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Inequivalent Representations of a q-Oscillator Algebra in a Quantum q-Gas

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Abstract

We study the consequences of inequivalent representations of a q-oscillator algebra on a quantum q-gas. As in the "fundamental" representation of the algebra, the q-gas presents the Bose-Einstein condensation phenomenum and a λ-point transition. The virial expansion and the critical temperature of condensation are very sensible to the representation chosen; instead, the discontinuity in the λ-point transition is unaffected.

Key-words: Statistical mechanics; Ideal gases; Bose-Einstein condensation; Virial expansion; λ-point transition; Algebras; Quantum groups; q-Oscillators.

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Since the connection between q-oscillators and quantum algebras was established [1, 2], the interest in q-oscillators has increased both in physical and mathematical literature. In this letter we discuss the consequences of choosing different representations of a q-oscillator algebra on a system which generalizes the ideal quantum boson gas. The interest on this problem comes from the role played by the theory of ideal gases in many different phenomena.

Let us consider the algebra generated by a, a^+ and N satisfying

$$[N, a^+] = a^+ \ , \ [N, a] = -a, \tag{1}$$
$$aa^+ - qa^+a = q^{-N}$$

where $q \in \mathbb{R}^+$.

Assuming that a, a^+ are mutually adjoint, $N = N^+$ and the spectrum is non-degenerate, representations of algebra (1) in a Hilbert space $\mathcal{H}$ were built [3].

Denoting the normalized basis vectors by $|n>$, in ref.[3], for $q > 1$ (which is the case we shall be interested in this letter) the following representations were obtained:

$$\begin{aligned} a^+|n> &= q^{-\nu_0/2}[n+1]^{1/2}|n+1>, \\ a|n> &= q^{-\nu_0/2}[n]|n-1>, \\ N|n> &= (\nu_0+n)|n>\ , \end{aligned} \tag{2}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and ν_0 is a real free parameter which goes to zero when $q \to 1$. When $\nu_0 = 0$, N is interpreted as the usual particle number operator for the state $|n>$. This is not anymore the case for $\nu_0 \neq 0$; its eingenvalue can now be interpreted as the sum of the number of particles n, in the state $|n>$, plus a background effect ν_0.[1] We define here the the operator $\hat{N} = N - \nu_0$, which is now the number operator, $\hat{N}|n>= n|n>$, for the representations in (2) characterized by ν_0.

In our case ($q > 1$), as ν_0 is the lowest bound of the spectrum of N, it classifies inequivalent representations of the algebra (1) [3]. In fact, it can be easily verified that [5]

$$\mathcal{C} = q^{-N}([N] - a^+a) \tag{3}$$

is a Casimir operator for the algebra (1) and in the representation (2) one has

$$\mathcal{C}|n>= q^{-\nu_0}[\nu_0]|n>\ . \tag{4}$$

[1] A similar effect appears in the reparametrization ghosts of string theories [4].

As (3) is different from zero only for $q \neq 1$, one sees from (4) that when $q = 1$, ν_0 is necessarily zero. On the other hand, it should be stressed that the q-Fock representation for which the relations

$$aa^+ - qa^+a = q^{-N} \ , \ aa^+ - q^{-1}a^+a = q^N, \tag{5}$$

are simultaneously verified, is given in (2) when $\nu_0 = 0$.

We are now going to analyse the behaviour of the ideal quantum q-gas with respect to the more general representations ($\nu_0 \neq 0$) (2) of the q-oscillator algebra (1). For that sake, let us consider an ideal deformed system described by the Hamiltonian

$$H = \sum_i \omega_i \ a_i^+ a_i = \sum_i \ \omega_i([N_i] - q^{N_i}\mathcal{C}_i) \tag{6}$$

where a_i and a_i^+ are interpreted as annihilation and creation operators of particles in levels i with energy ω_i and N_i is an operator that can be interpreted as the number operator of particles in levels i when $\nu_0 = 0$. a_i, a_i^+ and N_i satisfy algebra (1) and commute for different levels. With μ the chemical potential, $\hat{N} = \sum_i \hat{N}_i$ and Ω the grand canonical potential, the grand canonical partition function is given by

$$Z = Tr \exp[-\beta(H - \mu\hat{N})] = exp(-\beta\Omega), \tag{7}$$

where $\beta = 1/kT$, with k the Boltzmann constant.

As Z factorizes for the above system, the grand canonical potential is given by a sum over single level partition functions [6]

$$\Omega = -\frac{1}{\beta}\sum_i \ln Z_i^0(\omega_i, \beta, \mu), \tag{8}$$

with

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i q^{-\nu_0}[n] - \mu n)} \tag{9}$$

where we are assuming the same ν_0 for all levels. This means that in all levels the spectrum has the same lowest bound, which is a sensible assumption.

According to the usual procedure the system is enclosed in a large d-dimensional volume V and the sum over levels is replaced by an integral over $\vec{p}$-space. Assuming that the energy spectrum of the q-particles follows the dispersion law $\omega_i \to \gamma p^\alpha$, the grand canonical potential becomes

$$\Omega = \frac{-V}{h^d\beta}\int d^dp \ln \sum_{n=0}^{\infty} e^{-\beta(\gamma_q p^\alpha [n] - \mu n)}, \tag{10}$$

where $\gamma_q = q^{-\nu_0}\gamma$ and for $\alpha = 1(2)$ one recovers the ultrarelativistic (non-relativistic) case, with $\gamma = 1(1/2m)$.

The pressure $P = -\Omega/V$ and the density $n = \partial P/\partial\mu|_{T,V}$ are then:

$$\begin{aligned} P(T,z) &= \beta^{-1} \wedge_q^{-d} Y_q(z), \\ n(T,z) &= \wedge_q^{-d}\, y_q(z), \end{aligned} \tag{11}$$

where $z = exp(\beta\mu)$ is the fugacity and $\wedge_q^{-d} = \frac{\pi^{d/2}\Gamma\left(\frac{d}{\alpha}+1\right)}{\Gamma(\frac{d}{2}+1)h^d(\beta\gamma_q)^{d/\alpha}}$ is the modified thermal wavelength. The functions $Y_q(z)$ and $y_q(z)$ are respectively

$$\begin{aligned} Y_q(z) &= \frac{1}{\Gamma\left(\frac{d}{\alpha}+1\right)} \int_0^\infty d\eta\; \eta^{d/\alpha} \frac{\sum_{n=0}^\infty [n] z^n e^{-[n]\eta}}{\sum_{n=0}^\infty z^n\; e^{-[n]\eta}} \\ y_q(z) &= \frac{1}{\Gamma\left(\frac{d}{\alpha}+1\right)} \int_0^\infty d\eta\; \eta^{d/\alpha} \\ &\left[\frac{\sum_{n=0}^\infty [n]n\; z^n\; e^{-[n]\eta}}{\sum_{n=0}^\infty z^n\; e^{-[n]\eta}} - \frac{\sum_{n,m=0}^\infty [n]m\; z^{n+m}\; e^{-([n]+[m])\eta}}{\left(\sum_{n=0}^\infty z^n\; e^{-[n]\eta}\right)^2} \right] \end{aligned} \tag{12}$$

where $\eta = \beta\gamma_q p^\alpha$. Notice that the modified thermal wavelength depends now on q and ν_0 through γ_q, $\wedge_q = q^{-\nu_0/\alpha}\wedge$ with $\wedge$ the usual thermal wavelength.

We now consider the high-temperature (or low-density) [6] approximation for large q [7]. Assuming that the fugacity is small compared to one we obtain the virial expansion for the equation of state

$$\begin{aligned} P = \tfrac{n}{\beta}\Big[1 - q^{-\nu_0 d/\alpha}\left(-\tfrac{1}{2^{d/\alpha+1}} + \tfrac{1}{[2]^{d/\alpha}}\right)(n\wedge^d) \\ +q^{-2\nu_0 d/\alpha} 2\left(\tfrac{1}{2^{2d/\alpha+1}} - \tfrac{1}{3^{d/\alpha+1}} + \tfrac{2}{[2]^{2d/\alpha}} - \tfrac{2}{2^{d/\alpha}[2]^{d/\alpha}} + \tfrac{1}{(1+[2])^{d/\alpha}}\right)(n\wedge^d)^2 + \cdots\Big]. \end{aligned} \tag{13}$$

The virial expansion (13) deserves some comments. For $q \to \infty$ q, the q-gas behaves exactly like an ideal Fermi-gas only if $\nu_0 = 0$. When $\nu_0 < 0$ ($\nu_0 > 0$) the pressure is higher (lower) than the ideal Fermi gas. When $\nu_0 = 0$ we reobtain the virial expansion of ref. [8] and for $[n]_A \equiv (q^{2n} - 1)/(q^2 - 1)$, the one of ref. [7].

Let us now study the Bose-Einstein condensation [9] for the highly deformed case where, as shown by Matheus-Valle [8], similarly as in the case of ref. [10] (with $[n]_A$), in order to reach a given accuracy in the integrals (12) the number of terms to be kept depend on the value of q. As usual, when $z \to 1$ (or $T \to T_c$, T_c being the critical temperature) one has to take into account the zero-point energy and single out its contribution in (12). In addition, inspection of eq. (9) clearly shows that when $\omega_i = 0$ the effect of the deformation is cancelled; therefore series (9) cannot be approximated by a polynomial for the zero energy level. Keeping n constant we now consider lower temperatures: $n\wedge_q^\nu$ increases and so does z. When z reaches 1, the temperature attains its critical value T_c^q, defined by $n^{1/d}\wedge_c^q = y_q^{1/d}(1)$ or

$$T_c^q = \frac{\gamma_q \Gamma^{\alpha/d}\left(\frac{d}{2}+1\right) h^\alpha n^{\alpha/d}}{k\pi^{\alpha/2}\Gamma^{\alpha/d}\left(\frac{d}{\alpha}+1\right) y_q^{\alpha/d}(1)}. \tag{14}$$

Comparing T_c^q with the critical temperature for the non-deformed ideal gas of the same density n, we find

$$\frac{T_c^q}{T_c} = \left(\frac{2.61}{y_q(1)}\right)^{\alpha/d} q^{-\nu_0}. \tag{15}$$

For $\nu_0 = 0$ we recover the results of refs. [10, 8], the critical temperature increasing with respect to the non-deformed case, but associated to the different representations of q-oscillator (1) (different background “charges” ν_0) a different phenomenum appears. For $\nu_0 > 0$ ($\nu_0 < 0$) the ratio T_c^q/T_c decreases (increases).

Similarly to the non-deformed case [11] the basic equations are

$$P(T,z) = \beta^{-1}\wedge_q^{-d} Y_q(z) \tag{16.a}$$

$$n(T,z) = \frac{1}{V}\frac{z}{1-z} + \wedge_q^{-d} y_q(z), \tag{16.b}$$

where the first term on the right-hand side of (16b), which is due to the contribution of zero energy, is relevant only for $T \leq T_c^q$. In this region z remains equal to one, as in the standard case.

The specific heat per particle, C_V, defined as

$$\frac{C_V}{k} = \frac{1}{kn}\frac{\partial \tilde{e}}{\partial T}\Big|_n , \tag{17}$$

where $\tilde{e}$ is the energy density (internal energy per volume) is

$$C_V/k = \frac{d}{\alpha}\left(\frac{d}{\alpha}+1\right)(\wedge_q^d n)^{-1} Y_q(z) - \left(\frac{d}{\alpha}\right)^2 \frac{y_q(z)}{z y_q'(z)}, \; T > T_c^q , \tag{18.a}$$

$$C_V/k = \frac{d}{\alpha}\left(\frac{d}{\alpha}+1\right)(\wedge_q^d n)^{-1} Y_q(1) \; , \; T < T_c^q. \tag{18.b}$$

It is interesting to observe that the presence of the parameter ν_0, which characterizes the different representations of the q-oscillators algebra (1), changes the shape of the specific heat C_V in (18) but leaves the λ-point discontinuity invariant [10].

In summary we have analysed the role played by different representations of the q-oscillator algebra (1). We find that the high-temperature (or low density) regime is strongly representation dependent. As in the "fundamental" representation ($\nu_0 = 0$), the q-gas presents Bose-Einstein condensation but now the critical temperature depends on the representation under consideration. Finally, we also find a λ-point transition with a discontinuity independent of the value of ν_0.

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