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Cylindrical Collapse and Gravitational Waves

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Abstract

We study the matching conditions for a collapsing anisotropic cylindrical perfect fluid, and we show that its radial pressure is non zero on the surface of the cylinder and proportional to the time dependent part of the field produced by the collapsing fluid. This result resembles the one that arises for the radiation - though non-gravitational - in the spherically symmetric collapsing dissipative fluid, in the diffusion approximation.

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1 Introduction

Spherical gravitational collapse of a dissipative fluid produces outgoing radiation which can be modeled with Vaidya spacetime [1]. By using Darmois matching conditions [2], it can be proved that if dissipation within the sphere is described in the diffusion approximation with heat flux, then the pressure on the surface of the collapsing sphere is non zero due to the continuity of the radial flux of momentum [1] (see also [3, 4] and references therein). Indeed, the fact that heat flux does not contribute to the fluid radial pressure within the sphere implies that the pressure does not vanish on the inner part of the boundary surface, because there is radiation pressure on the outer part of that surface. Of course, in the streaming out approximation (i.e. when dissipation within the sphere is described by a radially outgoing null fluid), the fluid radial pressure is continuous across the boundary surface, provided the energy density of such null fluid is continuous too.

It is generally accepted that gravitational waves carry energy, so a source radiating them should lose mass [5, 6, 7]. On the other hand such radiation does not contribute to fluid pressure of the source. Then if one matches a cylindrical non-dissipative fluid to an exterior containing gravitational waves, one may expect that gravitational radiation will exert a non zero pressure on its collapsing surface, like null radiation induces a non-vanishing radial pressure on the boundary surface of a dissipating sphere in the diffusion approximation.

To analyze this problem we studied the collapse of a cylindrical anisotropic perfect fluid source by also using the Darmois matching conditions. For the exterior we assumed the cylindrical vacuum spacetime in Einstein-Rosen coordinates [8].

2 Collapsing perfect fluid cylinder

We consider a collapsing cylinder filled with anisotropic non-dissipative fluid bounded by a cylindrical surface Σ and with energy momentum tensor given by

$$T^{-}_{\alpha\beta} = (\mu + P_r)V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_\phi - P_r)K_\alpha K_\beta + (P_z - P_r)S_\alpha S_\beta, \tag{1}$$

where μ is the energy density, P_r, P_z and P_ϕ are the principal stresses and V_α, K_α and S_α are vectors satisfying

$$V^\alpha V_\alpha = -1, \;\; K^\alpha K_\alpha = S^\alpha S_\alpha = 1, \;\; V^\alpha K_\alpha = V^\alpha S_\alpha = K^\alpha S_\alpha = 0. \tag{2}$$

We assume the general time dependent cylindrically symmetric metric

$$ds^2_- = -A^2(dt^2 - dr^2) + B^2 dz^2 + C^2 d\phi^2, \tag{3}$$

where A, B and C are functions of t and r. To represent cylindrical symmetry, we impose the following ranges on the coordinates

$$-\infty \leq t \leq \infty, \;\; 0 \leq r, \;\; -\infty < z < \infty, \;\; 0 \leq \phi \leq 2\pi. \tag{4}$$

We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = z$ and $x^3 = \phi$ and we choose the fluid to be comoving in this coordinate system, hence from (2) and (3)

$$V_\alpha = -A\delta^0_\alpha, \;\; K_\alpha = C\delta^3_\alpha, \;\; S_\alpha = B\delta^2_\alpha. \tag{5}$$

The Einstein field equations, $G_{\alpha\beta} = \kappa T_{\alpha\beta}$ for (1), (3) and (5) reduce to five non zero components, but we shall need only the following ones

$$G^{-}_{11} = -\frac{B_{,tt}}{B} - \frac{C_{,tt}}{C} + \frac{A_{,t}}{A}\frac{B_{,t}}{B} + \frac{A_{,t}}{A}\frac{C_{,t}}{C} - \frac{B_{,t}}{B}\frac{C_{,t}}{C} + \frac{A_{,r}}{A}\frac{B_{,r}}{B} + \frac{A_{,r}}{A}\frac{C_{,r}}{C} + \frac{B_{,r}}{B}\frac{C_{,r}}{C} = \kappa P_r A^2, \tag{6}$$

$$G^{-}_{01} = -\frac{B_{,tr}}{B} + \frac{B_{,t}}{B}\frac{A_{,r}}{A} + \frac{A_{,t}}{A}\frac{B_{,r}}{B} - \frac{C_{,tr}}{C} + \frac{C_{,t}}{C}\frac{A_{,r}}{A} + \frac{A_{,t}}{A}\frac{C_{,r}}{C} = 0. \tag{7}$$

For the exterior vacuum spacetime of the cylindrical surface Σ we take the metric in Einstein-Rosen coordinates [8],

$$ds^2_{+} = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dz^2 + e^{-2\psi}R^2 d\phi^2, \tag{8}$$

where γ and ψ are functions of T and R and for the field equations $R_{\alpha\beta} = 0$ we have the gravitational wave field

$$\psi_{,TT} - \psi_{,RR} - \frac{\psi_{,R}}{R} = 0, \tag{9}$$

and

$$\gamma_{,T} = 2R\psi_{,T}\psi_{,R}, \quad \gamma_{,R} = R(\psi^2_{,T} + \psi^2_{,R}). \tag{10}$$

3 Junction conditions for the collapsing perfect fluid cylinder

We take the Darmois junction conditions [2, 9], so we suppose that the first fundamental form which Σ inherits from the interior metric (3) must be the same as the one it inherits from the exterior metric (8); and similarly, the inherited second fundamental form must be the same. The conditions are necessary and sufficient for a smooth matching without a surface layer.

The equations of Σ may be written

$$f_{-} = r - r_{\Sigma} = 0, \tag{11}$$

$$f_{+} = R - R_{\Sigma}(T) = 0, \tag{12}$$

where f_{-} refers to the spacetime interior of Σ and f_{+} to the spacetime exterior, and r_{Σ} is a constant because Σ is a comoving surface forming the boundary of the fluid. To apply the junction conditions we must arrange that Σ has the same parametrisation whether it is considered as embedded in f_{-} or in f_{+}.

Using (11) in (3) we have for the metric on Σ

$$ds^2 \stackrel{\Sigma}{=} -d\tau^2 + B^2 dz^2 + C^2 d\phi^2, \tag{13}$$

where we define the time coordinate τ only on Σ by

$$d\tau \stackrel{\Sigma}{=} A dt, \tag{14}$$

and $\overset{\Sigma}{=}$ means that both sides of the equation are evaluated on Σ. We shall take $\xi^0 = \tau$, $\xi^2 = z$ and $\xi^3 = \phi$ as the parameters on Σ.

For the exterior metric (8) using (12) reduces on Σ to

$$ds^2 \overset{\Sigma}{=} -e^{2(\gamma-\psi)}\left[1-\left(\frac{dR}{dT}\right)^2\right]dT^2 + e^{2\psi}dz^2 + e^{-2\psi}R^2 d\phi^2, \tag{15}$$

and is the same as (13) if

$$e^{\gamma-\psi}\left[1-\left(\frac{dR}{dT}\right)^2\right]^{1/2} dT \overset{\Sigma}{=} d\tau, \tag{16}$$

$$e^{\psi} \overset{\Sigma}{=} B, \tag{17}$$

$$e^{-\psi}R \overset{\Sigma}{=} C, \tag{18}$$

where we assume on Σ

$$1-\left(\frac{dR}{dT}\right)^2 > 0, \tag{19}$$

so that T is a timelike coordinate.

Equations (14) and (16-18) are the conditions on the interior and exterior metrics imposed by the continuity of the first fundamental form on Σ.

We turn now to the second fundamental form on Σ. We need the outward unit normals to Σ in f_- and f_+; these come from (11) and (12) and are

$$n_\alpha^- \overset{\Sigma}{=} (0, A, 0, 0), \tag{20}$$

$$n_\alpha^+ \overset{\Sigma}{=} e^{\gamma-\psi}\left[1-\left(\frac{dR}{dT}\right)^2\right]^{-1/2}\left(-\frac{dR}{dT}, 1, 0, 0\right)$$
$$\overset{\Sigma}{=} e^{2(\gamma-\psi)}(-\dot{R}, \dot{T}, 0, 0), \tag{21}$$

where the dot stands for differentiation with respect to τ introduced by means of (16). Both unit vectors (20) and (21) are spacelike provided (19) is satisfied.

The second fundamental form of Σ is

$$K_{ab}d\xi^a d\xi^b, \quad a, b = 0, 2, 3, \tag{22}$$

where K_{ab} is the extrinsic curvature given on the two sides by

$$K_{ab}^{\mp} = -n_\alpha^{\mp}\left(\frac{\partial^2 x^\alpha}{\partial\xi^a\partial\xi^b} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial\xi^a}\frac{\partial x^\gamma}{\partial\xi^b}\right). \tag{23}$$

The Christoffel symbols are to be calculated from the appropriate exterior or interior metric, (3) or (8), $n_\alpha^{\mp}$ are given by (20) and (21), and x^α refers to the equation of Σ in f_- or f_+, namely (11) or (12). The non zero $K_{ab}^{\mp}$ are as follows

$$K_{00}^- \overset{\Sigma}{=} -\frac{A_{,r}}{A^2}, \tag{24}$$

$$K^-_{22} \stackrel{\Sigma}{=} \frac{BB_{,r}}{A}, \tag{25}$$

$$K^-_{33} \stackrel{\Sigma}{=} \frac{CC_{,r}}{A}, \tag{26}$$

$$K^+_{00} \stackrel{\Sigma}{=} e^{2(\gamma-\psi)} \Big\{ \ddot{T}\dot{R} - \ddot{R}\dot{T}$$
$$-(\dot{T}^2 - \dot{R}^2) \left[\dot{R}(\gamma_{,T} - \psi_{,T}) + \dot{T}(\gamma_{,R} - \psi_{,R}) \right] \Big\}, \tag{27}$$

$$K^+_{22} \stackrel{\Sigma}{=} e^{2\psi}(\dot{R}\psi_{,T} + \dot{T}\psi_{,R}), \tag{28}$$

$$K^+_{33} \stackrel{\Sigma}{=} -e^{-2\psi} R^2 \left(\dot{R}\psi_{,T} + \dot{T}\psi_{,R} - \frac{\dot{T}}{R} \right). \tag{29}$$

The complete junction conditions consist of (14) and (16-18) together with the continuity of K_{ab} across Σ.

4 Results

In this section, we use the interior and exterior field equations to write the boundary conditions in a concise form. In order to do that we first derive some useful formulae.

From (16) we have that

$$e^{2(\gamma-\psi)}(\dot{T}^2 - \dot{R}^2) \stackrel{\Sigma}{=} 1, \tag{30}$$

and from (17) and (18)

$$R \stackrel{\Sigma}{=} BC. \tag{31}$$

Using (14) we can differentiate (31) yielding

$$\dot{R} \stackrel{\Sigma}{=} \frac{(BC)_{,t}}{A}, \tag{32}$$

and from the continuity of K_{22} and K_{33} with (17) and (18) we obtain

$$\dot{T} \stackrel{\Sigma}{=} \frac{(BC)_{,r}}{A}. \tag{33}$$

Now differentiating (32) and (33) with (6), (7) and (14) we can write the relation

$$\ddot{T}\dot{R} - \ddot{R}\dot{T} \stackrel{\Sigma}{=} \frac{1}{A^4} \{ (BC)_{,t} [B_{,t}(AC)_{,r} + C_{,t}(AB)_{,r}]$$
$$+(BC)_{,r} \left[+\kappa P_r A^3 BC - AB_{,t}C_{,t} - A_{,r}(BC)_{,r} - AB_{,r}C_{,r} \right] \Big\}. \tag{34}$$

From the continuity of K_{00} and K_{22} and with (10), (17), (30), (31), (33) and (34) we get the expression

$$\frac{1}{A^4} \left[(B_{,t}C_{,r} - B_{,r}C_{,t})^2 + \kappa P_r A^2 (BC)^2_{,r} \right] \stackrel{\Sigma}{=} (\dot{T}^2 - \dot{R}^2)^2 \psi^2_{,T}. \tag{35}$$

Differentiating (17) and (18) with respect to (14) and (16) we obtain

$$e^{2\psi-\gamma}\left[1-\left(\frac{\dot{R}}{\dot{T}}\right)^2\right]^{-1/2}\psi_{,T} \overset{\Sigma}{=} \frac{B_{,t}}{A}, \tag{36}$$

$$e^{-\gamma}\left[1-\left(\frac{\dot{R}}{\dot{T}}\right)^2\right]^{-1/2}\left(\frac{\dot{R}}{\dot{T}}-R\psi_{,T}\right) \overset{\Sigma}{=} \frac{C_{,t}}{A}. \tag{37}$$

Then from the continuity of K_{22} and K_{33} with (36) and (37) we have

$$\frac{1}{A^2}(B_{,t}C_{,r}-B_{,r}C_{,t}) \overset{\Sigma}{=} (\dot{T}^2-\dot{R}^2)\psi_{,T}-\dot{T}\dot{R}\psi_{,R}. \tag{38}$$

Finally substituting (38) into (35) and considering (10) and (33) we obtain

$$\kappa P_r \overset{\Sigma}{=} e^{2(\psi-\gamma)}\psi_{,R}^2\left(2\frac{\psi_T}{\psi_{,R}}v-\frac{v^2}{1-v^2}\right), \tag{39}$$

where $v \overset{\Sigma}{=} dR/dT$ denotes the radial velocity of the boundary surface. The result (39)shows that the radial pressure P_r on the surface Σ of the collapsing perfect fluid is non zero. The reason for this might be due to the flux of momentum of the gravitational wave emerging from the cylinder. If the cylindrical fluid source is static then $v \overset{\Sigma}{=} 0$ and $P_r \overset{\Sigma}{=} 0$ as expected.

However, it might be argued that $P_r \overset{\Sigma}{=} 0$ is always satisfied in cylindrical collapse, leading to an equation for v, which reads

$$v \overset{\Sigma}{=} \frac{1}{x}[-1 \pm (1+x^2)^{1/2}], \tag{40}$$

with $x = 4\psi_{,T}/\psi_{,R}$. By means of an example we shall prove in the next section that this is not the case.

5 A pulse solution

Let us now consider a cylindrical source which is static for a period of time until it starts contracting and emits a sharp pulse of radiation travelling outward from the axis. Then, the function ψ can be written as [10]

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{T-R}\frac{f(T')dT'}{[(T-T')^2-R^2]^{1/2}}+\psi_{st}, \tag{41}$$

In (41) ψ_{st} represents the Levi-Civita static solution [11], and $f(T)$ is a function of time representing the strength of the source of the wave and it is assumed to be of the form

$$f(T) = f_0\delta(T), \tag{42}$$

where f_0 is a constant and $\delta(T)$ is the Dirac delta function. It can be shown that (41) satisfies the wave equation (9). Then we get

$$\psi = \psi_{st}, \quad T < R; \tag{43}$$

$$\psi = \frac{f_0}{2\pi(T^2 - R^2)^{1/2}} + \psi_{st}, \quad T > R. \tag{44}$$

The function ψ, as well as its derivatives, is regular everywhere except at the wave front determined by the surface $T = R$, followed by a tail decreasing with T. Using $\psi_{st} = \alpha - \beta \ln R$, with α and β being constants, and (44) we obtain

$$x \stackrel{\Sigma}{=} \frac{4f_0 TR}{2\pi\beta(T^2 - R^2)^{3/2} - f_0 R^2}. \tag{45}$$

From (45) we see that as T increases, the system tends asymptotically to a static situation, and for a sufficiently large values of T we have

$$x \stackrel{\Sigma}{\approx} \frac{2f_0 R}{\pi\beta T^2}. \tag{46}$$

which is a positive quantity. On the other hand for values of T larger than, but sufficiently close to R, we have that x is negative. This implies that for some value of T (say T_0) the denominator in (45) vanishes and x tends to $\mp\infty$ as T goes from $T_0 - 0$ to $T_0 + 0$. This in turn implies, because of (40) (where only the upper sign before the square root in (40) has to be considered because $v^2 < 1$), that in the infinitesimal time interval $(T_0 - 0, T_0 + 0)$ the velocity v changes from -1 to $+1$. This of course is impossible. Therefore, condition (40) cannot be satisfied in this example and $P_r \neq 0$ on Σ.

It is worth noticing that for sufficiently large values of T, $\Psi_{,R}\Psi_{,T} > 0$ on Σ. This quantity, which is related to the rate of change of the C energy, is shown to be negative in [7]. However that proof is valid only for large values of R, which of course is not the case here.

It can be proved that, in general, the static exterior spacetime, i. e. Levi-Civita spacetime, cannot be matched to a collapsing cylinder with source (1). Indeed, considering that the exterior spacetime is static, $\psi(R)$, then we have from (17), (33) and the continuity of K_{22} that $B \stackrel{\Sigma}{=} B(r)$ and $C \stackrel{\Sigma}{=} C(r)$, which in turn implies because of (31) that $R \stackrel{\Sigma}{=}$ constant showing that the source cannot be collapsing. So we can state that if a cylinder with source, as described in (1), is collapsing, it will *always* produce a time-dependent field outside the source. This result has been shown for the particular case of a collapsing dust cylinder [12].

6 Conclusion

The main result of this letter is that the radial pressure on the surface of a collapsing anisotropic perfect fluid cylinder is non zero (39). If the system is static we reobtain the usual result that the radial pressure is zero on the boundary surface (the same happens in the slowly evolving case considered by Bondi [13]).

The physical interpretation of the non-vanishing surface pressure might be justified through the continuity of the radial flux of momentum across the boundary surface. However, based on this interpretation what follows is that the collapsing dust cylinder ($P_r = 0$) should not radiate gravitational waves, the exterior spacetime then belongs to the class of time dependent systems without news, as discussed by Bondi et al in [6]. Parenthetically Bondi suggests that pressure-free dust would be the most clear cut example of such non-radiating time dependent systems.

Finally, we also proved, following the matching conditions, that a cylinder filled with anisotropic perfect fluid always produces a time-dependent field outside the source when undergoing collapse.

References

[1] Santos N O 1985 *Mon. Not. R. Astron. Soc.* **216** 403

[2] Darmois G 1927 *Mémorial des Sciences Mathématiques*, Gauthier-Villars, Paris, Fasc. 25

[3] Bonnor W B, de Oliveira A K G and Santos N O 1989 *Phys. Rep.* **181** 269

[4] Herrera L and Santos N O 1997 *Phys. Rep.* **286** 53.

[5] Marder L 1958 *Proc. Roy. Soc. Lond.* **A244** 524

[6] Bondi H, van der Burg M G J and Metzner A W K 1962 *Proc. Roy. Soc. Lond.* **A269** 21

[7] Apostolatos T A and Thorne K S 1992 *Phys. Rev. D* **46** 2435

[8] Einstein A and Rosen N 1937 *J. Franklin Inst.* **223** 43

[9] Bonnor W B and Vickers P A 1981 *Gen. Rel. Grav.* **13** 29

[10] Carmeli M 1982 *Classical Fields: General Relativity and Gauge Theory*, John Wiley & Sons, New York

[11] Levi-Civita T 1919 *Rend. Acc. Lincei* **28** 101

[12] Gutti S, Singh T P, Sundararajan P A and Vaz C 2002 arXiv:gr-qc/0212089

[13] Bondi H 1990 *Proc. Roy. Soc. Lond.* **A427** 259