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STATIC, AXIALLY SYMMETRIC VACUUM FIELDS

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ABSTRACT

A method to obtain a class of exact Weyl solutions to the vacuum field is presented, in which the gravitational potentials are obtained through integrations of the gravitational sources along the axis of symmetry. An exact solution is obtained, with a five-fold infinity of independent parameters. These parameters have simple physical interpretation, and can be adjusted to promptly give approximate solutions to more intricate gravitational situations.

1. INTRODUCTION

In a pioneer work, Weyl[1] showed that static spacetimes with rotational symmetry about z-axis are conveniently described by

$$ds^2 = e^{2N}\, dt^2 - e^{-2N} \left[e^{2K}(dz^2 + d\rho^2) + \rho^2\, d\phi^2\right] \quad , \tag{1}$$

with N and K both functions of z and ρ. The independent Einstein's vacuum field equations are then

$$N_{,zz} + N_{,\rho\rho} + \rho^{-1} N_{,\rho} = 0 \quad , \tag{2}$$

$$K_{,z} = 2\rho\, N_{,z} N_{,\rho} \quad , \qquad K_{,\rho} = \rho(N^2_{,\rho} - N^2_{,z}) \quad , \tag{3}$$

where a subscripted comma means derivative. A class of solutions to (2) is

$$N(z,\rho) = -\int_{-\infty}^{+\infty} \sigma(\zeta)\, r_\zeta^{-1}\, d\zeta \quad , \quad r_\zeta = \left[\rho^2+(z-\zeta)^2\right]^{1/2} \quad . \tag{4}$$

The arbitrary function $\sigma(\zeta)$ is interpreted, in Newtonian mechanics, as linear density of matter along z-axis. If one substitutes (4) into (3), one obtains equations which give the z- and ρ-derivative of $K(z,\rho)$ in terms of $\sigma(\zeta)$. However, one finds difficulty in integrating these partial differential equations for arbitrary $\sigma(\zeta)$. It is the main purpose of this paper to present an expression for $K(z,\rho)$ which plays the same role as (4) does for $N(z,\rho)$.

2. CLASS OF SOLUTIONS

One starts from Curzon's[2] three-parametric solution

$$N(z,\rho) = -m_A r_A^{-1} - m_B r_B^{-1} \quad , \tag{5}$$

$$K(z,\rho) = -\frac{1}{2}\rho^2(m_A^2\, r_A^{-4} + m_B^2\, r_B^{-4}) - 2m_A m_B(2a)^{-2}\{1-\left[\rho^2+(z-a)(z+a)\right] r_A^{-1}\, r_B^{-1}\} \quad , \tag{6}$$

$$r_A = \left[\rho^2 + (z-a)^2\right]^{1/2} \quad , \quad r_B = \left[\rho^2+(z+a)^2\right]^{1/2} \quad . \tag{7}$$

This solution corresponds to the field of a gravitational singularity[3] m_A at $(a,0)$ and another m_B at $(-a,0)$.

Generalization for an arbitrary number of singularities m_i at positions ζ_i along z-axis is straightforward, since the parameters m_i are independent :

$$N(z,\rho) = -\sum_i m_i\, r_i^{-1} \quad , \quad r_i = (\rho^2+z_i^2)^{1/2} \quad , \quad z_i = z-\zeta_i \quad , \tag{8}$$

$$K(z,\rho) = -\frac{1}{2}\rho^2 \sum_i m_i^2 r_i^{-4}$$

$$- 2 \sum_{i<j} \sum_j m_i m_j (\zeta_i - \zeta_j)^{-2} \left[1-(\rho^2+z_i z_j) r_i^{-1} r_j^{-1}\right] . \quad (9)$$

One conveniently reexpresses (9) in the more compact form

$$K(z,\rho) = -\rho^2 \sum_i m_i r_i^{-1} \sum_j m_j r_j^{-1} (\rho^2+z_i z_j+r_i r_j)^{-1} , \quad (10)$$

so that a further generalization becomes trivial, for continuous source distribution along z-axis:

$$N(z,\rho) = -\int \sigma(\zeta) r_\zeta^{-1} d\zeta , \quad r_\zeta = (\rho^2+z_\zeta^2)^{1/2} , \quad z_\zeta = z-\zeta , \quad (11)$$

$$K(z,\rho) = -\rho^2 \iint \sigma(\zeta)\sigma(\zeta') r_\zeta^{-1} r_{\zeta'}^{-1} (\rho^2-z_\zeta z_{\zeta'}+r_\zeta r_{\zeta'})^{-1} d\zeta d\zeta' , \quad (12)$$

all integrations ranging from $-\infty$ to $+\infty$.

A number of expressions equivalent to (11) and (12) may be obtained, one of the most convenient being

$$N(z,\rho) = -\int \frac{\sigma(\zeta)}{\sin\theta_\zeta} d\theta_\zeta , \quad \cos\theta_\zeta = z_\zeta / r_\zeta , \quad (13)$$

$$K(z,\rho) = -\iint \frac{\sigma(\zeta)\sigma(\zeta')}{1+\cos(\theta_\zeta-\theta_{\zeta'})} d\theta_\zeta d\theta_{\zeta'} ; \quad (14)$$

the integrations now range from 0 to π .

3. EXACT SOLUTIONS

A few exact solutions to these equations is found in the literature. For $\sigma = \sigma_0$ = const along all z-axis, one finds the cylindrically symmetric field of Levi-Civita[4]. The Curzon monopole solution[5,7] is obtained when $\sigma(\zeta) = m\delta(\zeta)$, where δ is the Dirac delta distribution. For $\sigma(\zeta) = \frac{1}{2}\theta(m-\zeta)\theta(\zeta-m)$ one finds, after a coordinate transformation, the spherically symmetric Schwarzschild field[1,5,6,7]; θ is a step distribution, with values 0 and 1 for negative and positive values of the argument, respectively. The two-parametric "rod" field[3,6] arises from $\sigma(\zeta) = \sigma_0\theta(a-\zeta)\theta(\zeta+a)$, σ_0 = const. Finally, for $\sigma(\zeta) = \sum_i m_i\delta(\zeta-\zeta_i)$ one obtains the solution (8), (9), which generalizes Curzon's[2] field.

All these solutions can be generated by

$$\sigma(\zeta) = \sum_i \left[m_i\delta(\zeta-\zeta_i) + \sigma_i\theta(\zeta_i^*-\zeta)\theta(\zeta-\zeta_{*i})\right] \quad , \sigma_i = \text{const} \quad , \tag{15}$$

which corresponds to an arbitrary number of singularities m_i, each at position ζ_i , and to an arbitrary number of uniform distributions σ_i, each between positions ζ_i^* and $\zeta_{*i} < \zeta_i^*$. For this $\sigma(\zeta)$ one obtains

$$N(z,\rho) = -\sum_i \left[M_i \csc 2\alpha_i + \sigma_i \log\left(\frac{\tan\alpha_i^*}{\tan\alpha_{*i}}\right)\right] \quad , \tag{16}$$

$$K(z,\rho) = -\sum_{i,j}\left\{ \frac{1}{2} M_iM_j\sec^2(\alpha_i-\alpha_j)+2M_i\sigma_j\left[\tan(\alpha_i-\alpha_j^*)-\tan(\alpha_i-\alpha_{*j})\right]\right.$$

$$\left. +2\sigma_i\sigma_j \log\left[\frac{\cos(\alpha_i^*-\alpha_{*j})\cos(\alpha_{*i}-\alpha_j^*)}{\cos(\alpha_i^*-\alpha_j^*)\ \cos(\alpha_{*i}-\alpha_{*j})}\right] \right\} \quad , \tag{17}$$

$$M_i = m_i\rho\left[\rho^2+(z-\zeta_i)^2\right]^{-1} \quad , \quad \alpha_i(\zeta_i) = \frac{1}{2}\cot^{-1}\left[\rho^{-1}(z-\zeta_i)\right] \quad , \tag{18}$$

with analogous expressions for $\alpha_i^*(\zeta_i^*)$ and $\alpha_{*i}(\zeta_{*i})$.

4. APPROXIMATE SOLUTIONS

The solution (16)-(18) is also valid when the σ_i's superpose. A method is then suggested, to obtain approximate solu tions in cases where $\sigma(\zeta)$ makes difficult the exact integrations of (13) and (14). One decomposes the exact distribution $\sigma(\zeta)$ into any apropriate combination of uniform distributions σ_i and singularities m_i, in the form (15); an approximate solution is then readily obtained by mere substitution of the values of the i,j-indexed parameters into eqs. (16)-(18).

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