)^{1/2} \lim_{n\to\infty} \left\{ \left(\frac{1}{\varepsilon\ N_n} \right)^{1/2} \exp \left(\frac{im}{2\varepsilon\hbar} (A_n x''^2 - 2B_n x'' x' + C_n x'^2) \right) \right\} \tag{9}$$

The coefficients N_n, A_n, B_n and C_n are shown in what follows and calculated in the limit $n\to\infty$.

The normalization factor N_n satisfies the recurrence relation

$$N_k = \left(P^-_{k+1,k} + P^-_{k,k-1} \right) N_{k-1} - P^{+2}_{k,k-1} N_{k-2} \tag{10}$$

with the initial conditions

$$N_{-1} = 0 \quad \text{and} \quad N_0 = 1 \tag{11}$$

and

$$P^{\pm}_{k,k-1} = 1 \pm \frac{\varepsilon^2}{4} \omega^2_{k,k-1} \tag{12}$$

Let $M(t_{k+1}) = \varepsilon N_k$. (13)

Then, taking the limit $n \to \infty$, the finite-difference equation (10) and the initial conditions (11) take the following forms respectively:

$$\ddot{M} + \omega^2(t)M = 0 \tag{14}$$

$$M(t') = 0 \quad \text{and} \quad \dot{M}(t') = 1 \tag{15}$$

A solution of the form

$$M(t) = s(t) \sin(\gamma(t) - \gamma(t')) \tag{16}$$

automatically satisfies the condition $M(t')=0$. Also eq.(14) is satisfies if

$$\ddot{s} - s\dot{\gamma}^2 + \omega^2 s = 0 \tag{17}$$

and

$$s^2 \dot{\gamma} = \text{constant} \tag{18}$$

The remaining boundary condition is satisfied if

$$s(t')\dot{\gamma}(t') = 1 \tag{19}$$

The equations 17-19 can be recast in the form

$$\ddot{s} - \frac{s^2(t')}{s^3} + \omega^2 s = 0 \tag{20}$$

and

$$s^2\dot{\gamma} = s(t') \tag{21}$$

From (13) we have

$$\lim_{n\to\infty} \varepsilon N_n = M(t'') = s(t'')\sin(\gamma(t'') - \gamma(t')) \tag{22}$$

and using (19) and (21)

$$\lim_{n\to\infty} \varepsilon N_n = M(t'') = \frac{\sin[\gamma(t'')-\gamma(t')]}{\left[\dot{\gamma}(t'')\dot{\gamma}(t')\right]^{1/2}} \tag{23}$$

The coefficient A_n appearing in eq.(9) can be written as:

$$A_n = \frac{P^2_{n+1,n} N_n - P^2_{n+1,n} N_{n-1}}{N_n} \tag{24}$$

and in the limit $n\to\infty$

$$\frac{A_n}{\varepsilon} = \frac{\dot{M}(t'')}{M(t'')}$$

$$= \frac{\dot{s}(t'')}{s(t'')} + \dot{\gamma}(t'')\,\mathrm{cotg}\,[\gamma(t'') - \gamma(t')] \tag{25}$$

Next,

$$B_n = \frac{P^+_{n+1,n} P^+_{n,n-1} \cdots P^+_{1,0}}{N_n} \tag{26}$$

so that from (23)

$$\lim_{n\to\infty} \frac{Bn}{\varepsilon} = \frac{1}{M(t'')} \tag{27}$$

Finally the coefficient C_n is given by

$$C_n = P^-_{1,0} - \frac{P^{+2}_{1,0}}{N_1} - \frac{P^{+2}_{1,0}\, P^{+2}_{2,1}}{N_1 N_2} - \frac{P^{+2}_{3,0}\, P^{+2}_{2,1}\, P^{+2}_{3,2}}{N_2 N_3} - \dots - \frac{P^{+2}_{1,0}\, P^{+2}_{2,1} \dots P^{+2}_{n,n-1}}{N_{n-1} N_n}$$

$$= P^-_{1,0} - \frac{P^{+2}_{1,0}}{N_1}\left(1 + \frac{P^{+2}_{2,1}\, N_0}{N_2}\left(1 + \frac{P^{+2}_{3,2} N_1}{N_3}\left(1 + \dots + \frac{P^{+2}_{n,n-1} N_{n-2}}{N_n}\right) \dots\right)\right) \tag{28}$$

We want to write the right-hand side of eq.(28) as a feaction. To this aim, the trick consists in first defining the transformation

$$s_p(\omega) = 1 + a_p\, \omega \tag{29}$$

$$= \frac{1}{1 - \dfrac{a_p}{a_p + \dfrac{1}{\omega}}} \tag{30}$$

$$p = 1, 2, \dots$$

and second applying successively this kind of transformation and taking $\omega=0$, so that:

$$s_1 s_2 \dots s_n(0) = 1 + a_1(1 + a_2(1 + \dots + a_{n-1}) \dots) \tag{31}$$

$$= \cfrac{1}{1 - \cfrac{a_1}{a_1 + 1 - \cfrac{a_2}{a_2 + 1 - \ddots - \cfrac{a_{n-1}}{a_{n-1} + 1}}}} \tag{32}$$

Now, comparing eqs.(28) and (31), and using eq.(32) and the recurrence relation (10), it is easy to rewrite C_n as a fraction

$$C_n = P^-_{1,0} - \cfrac{P^{+2}_{1,0}}{P^-_{2,1} + P^-_{1,0} - \cfrac{P^{+2}_{2,1}}{P^-_{3,2} + P^-_{2,1} - \ddots - \cfrac{P^{+2}_{n,n-1}}{P^-_{n+1,n} - P^-_{n,n-1}}}} \tag{33}$$

This complicated looking expression can be evaluated very simply. First observe that if we use the recurrence relation (10) in (24) we get

$$A_n = P^-_{n+1,n} - \cfrac{P^{+2}_{n+1,n}}{P^-_{n+1,n} + P_{n,n-1} - \cfrac{P^{+2}_{n,n-1}}{(P^-_{n,n-1} + P^-_{n-1,n-2}) - \ddots - \cfrac{P^{+2}_{2,1}}{P^-_{2,1} + P^-_{1,0}}}} \tag{34}$$

By comparing (33) and (34) we see that C_n is exactly what we obtain by reversing time in expression (34), i.e., by making the following exchanges

$$0 \leftrightarrow n+1 \quad , \quad 1 \leftrightarrow n, \ldots$$

Thus in the limit $n\to\infty$ we obtain C_n by taking A_n as in (25) and exchanging t" and t' and keeping in mind that the exchange above imply $\frac{d}{dt} \to -\frac{d}{dt}$.

Hence when $n\to\infty$,

$$\frac{C_n}{\varepsilon} = -\frac{\dot{s}(t')}{s(t')} + \dot{\gamma}(t') \operatorname{cotg}\left[\gamma(t") - \gamma(t')\right] \qquad (35)$$

This gives the final result for the propagator as

$$K(x",t";x',t') = \left(\frac{m}{2\pi i\hbar}\right)^{1/2} \cdot \frac{(\dot{\gamma}(t")\dot{\gamma}(t'))^{1/2}}{\sin\left[\gamma(t") - \gamma(t')\right]}$$

$$\exp\frac{im}{2\hbar}\left\{\left(\frac{\dot{s}(t")}{s(t")} + \dot{\gamma}(t") \operatorname{cotg}\left[\gamma(t") - \gamma(t')\right]\right) x"^2\right.$$

$$+ \left(-\frac{\dot{s}(t')}{s(t')} + \dot{\gamma}(t') \operatorname{cotg}\left[\gamma(t") - \gamma(t')\right]\right) x'^2$$

$$\left. - 2\,\frac{\left[\dot{\gamma}(t")\,\dot{\gamma}(t')\right]^{1/2}}{\sin\left[\gamma(t") - \gamma(t')\right]} x"\,x'\right\} \qquad (36)$$

Where $s(t)$, $\gamma(t)$ are to be obtained by solving eqs.(20) and (21), once $\omega(t)$ given.

ACKNOWLEDGEMENTS:

We are glad to thank Prof. L.C.Gomes and Prof. R.Lobo for helpful discussions.

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