HYPERBOLIC BIRKHOFF CENTERS

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ABSTRACT. The purpose of this paper is to show that if f is a diffeomorphism of a compact manifold whose Birkhoff center, $c(f)$, is hyperbolic and has no cycles, then f satisfies Axiom A and is Ω-stable. To obtain a filtration for $c(f)$, the concept of an isolated set for a homeomorphism of a compact metric space is introduced. As a partial converse it is proved that if $c(f)$ is hyperbolic and f is Ω-stable, then $c(f)$ has the no cycle property. It is also given a characterization of Ω-stability when $c(f)$ is finite.

INTRODUCTION

In this paper we consider a diffeomorphism f of a compact manifold M whose Birkhoff center is hyperbolic. We prove that if the Birkhoff center of f is hyperbolic and has no cycles, then f satisfies Axiom A and is Ω-stable.

We denote by $\Omega = \Omega(f)$ the set of non-wandering points of f. Smale's Axiom A requires:

(a) Ω has a hyperbolic structure.

(b) The periodic points are dense in Ω.

In Smale's paper about Ω-stability [9], he proved that if f satisfies Axiom A and has no cycles, then f is Ω-stable (for easier proof see [6]). Later on, Newhouse [5] showed that if the closure of the α-limit set of f, denoted by L^-, is hyperbolic and does not admit cycles, then f satisfies Axiom A and has no cycles, and therefore f is Ω-stable. In this paper we obtain the same conclusions by requiring the condition of hyperbolicity and the property of having no cycles for a subset of L^-, the Birkhoff center of

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f, denoted by $c(f)$. We define $c(f)$ to be the closure of the set $\{x \in M \mid x \in \alpha(x) \cap \omega(x)\}$, where $\alpha(x)$ and $\omega(x)$ are the α-limit and ω-limit sets of x for f. Our main result is the following theorem:

Theorem B - If $c(f)$ is hyperbolic and has no cycles, then f satisfies Axiom A and has no cycles. In particular f is Ω-stable.

We also prove, following [7], a partial converse of this result.

Theorem C - If $c(f)$ is hyperbolic and f is Ω-stable, then $c(f)$ has the no cycle property. In particular, $c(f) = \Omega(f)$.

In order to prove Theorem B, we initially consider a homeomorphism f of a compact metric space K. We introduce some concepts which generalize others used in the study of a diffeomorphism of a compact manifold. In this context, the basic concept introduced is the one of isolated set for a homeomorphism. As we show in Proposition (1.2), isolated sets have a fundamental property valid, in the case of a diffeomorphism, for hyperbolic sets with local product structure: if there exists a decomposition having no cycles, we may obtain a filtration for the elements of this decomposition. We show then that if $c(f)$ is isolated and satisfies the above hypothesis, then $c(f)$ coincides with the non-wandering set of f, $\Omega(f)$. More explicitly, we say that a compact set $\Lambda \subset K$ is isolated for f if it is invariant by f (i.e., $f(\Lambda) = \Lambda$) and there exists a neighborhood U of Λ such that Λ is the maximal invariant set for f.

in U; that is, $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. We then prove the following theorem:

Theorem A - If $c(f)$ is an isolated set for f and admits a decomposition $c(f) = \Lambda_1 \cup \ldots \cup \Lambda_\ell$ into isolated sets having no cycles, then $c(f) = \Omega(f)$.

From this theorem we prove the main result of the paper, Theorem B, as stated above. The idea of the proof is as follows. Let f be a C^r diffeomorphism, $r \geq 1$, of a compact C^∞ manifold without boundary, whose Birkhoff center $c(f)$ is hyperbolic. From Anosov's Closing Lemma we can show that $c(f)$ coincides with the closure of the set of periodic points of f, denoted by $\bar{P}$. Then using Newhouse's theorem [5] relative to the decomposition of $\bar{P}$ into basic sets, we obtain a decomposition of $c(f)$. Theorem B then follows, showing that this decomposition satisfies the hypothesis of Theorem A.

In Section 1 we establish some notation and introduce the definitions in the context of compact metric spaces. In particular, we define an isolated set for a homeomorphism of a compact metric space, which generalizes the idea of hyperbolic sets having local product structure for a diffeomorphism of a compact manifold. In analogy to the invariant manifolds of a hyperbolic set [3], we associate invariant spaces to an isolated set. We obtain, in this context, well known results for diffeomorphisms, including a filtration lemma, that we use in the proof of Theorem A.

In Section 2 we restrict our considerations to diffeomorphisms of a C^∞ compact manifold. We restate the defi-

nition of a hyperbolic set having local product structure for a diffeomorphism and recall some known results. These results enable us to show that such sets are isolated. Also the invariant manifolds coincide with the invariant spaces as defined in Section 1. We prove Theorem B making use of Anosov's Closing Lemma, Newhouse's theorem for the decomposition of $\bar{P}$ and Theorem A.

In Section 3 we prove Theorem C. For this purpose, we introduce the concept of $\bar{P}$-stability; that is, the stability of the closure of the set of periodic points of a diffeomorphism f. We prove that if $\bar{P} = \overline{Per(f)}$ is hyperbolic and f is $\bar{P}$-stable, then $\bar{P}$ has the no cycle property. This result is similar to the one known as Ω-explosion. In fact, the proof of the above result is essentially the same given by Palis in [7].

We observe that to prove our main result, we need to consider the restriction of a diffeomorphism to certain invariant subsets of the ambient manifold, which do not have in general a differentiable structure. This is the main reason to study isolated sets for homeomorphisms in the first part of the paper, where Theorem A is proved.

We also point out that Theorem B implies a positive answer to a question posed by Newhouse in [5].

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1. ISOLATED SETS

In this section we will define isolated sets for a homeomorphism of a compact metric space K, which generalize the idea of hyperbolic sets with local product structure of a diffeomorphism. In analogy to the invariant manifolds of an hyperbolic set for a diffeomorphism, we will associate two invariant spaces, called the unstable and the stable space, to each isolated set for a homeomorphism. We can therefore consider the notion of cycles in a disjoint family of isolated sets.

The basic result of this section is Proposition (1.2). From it we obtain a Filtration Lemma (1.8) and Lemma (1.9) which will be used to prove Theorem A.

Here f is a homeomorphism of a compact metric space K. We denote by P the set of periodic points of f and by Ω the set of non-wandering points of f. That is,

$$\Omega = \{x \in K \mid \text{given a neighborhood } U \subset K \text{ of } x \text{ and } n_o > 0, \text{ there exists } n \in Z \text{ with } |n| > n_o \text{ such that } f^n(U) \cap U \neq \phi\}.$$

For a subset $D \subset K$, $\bar{D}$ will denote its closure in K, and int D will denote its interior in K. For each $x \in K$, we denote by $\mathcal{O}(x)$ the orbit of x, and by $\mathcal{O}_-(x)$ and $\mathcal{O}_+(x)$ the negative and positive orbit of x, respectively. Let $x,y \in K$. We say that $y \in \alpha(x)$ if there is a sequence of integers $n_i \to \infty$ such that $f^{-n_i}(x) \to y$ as $i \to \infty$. In a similar way, we define $\omega(x)$ to be the set of $y \in K$ such that there is a sequence of integers $n_i \to \infty$ such that

$f^{n_i}(x) \to y$ as $i \to \infty$. The set $\alpha(x)$ is called the α-limit set of x and $\omega(x)$ is the ω-limit set of x. Notice that $\alpha(x)$ (or $\omega(x)$) is closed, non-empty and invariant under f, that is, $f(\alpha(x)) = \alpha(x)$, for every x in K.

Definition - A compact subset $\Lambda \subset K$ is isolated for the homeomorphism $f: K \to K$ if it is invariant under f, that is $f(\Lambda) = \Lambda$, and if there is a neighborhood U of Λ such that $\bigcap_{n\in Z} f^n(U) = \Lambda$.

In the sequal $\Lambda \subset K$ is assumed to be an isolated set for f. As in the case of hyperbolic sets for a diffeomorphism, we associate to an isolated set Λ, invariant spaces, $W^s(\Lambda)$ and $W^u(\Lambda)$, called the stable space and unstable space of Λ, respectively, defined as follows:

$$W^u(\Lambda) = \{x \in K \mid \alpha(x) \subset \Lambda\} \quad \text{and}$$

$$W^s(\Lambda) = \{x \in K \mid \omega(x) \subset \Lambda\}.$$

If U is a compact neighborhood of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$, we define $W^s_{loc}(\Lambda,U) = \bigcap_{n\geq 0} f^{-n}(U)$ and $W^u_{loc}(\Lambda,U) = \bigcap_{n\ 0} f^n(U)$. Thus we have $W^s(\Lambda) = \bigcup_{n\geq 0} f^{-n}(W^s_{loc}(\Lambda,U))$, $W^u(\Lambda) = \bigcup_{n\geq 0} f^n(W^u_{loc}(\Lambda,U))$ and $W^s_{loc}(\Lambda,U) \cap W^u_{loc}(\Lambda,U) = \Lambda$.

We consider in $W^\sigma_{loc}(\Lambda,U)$, $\sigma = s,u$, the topology induced by the metric in K.

Let U be a compact neighborhood of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. If V is a compact neighborhood of Λ in $W^s_{loc}(\Lambda,U)$ such that $f(V) \subset V$, we call the set $D = \overline{V - f(V)}$ a fundamental domain for $W^s_{loc}(\Lambda,U)$. In a similar way we

define a fundamental domain for $W^u_{loc}(\Lambda,U)$. We observe that if $x \in W^s(\Lambda)-\Lambda$, then there exists a integer k such that $f^k(x) \in D$. If $D \cap \Lambda = \phi$ we call D a proper fundamental domain.

(1.1) Proposition - If Λ is an isolated set for f, then $D^s = \overline{W^s_{loc}(\Lambda,U) - f(W^s_{loc}(\Lambda,U))}$ is a proper fundamental domain for $W^s_{loc}(\Lambda,U)$.

Proof: We need only to prove that Λ is contained in the interior of $f(W^s_{loc}(\Lambda,U))$ in $W^s_{loc}(\Lambda,U)$. In fact $f(W^s_{loc}(\Lambda,U)) = f(U) \cap W^s_{loc}(\Lambda,U)$, thus $\Lambda \subset f(\text{int } U) \cap$ $\cap W^s_{loc}(\Lambda,U) \subset f(W^s_{loc}(\Lambda,U))$ and $f(\text{int } U) \cap W^s_{loc}(\Lambda,U)$ is an open neighborhood of Λ in $W^s_{loc}(\Lambda,U)$. So we have $D^s \cap \Lambda = \phi$. //

The following fact is a basic one for our study since it enables us to obtain several results, including a filtration lemma, which we will use in the proof of Theorem A. Notice that it generalizes an analogous result for hyperbolic sets with local product structure for a diffeomorphism [4].

(1.2) Proposition - Let V be a neighborhood of D^s in K. Then $U' = W^u_{loc}(\Lambda,U) \cup \mathcal{O}_+(V)$ is a neighborhood of Λ in K, where $\mathcal{O}_+ V = \bigcup_{n\geq 0} f^n(V)$.

Proof: Let us suppose the statement is false. Then if W is a neighborhood of Λ in K, we have $W \not\subset U'$, that is, there is a sequence $x_n \in U-U'$ which converges to x in Λ. Since $x_n \notin W^u_{loc}(\Lambda,U) = \bigcap_{n\geq 0} f^n(U)$ we have $\mathcal{O}_-(x_n) \not\subset U$. For each n, let $m_n \geq 0$ be such that $f^{-k}(x_n) \in U$ for $0 \leq k \leq m_n$,

but $f^{-(m_n+1)}(x_n) \notin U$. Thus $y_n = f^{-m_n}(x_n) \in U-f(U)$. We assert that $m_n \to \infty$ as $n \to \infty$. Indeed, given $N \in Z^+$ take $M \geq 0$ such that $x_n \in \bigcap_{k\geq 0}^{N} f^k(U)$ for every $n \geq M$. Then $f^{-k}(x_n) \in U$ for $a \leq k \leq N$ and every $n \geq M$, thus $m_n > N$ for every $n \geq M$ proving the statement. We can assume that y_n converges to $y \in U$. We now claim that $y \in W^s(\Lambda)-\Lambda$. It is clear that $y \notin \Lambda$ since $\Lambda \subset \text{int } f(V)$, thus all we need to prove is $y \in W^s(\Lambda)$. But if $y \notin W^s(\Lambda)$ there exists $k > 0$ such that $f^k(y) \notin U$, so for n sufficiently large we have $f^k(y_n) = f^{-m_n+k}(x_n) \notin U$. Since $m_n \to \infty$ as $n \to \infty$, $0 < m_n-k \leq m_n$ for n sufficiently large, which gives a contradiction. Thus $y \in W^s(\Lambda)$ and we can choose $k_o \in Z$ such that $f^{k_o}(y) \in D^s$. Then $f^{k_o}(y_n) = f^{-m_n+k_o}(x_n) \in V$ when n is large and so $x_n \in U'$ contradicting our assumption. //

(1.3) Lemma - Let $\Lambda_1, \Lambda_2 \subset K$ be isolated sets for f such that $\Lambda_1 \cap \Lambda_2 = \phi$. Then

(i) $\hat{\overline{W^u(\Lambda_1)}} \cap W^s(\Lambda_2) \neq \phi \Rightarrow \hat{\overline{W^u(\Lambda_1)}} \cap \hat{W}^s(\Lambda_2) \neq \phi$

(ii) $\hat{\overline{W^u(\Lambda_1)}} \cap \hat{W}^u(\Lambda_2) \neq \phi \Rightarrow \hat{\overline{W^u(\Lambda_1)}} \cap \hat{W}^s(\Lambda_2) \neq \phi$

where

$$\hat{\overline{W^\sigma(\Lambda_i)}} = \overline{W^\sigma(\Lambda_i)} - \Lambda_i$$

$$\hat{W}^\sigma(\Lambda_i) = W^\sigma(\Lambda_i) - \Lambda_i \quad \text{for } i=1,2 \text{ and } \sigma=s,u.$$

Proof: Let U be a compact neighborhood of Λ_2 such that $\bigcap_{n\in Z} f^n(U) = \Lambda_2$, and let $D^s = D^s(\Lambda_2,U)$ be a proper fundamental domain for $W^s_{loc}(\Lambda_2,U)$.

(i) Let $x_n \in W^u(\Lambda_1)$ such that $x_n \to x \in \overline{\hat{W}^u(\Lambda_1)} \cap \Lambda_2$, and let $F = \overline{\bigcup_{n\geq 0} \mathcal{O}(x_n)}$. Then $F \subset \overline{W^u(\Lambda_1)}$ and so we need only to prove that $F \cap D^s \neq \phi$.

Given any neighborhood V of D^s, with $V \cap \Lambda_2 = \phi$, let $U' = W^u_{loc}(\Lambda_2, U) \cup \mathcal{O}_+(V)$ and $k \geq 0$ such that $x_k \in U'$. Since $x_k \notin W^u(\Lambda_2)$ (recall we have $x_k \in W^u(\Lambda_1)$) there is $n_k \geq 0$ such that $x_k \in f^{n_k}(V)$. Thus $\bigcup_{n\geq 0} \mathcal{O}(x_n) \cap V \neq \phi$ for every V, which implies $F \cap D^s \neq \phi$.

(ii) From (i) it suffices to prove that if $\overline{\hat{W}^u(\Lambda_1)} \cap \hat{W}^u(\Lambda_2) \neq \phi$. Then $\overline{\hat{W}^u(\Lambda_1)} \cap \Lambda_2 \neq \phi$. If $x \in \overline{\hat{W}^u(\Lambda_1)} \cap \hat{W}^u(\Lambda_2)$, then $\alpha(x) \subset \overline{W^u(\Lambda_1)} \cap \Lambda_2$, thus $\alpha(x) \subset \overline{\hat{W}^u(\Lambda_1)} \cap \Lambda_2$ since $\Lambda_1 \cap \Lambda_2 = \phi$. //

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell \subset K$ be disjoint isolated sets for f.

Definition - A cycle for the family $\{\Lambda_i\}_{i=1,\ldots,\ell}$ is a sequence $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ such that $\Lambda_{i_1} = \Lambda_{i_k}$ and $\hat{W}^u(\Lambda_{i_j}) \cap \cap \hat{W}^s(\Lambda_{i_{j+1}}) \neq \phi$ for $1 \leq j < k$.

Remark: The sequence $\Lambda_{i_1}, \ldots, \Lambda_{i_k} = \Lambda_{i_1}$ is a cycle if there are points $x_1, \ldots, x_k$ such that $x_i \notin \bigcup_{i=1}^{k} \Lambda_{i_j}$, $\alpha(x_i) \subset \Lambda_{i_j}$ and $\omega(x) \subset \Lambda_{i_{j+1}}$ for $1 \leq j < k$.

(1.4) Lemma - Let $\{\Lambda_1, \ldots, \Lambda_\ell\}$ be a family of disjoint isolated sets for f having no cycles such that $K \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$. Then if $\overline{\hat{W}^u(\Lambda_i)} \cap W^s(\Lambda_j) \neq \phi$ there exists a sequence $\Lambda_j = \Lambda_{j_o}, \ldots, \Lambda_{j_k} = \Lambda_i$ such that $\hat{W}^u(\Lambda_{i_\alpha}) \cap \hat{W}^s(\Lambda_{i_{\alpha-1}}) \neq \phi$ for $1 \leq \alpha \leq k$.

Proof: Lemma (1.3) implies that $\overline{\hat{W}^u(\Lambda_i)} \cap W^s(\Lambda_{j_o}) \neq \phi$. Let $x \in \overline{\hat{W}^u(\Lambda_i)} \cap \hat{W}^s(\Lambda_{j_o})$. Since $K \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$, there is $1 \leq j_1 \leq \ell$ such that $x \in W^u(\Lambda_{j_1})$. Then $x \in \hat{W}^u(\Lambda_{j_1}) \cap \cap \hat{W}^s(\Lambda_{j_o})$. But $x \in \overline{\hat{W}^u(\Lambda_i)} \cap \hat{W}^u(\Lambda_{j_1})$ and so, by Lemma (1.3) (ii), we have $\overline{\hat{W}^u(\Lambda_i)} \cap W^s(\Lambda_{j_1}) \neq \phi$. Now assume $\Lambda_{j_1},\ldots,\Lambda_{j_p}$ are defined for $p < \ell$ such that $\Lambda_{j_\alpha} \neq \Lambda_i$ for $1 \leq \alpha < p$, $\hat{W}^u(\Lambda_{j_\alpha}) \cap \hat{W}^s(\Lambda_{j_{\alpha-1}}) \neq \phi$ for $1 \leq \alpha \leq p$ and $\overline{\hat{W}^u(\Lambda_i)} \cap \cap \hat{W}^s(\Lambda_{j_p}) \neq \phi$. If $j_p \neq i$ we can obtain $\Lambda_{j_{p+1}}$ by the same process used to get Λ_{j_1}. Notice that the assumption of no cycles implies that $\Lambda_{j_\alpha} \neq \Lambda_{j_\beta}$ if $\alpha \neq \beta$. Since $1 \leq j_\alpha \leq \ell$ we must have $j_\alpha = i$ for some α which proves the lemma.

(1.5) Lemma - Let Λ be an isolated set for f and $x \in K$. If $\alpha(x) \cap \Lambda \neq \phi$ but $x \notin W^u(\Lambda)$, then

(i) $\alpha(x) \cap \hat{W}^s(\Lambda) \neq \phi$

(ii) $\alpha(x) \cap \hat{W}^u(\Lambda) \neq \phi$.

Proof: (i) Let U be a compact neighborhood of Λ such that $\bigcap_{x\in\mathbb{Z}} f^n(U) = \Lambda$, and let $D^s = D^s(\Lambda,U)$ be a proper fundamental domain for $W^s_{loc}(\Lambda,U)$. It suffices to prove that given a neighborhood V of D^s such that $V \cap \Lambda = \phi$, we have $\mathcal{O}_-(x) \cap V \neq \phi$. By Proposition (1.2) we know that $U' = W^u_{loc}(\Lambda,U) \cup \mathcal{O}_+V$ is a neighborhood of Λ. Since $\alpha(x) \cap \cap \Lambda \neq \phi$, there exists $k > 0$ such that $f^{-k}(x) \in U'$. But $x \notin W^u(\Lambda)$, thus $f^{-k}(x) \notin W^u_{loc}(\Lambda,U)$ and so $f^{-k}(x) \in f^n(V)$ for some $n \geq 0$, which implies $\mathcal{O}_-(x) \cap V \neq \phi$.

(ii) Let U be as above, such that $x \notin U$. Since $f(W^s_{loc}(\Lambda,U) \subset W^s_{loc}(\Lambda,U)$ we have $\mathcal{O}_-(x) \cap W^s_{loc}(\Lambda,U) \neq \phi$.

Let $x_k = f^{-n_k}(x) \in U$, with $n_k \to \infty$ as $k \to \infty$, such that $x_k \to z \in \Lambda$. Since $x_k \notin W^s_{loc}(\Lambda, U)$ we have $\mathcal{O}_+(x_k) \not\subset U$. For each $k > 0$, let $m_k \geq 0$ be such that $f^m(x_k) \in U$ for $0 \leq m \leq m_k$ but $f^{m_k+1}(x_k) \notin U$. Then $y_k = f^{m_k}(x_k) \in (U - f^{-1}(U))$ and $m_k - n_k < 0$, that is $y_k \in \mathcal{O}_-(x)$ for if not we would have $0 \leq n_k \leq m_k$ and so $f^{n_k}(x_k) = x \in U$, which is a contradiction. We state now that $m_k \to \infty$ as $k \to \infty$. Indeed, given $N \in Z^+$, there is $k_o > 0$ such that $x_k \in \bigcap_{n \geq 0}^{N} f^{-n}(U)$ for every $k \geq k_o$, since x_k converges to Λ. Then $f^n(x_k) \in U$ for $0 \leq n \leq N$ and so $m_k \geq N$ for every $k \geq k_o$. We can assume that y_k converges to $y \in \overline{U - f^{-1}(U)}$, and we state that $y \in \hat{W}^u(\Lambda)$. It suffices to prove that $y \in W^u(\Lambda)$, since $\Lambda \subset \text{int } f^{-1}(U)$. If $y \notin W^u(\Lambda)$, there exists $j > 0$ such that $f^{-j}(y) \notin U$. Thus, there exists $k_o > 0$ such that for every $k \geq k_o$ we have $m_k - j > 0$ and $f^{m_k - j}(x_k) = f^{-j}(y_k) \notin U$, which is a contradiction since $0 < m_k - j \leq m_k$. //

From this lemma we obtain immediately the following result.

(1.6) Corollary - If $\{\Lambda_1, \ldots, \Lambda_\ell\}$ is a family of disjoint isolated sets for f such that $\alpha(x) \subset \bigcup_{i=1}^{\ell} \Lambda_i$ for every $x \in K$, then $\alpha(x) \subset \Lambda_i$ for some $i = 1, \ldots, \ell$, that is, $K = \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$.

In the proof of the Filtration Lemma below, we will need the following lemma that appears in [9].

(1.7) Lemma - Let F be a compact f-invariant subset of K

and Q be a compact neighborhood of F such that $\bigcap_{n\geq 0} f^n(Q) = F$. Then there is a compact neighborhood V of F such that $V \subset Q$ and $f(V) \subset \text{int } V$.

(1.8) Filtration Lemma - If $\{\Lambda_1,\ldots,\Lambda_\ell\}$ is a family of disjoint isolated sets for f having no cycles such that $\alpha(x) \subset \bigcup_{i=1}^{\ell} \Lambda_i$ for every $x \in K$, then

(a) There exists a filtration for the family, that is, there are compact sets $\phi = K_o \subset K_1 \subset \ldots \subset K_\ell = K$ such that $f(K_i) \subset \text{int } K_i$ and $\bigcap_{n\in\mathbb{Z}} f^n(K_i - K_{i-1}) = \Lambda_i$.

(b) $\Omega(f) \subset \bigcup_{i=1}^{\ell} \Lambda_i$.

Proof: (a) We define the following relation on $\{\Lambda_i\}_{i=1}^{\ell}$: $\Lambda_i \leq \Lambda_j$ if there is a sequence $\Lambda_j = \Lambda_{j_o},\ldots,\Lambda_{j_k} = \Lambda_i$ such that $\hat{W}^u(\Lambda_{j_{\alpha-1}}) \cap \hat{W}^s(\Lambda_{j_\alpha}) \neq \phi$ for $1 \leq \alpha \leq k$. By the assumption of no cycles this relation is a partial ordering. It induces a total ordering $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_\ell$ of the family where $\Lambda_i < \Lambda_j$ if $\Lambda_i \leq \Lambda_j$ but $\Lambda_i \neq \Lambda_j$. We define K_j by induction. Let $K_o = \phi$ and assume we have defined compact sets $\phi = K_o \subset K_1 \subset \ldots \subset K_j$ for $j < \ell$ such that

(i) $f(K_i) \subset \text{int } K_i$

(ii) $\bigcap_{n\in\mathbb{Z}} f^n(K_i - K_{i-1}) = \Lambda_i$ for $0 \leq i \leq j$.

Let $F = \bigcup_{i\leq j+1} W^u(\Lambda_i)$. We claim that F is closed and $F \cap \Lambda_k = \phi$ for $k > j+1$. Indeed, let (x_n) be a sequence in F converging to $x \in K$. We can assume that for every $n > 0$ we have $x_n \in W^u(\Lambda_i)$ for some fixed $i \leq j+1$. From Corollary (1.6) we know that there exists $1 \leq k \leq \ell$ such

that $\alpha(x) \subset \Lambda_k$; that is, $x \in W^u(\Lambda_k)$. Then $x \in \overline{W^u(\Lambda_i)} \cap$ $\cap W^u(\Lambda_k)$. By Lemma (1.4) we conclude that $\Lambda_k \leq \Lambda_i$ and so $k \leq i \leq j+1$; that is, $x \in F$. Clearly $F \cap \Lambda_k = \phi$ for $k > j+1$.

Now let $Q \subset K$ be a compact neighborhood of F such that $Q \cap (\bigcup_{i>j+1}^{\ell} \Lambda_i) = \phi$. Thus $\bigcap_{n\geq 0} f^n(Q) = F$ for if $x \in \bigcap_{n\geq 0} f^n(Q)$, then $f^{-n}(x) \in Q$ for every $n \geq 0$, that is, $\mathcal{O}_-(x) \subset Q$, and so $\alpha(x) \subset Q$. Thus $\alpha(x) \subset \Lambda_i$ for some $i \leq j+1$, so $x \in \bigcup_{i\leq j+1} W^u(\Lambda_i) = F$. Since F is invariant by f we can use Lemma (1.7) and obtain a compact neighborhood $V_j \subset Q$ of F such that $f(V_j) \subset \text{int}(V_j)$. Let $K_{j+1} = K_j \cup V_j$. Then we have $f(K_{j+1}) \subset \text{int } K_{j+1}$ and we need only show that $\bigcap_{n\in Z} f^n(K_{j+1}-K_j) = \Lambda_{j+1}$. First we observe that $S = \bigcap_{n\in\mathbb{Z}} f^n(K_{j+1}-K_j)$ is closed for if $x_k \in S$ is a sequence converging to $x \in K$, then $f^n(x) \in K_{j+1}$ for every $n \in Z$ since K_{j+1} is compact. Suppose, to achieve a contradiction, that there exists $n \in Z$ such that $f^n(x) \in K_j$. Then $f^{n+1}(x) \in \text{int}(K_j)$ and so $f^{n+1}(x_k) \in \text{int } K_j$ for large k, which contradicts the fact that $x_k \in S$. Thus $\alpha(x) \subset S$ for every $x \in S$, so $\alpha(x) \subset \Lambda_{j+1}$ for every $x \in S$; that is, $S \subset W^u(\Lambda_{j+1})$. Similarly $\omega(x) \subset S$ for every $x \in S$. Thus using the dual of Lemma (1.5) we can conclude that $S \subset W^s(\Lambda_{j+1})$. Then $S \subset W^u(\Lambda_{j+1}) \cap W^s(\Lambda_{j+1}) = \Lambda_{j+1}$. Now we observe that by Corollary (1.6), $K = \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$ and so $K_\ell = K$.

(b) It suffices to prove that if $y \in K - \bigcup_{i=1}^{\ell} \Lambda_i$, then there exist $y' \in o(y)$ and a neighborhood V of y' such that

$f^n(V) \cap V = \phi$ for every $n > 0$. Choose Λ_j such that $\alpha(y) \subset \Lambda_j \subset (\text{int } K_j) - K_{j-1}$. Take $y_1 \in \mathcal{O}(y)$ such that $\mathcal{O}_-(y_1) \subset K_j - K_{j-1}$. Since $y_1 \notin \Lambda_j$, there exists a number $n_o > 0$ such that $y_2 = f^{n_o}(y_1) \notin K_j - K_{j-1}$. But $\mathcal{O}_+(y_1) \subset K_j$, and thus $f^{n_o}(y_1) \in K_{j-1}$. Let $m_o \geq 0$ be such that $f^{-m}(y_2) \in K_{j-1}$ for $0 \leq m \leq m_o$ but $f^{-(m_o+1)}(y_2) \notin K_{j-1}$. Then $y' = f^{-m_o}(y_2) \in K_j - f(K_{j-1})$. Take a neighborhood $W \subset \text{int } K_{j-1}$ of $f(K_{j-1})$ and a neighborhood V of y' such that $W \cap V = \phi$ and $f(V) \subset W$. Thus we have $f^n(V) \subset \subset f^{n-1}(W) \subset W$ for every $n > 0$ and so $V \cap f^n(V) = \phi$ for $n > 0$. //

(1.9) Lemma - If $\{\Lambda_1, \ldots, \Lambda_\ell\}$ is a family of disjoint isolated sets for f having no cycles, and if $x \in K$ satisfies $\alpha(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$ (or $\omega(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$), then $x \in \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$.

Proof: First we will show that if $\alpha(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$ but $x \notin \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$, then we can get a cycle. Choose Λ_{i_1}, $1 \leq i_1 \leq \ell$ such that $\alpha(x) \cap \Lambda_{i_1} \neq \phi$. Since $\alpha(x) \subset \Lambda_{i_1}$, from Lemma (1.5)(i), we have $\alpha(x) \cap \hat{W}^s(\Lambda_{i_1}) \neq \phi$. But by assumption $\alpha(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$, so there exists $1 \leq i_2 \leq \ell$ such that $\hat{W}^u(\Lambda_{i_2}) \cap \hat{W}^s(\Lambda_{i_1}) \neq \phi$, and $\alpha(x) \cap \Lambda_{i_2} \neq \phi$. If $i_2 = i_1$ we have a cycle. If not, we repeat the process and obtain $1 \leq i_3 \leq \ell$ such that $\hat{W}^u(\Lambda_{i_3}) \cap \hat{W}^s(\Lambda_{i_2}) \neq \phi$ and $\alpha(x) \cap \Lambda_{i_3} \neq \phi$. Continuing as above, since there are only finitely many sets Λ_i, we eventually get a cycle. Similarly we prove the statement when $\omega(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$ using the dual statement in Lemma (1.5) (ii). //

(1.10) Lemma - Let $x \in K$. If $y \in \alpha(x)$, then given $\epsilon > 0$ there is a sequence $y = y_0, y_1, \ldots, y_k = y$ in $\alpha(x)$ such that $d(f(y_{j-1}), y_j) < \epsilon$ for $1 \leq j \leq k$.

Proof: Given $\epsilon > 0$, let $0 < \delta < \epsilon/2$ such that if $p,q \in K$ and $d(p,q) < \delta$, then $d(f(p),f(q)) < \epsilon/2$. Since $f^{-n}(x)$ converges to $\alpha(x)$ as $n \to \infty$, there is $n_0 > 0$ such that $d(f^{-n}(x),\alpha(x)) < \delta$, $\forall\, n \geq n_0$. On the other hand, since $y \in \alpha(x)$ there are n_1, $k \geq 0$ such that $n_1 - k \geq n_0$,

$$d(f^{-n_1}(x),y) < \delta$$

and

$$d(f^{-n_1+k}(x),y) < \delta.$$

Let $x' = f^{-n_1}(x)$. Since $n_1 - k \geq n_0$, if $1 \leq j \leq k$ then

$$d(f^j(x'),\alpha(x)) = d(f^{-n_1+j}(x),\alpha(x)) < \delta.$$

Thus there exists $y_j \in \alpha(x)$ such that

$$d(f^j(x'),y_j) = d(f^j(x'),\alpha(x)) < \delta,$$

so

$$d(f^{j+1}(x'),f(y_j)) < \epsilon/2.$$

Then putting $y_0 = y_k = y$ we get, for $1 \leq j \leq k$,

$$d(f(y_{j-1}),y_j) \leq d(f(y_{j-1}),f^j(x')) + d(f^j(x'),y_j) < \epsilon/2+\delta < \epsilon$$

proving the lemma. //

(1.11) Lemma - Let $(\Sigma_n)_{n\in Z^+}$ be a family of compact subsets of K, invariant by f such that if $n \geq m$ then $\Sigma_n \subset \Sigma_m$. For each $n \in Z^+$, let $x_n \in \Sigma_n$ and $y_n \in \alpha(x_n)$ be such that y_n converges to $y \in \Sigma = \bigcap_{n\in Z^+} \Sigma_n$. Then, given $\epsilon > 0$, there

is a sequence $y = y_0, y_1, \ldots, y_k = y$ in Σ such that $d(f(y_{j-1}), y_j) < \epsilon$ for $1 \leq j \leq k$.

Proof: Given $\epsilon > 0$, choose $0 < \delta < \epsilon/3$ such that if $p, q \in K$ and $d(p,q) < \delta$, then $d(f(p), f(q)) < \epsilon/3$. Taken $n_o \in \mathbb{Z}^+$ such that if $x \in \Sigma_{n_o}$, then there exists $y' \in \Sigma$ such that $d(x, y') < \delta$. Take $n > 0$ such that $y_n \in \alpha(x_n) \subset \subset \Sigma_{n_o}$ and $d(y, y_n) < \delta$. From Lemma (1.10) we know that there is a sequence $y_n = z_0, z_1, \ldots, z_k = y_n$ in $\alpha(x_n) \subset \subset \Sigma_{n_o}$ such that $d(f(z_{j-1}), z_j) < \epsilon/3$ for $1 \leq j \leq k$. Let $y_o = y_k = y$ and $y_j \in \Sigma$ be such that $d(y_j, z_j) < \delta$ for $1 \leq j \leq k$. Then $d(f(y_{j-1}), y_j) \leq d(f(y_{j-1}), f(z_{j-1})) + d(f(z_{j-1}), z_j) + d(z_j, y_j) < \epsilon$ for $1 \leq j \leq k$, which proves the lemma. //

Now we may proceed to the main theorem of this section. First we give a definition.

Definition - The Birkhoff center of f, $c(f)$, is the closure of the set of points in K whose orbits are recurrent in the past as well as in the future. That is, $c(f)$ is the closure of the set $\{x \in K \mid x \in \alpha(x) \cap \omega(x)\}$.

Theorem A - If $c(f)$ is an isolated set for f and admits a decomposition $c(f) = \Lambda_1 \cup \ldots \cup \Lambda_\ell$ into disjoint isolated sets having no cycles, then $c(f) = \Omega(f)$.

Proof: From Lemma (1.8) (filtration), it suffices to prove that $\alpha(x) \subset c(f)$ for every $x \in K$. Suppose, on the contrary, that there exists $x \in K$ such that $\alpha(x) \not\subset c(f)$. Let $\mathfrak{F}$ be the family of compact f-invariant subsets Σ of K, such that

there is an $x \in \Sigma$ with $\alpha(x) \not\subset c(f) \cap \Sigma$. We order $\mathfrak{F}$ by inclusion. Now suppose for the moment that $\mathfrak{F}$ possesses a minimal Σ. Then take $x \in \Sigma$ such that $\alpha(x) \not\subset c(f) \cap \Sigma$. Since $x \notin \alpha(x) \cap \omega(x)$, we have that either $\alpha(x)$ or $\omega(x)$ is a proper subset of Σ. Say $\alpha(x) \subsetneq \Sigma$. Since $\alpha(x)$ is a compact f-invariant set and Σ is a minimal, we conclude that $\alpha(x)$ is not in $\mathfrak{F}$, that is, $\alpha(x) \subset \bigcup_{i=1}^{\ell} W^u(\Lambda_i)$. Thus from Lemma (1.9) we have $\alpha(x) \subset c(f)$, which is a contradiction. A similar contradiction follows if $\omega(x) \subsetneq \Sigma$. Thus, to prove the theorem, it suffices to prove the following lemma:

Lemma - $\mathfrak{F}$ possesses a minimal Σ.

Proof: Let $\tilde{\mathfrak{F}} = \{\Sigma_\beta, \beta \in \mathcal{B}\}$ be any sub-family of $\mathfrak{F}$ totally ordered by inclusion. Let $\Sigma = \bigcap_{\beta \in \mathcal{B}} \Sigma_\beta$. We need to prove that $\Sigma \in \mathfrak{F}$. If $\Sigma \notin \mathfrak{F}$, then $\alpha(x) \subset c(f) \cap \Sigma$ for every $x \in \Sigma$. Thus by Lemma (1.8) the family $\{\Lambda_1 \cap \Sigma, \ldots, \Lambda_\ell \cap \Sigma\}$ admits a filtration and $\Omega(f/\Sigma) \subset \bigcup_{i=1}^{\ell} \Lambda_i \cap \Sigma$. For each $\beta \in \mathcal{B}$, take $x_\beta \in \Sigma_\beta$ such that $\alpha(x_\beta) \not\subset c(f) \cap \Sigma_\beta$. We know that $\alpha(x_\beta) \cap (\bigcup_{i=1}^{\ell} \Lambda_i) \neq \phi$ for every $\beta \in \mathcal{B}$. Thus we can suppose, without lost of generality, that $\alpha(x_\beta) \cap \Lambda_1 \neq \phi$ for every $\beta \in \mathcal{B}$. Let U be a compact neighborhood of Λ_1 such that $\bigcap_{n \in Z} f^n(U) = \Lambda_1$ and $U \cap (\bigcup_{i=2}^{\ell} \Lambda_i) = \phi$. Since $\alpha(x_\beta) \cap \Lambda_1 \neq \phi$ but $\alpha(x_\beta) \not\subset \Lambda_1$, we conclude from Lemma (1.5) that $\alpha(x_\beta) \cap D^s \neq \phi$, where $D^s = D^s(\Lambda_1, U) \subset U$ is a proper fundamental domain for $W^s_{loc}(\Lambda_1)$. For each $\beta \in \mathcal{B}$, choose $y_\beta \in \alpha(x_\beta) \cap D^s$. Let us take a subsequence y_{β_n} such that y_{β_n} converges to $y \in \Sigma \cap D^s$. Since $y \notin c(f) \cap \Sigma$ (recall

that $y \in D^s$ and $D^s \cap c(f) \neq \phi$) and Σ admits a filtration, there is a compact set $K' \subset \Sigma$ and a point $y' \in \mathcal{O}(y)$ such that $f(K') \subset \text{int } K'$ and $y' \in K' - f(K')$. Let $W \subset K'$ be an open set in Σ such that $y' \notin W$ and $f(K') \subset W$. Since $f(K')$ is compact, there is an $\varepsilon > 0$ such that if $z \in \Sigma$ and $d(z,z') < \varepsilon$ for some $z' \in f(K')$, then $z \in W$. From Lemma (1.11) we know that there exists a sequence $y' = y_0, y_1, \ldots, y_k = y'$ in Σ such that $d(f(y_{j-1}), y_j) < \varepsilon$ for $1 \leq j \leq k$. We claim that $y_j \in W$ for $1 \leq j \leq k$. Indeed, since $f(y_o) = f(y') \in f(K')$ and $d(f(y_o), y_1) < \varepsilon$ we conclude that $y_1 \in W \subset K'$. Let us suppose that we have proved that $y_j \in W$ for $1 \leq j \leq k$. Then $f(y_j) \in f(K')$ and $d(f(y_j), y_{j+1}) < \varepsilon$, so $y_{j+1} \in W$. Thus $y' \in f(K')$, which is a contradiction. Thus $\Sigma \in \mathfrak{F}$, proving the lemma.

We have concluded the proof of Theorem A. We will use it in the next section to prove our main result, Theorem B.

2. HYPERBOLIC BIRKHOFF CENTERS

In this section we will consider a diffeomorphism of a compact manifold M. We will make use of the notation already introduced in Section 1 together with the concepts of hyperbolic sets and their invariant manifolds. With the help of Theorem A and well known results, we will prove Theorem B. We observe that from [4], hyperbolic sets with local product structure are isolated in the sense of Section 1, which enable us to make use of Theorem A in this context.

Throughout f is assumed to be a C^r-diffeomorphism, $r \geq 1$, of a compact C^∞ manifold without boundary. A compact set $\Lambda \subset M$ is hyperbolic for f if Λ is invariant under f, that is, $f(\Lambda) = \Lambda$, and Tf leaves invariant a continuous splitting $T_\Lambda M = E^s + E^u$ expanding E^u and contracting E^s. That is

$$|Tf(v)| \leq \tau |v| \quad \text{if} \quad v \in E^s,$$

$$|Tf(v)| \leq \tau^{-1} |v| \quad \text{if} \quad v \in E^u,$$

for some constant τ, $0 < \tau < 1$, and some Riemannian metric on M. This Riemannian metric is said to be adapted to Λ.

Through each point p of Λ pass two smooth invariant manifolds, $W^s(p)$ and $W^u(p)$, tangent to E^s_p and E^u_p, respectively. These two invariant manifolds are called the stable manifold and the unstable manifold of p, respectively [3]. Let us define

$$W^s_\varepsilon(p) = \{y \in M \mid d(f^n(p), f^n(y)) \leq \varepsilon \quad \text{for} \quad n \geq 0\}$$

$$W^u_\varepsilon(p) = \{y \in M \mid d(f^{-n}(p), f^{-n}(y)) \leq \varepsilon \quad \text{for} \quad n \geq 0\}.$$

Then, $W^s(p) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(p))$ and $W^u(p) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon))$, where d is the topological metric on M induced by a Riemannian metric adapted to Λ. Recall that $W^s_\varepsilon(p)$ (respectively $W^u_\varepsilon(p)$) is an imbedded smooth manifold called the local stable (respectively unstable) manifold of "size ε" of p. The union $W^u(\Lambda) = \bigcup_{p \in \Lambda} W^u(p)$ is called the unstable manifold of Λ, and similarly $W^s(\Lambda) = \bigcup_{p \in \Lambda} W^s(p)$ is called the stable manifold of Λ.

Definition - A hyperbolic set Λ has a local product structure if, for some $\epsilon > 0$,

$$W^u_\epsilon(p) \cap W^s_\epsilon(q) \subset \Lambda$$

for all $p,q \in \Lambda$.

According to results in [4], if Λ is a hyperbolic set having a local product structure, then Λ is an isolated set for f in the sense of Section 1, and $W^u(\Lambda) = \{x \in M \mid \alpha(x) \subset \Lambda\}$. That is, the unstable manifold of Λ coincides with the unstable space associated to the isolated set Λ as in Section 1. Similarly for $W^s(\Lambda)$.

We denote by $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphism of M with the uniform C^r topology, $r \geq 1$. We call $f \in \mathrm{Diff}^r(M)$ Ω-stable if there exists a neighborhood N of f in $\mathrm{Diff}^r(M)$ such that for each $g \in N$, $g|\Omega(g)$ is topologically conjugate to $f|\Omega(f)$.

Definition - We say that f satisfies Axiom A if

(a) $\Omega = \Omega(f)$ has a hyperbolic structure,

(b) the set of periodic points is dense in Ω.

In [5], Newhouse proved that if $\bar{P}$ is hyperbolic, then $\bar{P}$ has a decomposition, $\bar{P} = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_\ell$ into disjoint hyperbolic sets having local product structure. From the observation above, $\{\Lambda_1, \ldots, \Lambda_\ell\}$ is a disjoint family of isolated sets.

Next we present a lemma which we will use in the proof of Theorem B. This lemma, first communicated by Anosov, can be proved using a stronger result showed by Bowen [1]. Newhouse

gives, in [5], a proof for the case $\Lambda = L^-$.

(2.1) Anosov's Closing Lemma - If Λ is a hyperbolic set for f, then $\bar{P} \supset \Omega(f|\Lambda)$.

We can now prove Theorem B.

Theorem B - If $c(f)$ is hyperbolic and has no cycles, then f satisfies Axiom A and has no cycles. In particular, f is Ω-stable.

Proof: Since $\bar{P} \subset c(f)$, $\bar{P}$ is hyperbolic. Thus $\bar{P}$ has a decomposition into disjoint isolates sets. So, by Theorem A, it suffices, to prove that $c(f) = \bar{P}$. But this fact is a direct consequence of Anosov's Closing Lemma and the fact that $c(f) = \Omega(f|c(f))$. In particular, from [9], f is Ω-stable. //

We recall that f is called a Kupka-Smale diffeomorphism if f satisfies:

(a) the periodic points of f are hyperbolic,

(b) the transversality condition.

If in addition $\Omega(f)$ is finite, f is said to be a Morse-Smale diffeomorphism.

As an immediate consequence of Theorem B, we have the following result.

(2.2) Corollary - If f is a Kupka-Smale diffeomorphism, and $c(f)$ is finite, then f is in fact a Morse-Smale diffeomorphism.

We observe that Theorem B is not true if we replace

$c(f)$ by the center of minimal sets; i.e., the closure of the minimal sets of f. An example is the time one map of Cherry's flow [2]. In this case the center of minimal sets is composed by the two fixed points of the diffeomorphism, which are hyperbolic without cycles, but does not coincides with the non-wandering set. In particular, this diffeomorphism is not Ω-stable.

3. A PARTIAL CONVERSE

In this section we prove Theorem C, which is a partial converse to our main result. We also give a characterization of Ω-stability when $c(f)$ is finite.

We first introduce the concept of $\bar{P}$-stability. Following the notation stablished in the preceding sections, $\bar{P} = \overline{Per(f)}$ will denote the closure of the set of periodic points of f.

We say that $f \in Diff^r(M)$, $r \geq 1$, is $\bar{P}$-stable if there exists a neighborhood N of f in $Diff^r(M)$ such that for each $g \in N$, $g|\overline{Per(g)}$ is topologically conjugate to $f|\overline{Per(f)}$.

Definition - Let $f \in Diff^r(M)$ with $\bar{P} = \overline{Per(f)}$ hyperbolic. An n-cycle $(n \geq 0)$ on $\bar{P}$ is a sequence of basic sets $\Lambda_0,\ldots,\Lambda_n,\Lambda_{n+1}$, such that $\hat{W}^u(\Lambda_i) \cap \hat{W}^s(\Lambda_{i+1}) = \emptyset$, $\Lambda_0 = \Lambda_{n+1}$ and otherwise $\Lambda_j \neq \Lambda_k$ for $j \neq k$. Here a basic set Λ is an hyperbolic set with local product structure such that $f|\Lambda$ is topologically transitive and the periodic points are dense in Λ.

(3.1) Theorem - If f is $\bar{P}$-stable and $\bar{P} = Per(f)$ is hyperbolic, then the decomposition of $\bar{P}$ into basic sets has the no cycle property.

Proof: We first prove the following assertion: if f is $\bar{P}$-stable and there is an n-cycle on $\bar{P}$, then there exists g near f in $Diff^r(M)$ such that $\overline{Per(g)}$ is hyperbolic for g and has an (n-1)-cycle.

From the stability of hyperbolic sets, and the $\bar{P}$-stability of f, we have that there exists a neighborhood V of $\bar{P}$ such that for g near f in $Diff^r(M)$, $\overline{Per(g)}$ is contained in V and is hyperbolic for g [3].

Let $\Lambda_0,\ldots,\Lambda_n,\Lambda_{n+1}$ be an n-cycle on $\bar{P} = \overline{Per(f)}$. From $W^s(\Lambda_i) = \bigcup_{x\in\Lambda_i} W^s(x)$ and $W^u(\Lambda_i) = \bigcup_{x\in\Lambda_i} W^u(x)$, we have $W^u(x_i) \cap W^s(y_{i+1}) \neq \phi$ for some $x_i, y_i \in \Lambda_i$ and $0 \leq i \leq n$. We claim that at least for one of the indices j, $\dim W^u(x_j) + \dim W^s(y_{j+1}) \geq \dim M$. For otherwise, if $r = \sum_{i=0}^{n} (\dim W^u(x_i) + \dim W^s(y_{i+1}))$ then $r < n \dim M$. But since $\Lambda_{n+1} = \Lambda_0$ and $\dim W^u(x_i) + \dim W^s(y_i) = \dim M$ we have $r = n \dim M$, thus reading a contradiction. Let $y \in W^u(x_j) \cap W^s(y_{j+1})$, the index j chosen as above. Clearly $y \notin \bar{P}$. Thus after a small perturbation of f near y we get f_1 near f in $Diff^r(M)$ so that, from the above observations, $\overline{Per(f_1)} = \bar{P}$, $f_1|\bar{P} = f|\bar{P}$, $W^u(x_j,f_1)$ and $W^s(y_{j+1},f_1)$ have a point of transversal intersection, and $W^u(x_i,f_1) \cap W^s(y_{i+1},f_1) \neq \phi$ for $0 \leq i \leq n$. We may assume that x_j, y_{j+1} are periodic points of f (and thus of f_1). Now, $f_1|_{\Lambda_{j+1}} = f|_{\Lambda_{j+1}}$ is topologically transitive and so we may further

assume $y_{j+1} = x_{j+1}$. Let $z \in W^u(x_{j+1},f_1) \cap W^s(y_{j+2},f_1)$. Since we have assumed that f is $\bar{P}$-stable, we have that $z \notin \overline{Per(f_1)} = \bar{P}$. Then, since closure $W^u(x_1,f_1) \supset W^u(x_{j+1},f_1)$ [8], by a small local perturbation of f_1 near z we get g near f in $Diff^r(M)$ with $Per(g) = \bar{P}$, $g|\bar{P} = f|\bar{P}$ and $W^u(x_j,g) \cap W^s(y_{j+2},g) \neq \phi$, thus producing an $(n-1)$-cycle on $\bar{P} = \overline{Per(g)}$ and hence proving our assertion.

The proof of the theorem goes now by induction on the length n of the cycle and the fact that if $\bar{P}$ has a zero-cycle then f is not $\bar{P}$-stable. For as above after a small local perturbation of f, say to f_1, $W^u(x_o,f_1)$ and $W^s(x_o,f_1)$ has a point of transversal intersection outside $\bar{P}$, that is a transversal homoclinic point. From [10], this point is acummulated by periodic points, and so f and f_1 are not $\bar{P}$-conjugate and thus f is not $\bar{P}$-stable. This finish the proof of the theorem.

From Lemma (2.1), the above theorem and Theorem B we get

Theorem C - If $c(f)$ is hyperbolic and f is Ω-stable, then $c(f)$ has the no cycle property. In particular, $c(f) = \Omega(f)$.

As a consequence of this theorem and Theorem B we get

(3.2) Corollary - Let $f \in Diff^r(M)$ and $c(f)$ be finite. Then f is Ω-stable iff $\Omega(f) = c(f)$ is hyperbolic and has the no cycle property.

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