CBPF-NF-028/81

ON THE CRITICAL FRONTIERS OF POTTS FERROMAGNETS

by

Aglaê C.N. de MAGALHÃES and C. TSALLIS

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Av. Wenceslau Braz, 71, fundos
22290 - R.J. - Rio de Janeiro - BRAZIL

ON THE CRITICAL FRONTIERS OF POTTS FERROMAGNETS

by

Aglaê C. N. de MAGALHÃES

and

Constantino TSALLIS

Centro Brasileiro de Pesquisas Físicas/CNPq

Av. Wenceslau Braz 71 - 22.290 - Rio de Janeiro - Brazil

ABSTRACT

We formulate a conjecture concerning the critical frontiers of q- state Potts ferromagnets on d- dimensional lattices ($d > 1$) which generalize a recent one stated for planar lattices. The present conjecture is verified within satisfactory accuracy (exactly in some cases) for all the lattices or arrays whose critical points are known. Its use leads to the prediction of: a) a considerable amount of new approximate critical points (26 on non-planar regular lattices, some others on Husimi trees and cacti); b) approximate critical frontiers for some 3- dimensional lattices; c) the possibly asymptotically exact critical point on regular lattices in the limit $d \to \infty$ for all $q \geqslant 1$; d) the possibly exact critical frontier for the pure Potts model on fully anisotropic Bethe lattices; e) the possibly exact critical frontier for the general quenched random-bond Potts ferromagnet (any $P(J)$) on isotropic Bethe lattices.

RÉSUMÉ

Nous formulons une conjecture sur les frontières critiques du modèle de Potts à q états ferromagnetique sur des réseaux d-dimensionnels ($d > 1$) qui généralise une proposition récente pour des réseaux plans. La conjecture présente est verifiée de façon satisfaisante (exacte dans certains cas) pour tous les réseaux dont les points critiques sont connus. Son utilisation nous permet de prédire: a) un nombre considérable de points critiques approchés (26 sur des réseaux réguliers non planaires, quelques uns sur des arbres de Husimi et des cacti); b) les frontières critiques approchées pour quelques réseaux tridimensionnels; c) le comportement assymptotique probablement exact du point critique pour des réseaux réguliers dans la limite $d \to \infty$ pour tout $q \geq 1$; d) la frontière critique probablement exacte pour le modèle de Potts pure sur des réseaux de Bethe completement anisotropes; e) la frontière critique probablement exacte pour le modèle trempé ferromagnetique de Potts à liaisons aléatoires distribuées selon une loi quelconque ($P(J)$) sur des réseaux de Bethe isotropes.

1. INTRODUCTION

Many efforts are presently being dedicated to the discussion of the critical properties of pure (as well as random) q- state Potts ferromagnets (whose Hamiltonian, for the particular case of first-neighbour interactions, is $\mathcal{H} = -qJ \sum_{<i,j>} \delta_{\sigma_i,\sigma_j}$; $\sigma_i = 1,2,\ldots,q \;\forall i$; $J > 0$). We have recently presented[1,2] a transformation which leads to a new type of quasi-universality for *planar* lattices, in the sense that the critical point depends on the particular lattice but, either exactly or within good accuracy, *not on the number of states* q ($1 \leq q \leq 4$). The purpose of the present paper is to exhibit how such transformation can be extended, on conjectural grounds, to *non planar* lattices. Before going on, let us recall that the limits $q \to 0$ and $q \to 1$ respectively correspond to tree-like[3,4] and standard[5] bond percolations, that $q = 2$ corresponds to the spin 1/2 Ising model, and that physical interpretations for $q = 3,4$ are discussed in refs.(6,7). Furthermore the phase transition is commonly believed to be a first order (continuous) one for $q > q_c(d)$ ($q \leq q_c(d)$), where $q_c(d)$ depends on the dimensionality d of the lattice; in particular, it has recently been proved[8] that, for $d \geq 2$, the transition is a first order one in the limit $q \to \infty$ (thus confirming the conjecture by Mittag and Stephen[9]). In what concerns $d = 2$, it is by now well established[10,11,12] that $q_c(2)=4$, whereas for $d = 3$ the situation is less clear in the sense that although quite an amount of evidence already exists (see refs. (13,14)) in favour of $q_c(3) \simeq 3$ (let us however mention that Jensen and Mouritsen's recent work[13] quite convincingly suggests that $q_c(3)$ is slightly smaller than 3); in what concerns d=4, $q_c(4)$ presumably satisfies[14] $2 \leq q_c(4) < 3$ (as a matter of fact it might well be that $q_c(d)$ monotonically and continuously decreases for d varying in the interval $[1,4]$, and equals 2 for all finite $d \geq 4$; in any case, in refs. (6,14) it is stated quite firmly that the $(q=3;\ d=4)$ case corresponds to a first order transition). A more delicate ambiguity appears in the limit $d \to \infty$: on one hand one could expect that the Landau theory prediction[9,11] (namely, *first order* phase transition for all $q > 2$ and all dimensionalities) becomes true for sufficiently high dimensionality, but on the other hand Southern and Thorpe[15] obtain, for a z- coordinated Bethe lattice (whose effective dimensionality on topological grounds can be considered, for $z > 2$, as being infinite[16]), a *continuous* phase transition for all values of q; as a possible solution of the paradox, one could speculate that the latent heat associated to $q > 2$ and $0 < d^{-1} << 1$ continuously vanishes at $d^{-1} = 0$ (thus becoming oncemore the transition a continuous one). The present work mainly concerns the region of the (q,d) space where the transition is a *continuous* one, or slightly first order (in the sense that the discrepancy between the two associated metastability points is small compared with the transition point itself).

2. FORMULATION OF THE CONJECTURE

We have used[2], for planar lattices, the variable[1]

$$s^{(q)} \equiv \frac{\ln\left[1 + (q-1)\, t^{(q)}\right]}{\ln q} \tag{1}$$

where

$$t^{(q)} \equiv \frac{1 - e^{-qJ/k_BT}}{1 + (q-1)\, e^{-qJ/k_BT}} \tag{2}$$

and verified, for a large set of lattices, that the ferromagnetic critical point $s_c \left(\equiv s(t_c) = s(t(T_c))\right)$, or critical frontier in more general cases, depends on the lattice but *practically not on the value of* q $(1 \leqslant q \leqslant 4)$. We intend here to generalize this to all dimensionalities (at least for $d>1$), by heuristically extending the variable s. Furthermore, because of the isomorphism[5] existing between the $q \to 1$ Potts model and bond percolation (see also ref.(1)) and by taking into account that $t^{(1)}$ corresponds to a probability, we impose that

$$s^{(1)} = t^{(1)} \qquad \forall d \qquad \forall t^{(1)} \tag{3}$$

We also impose, for all values of q, that

$$s(d \to \infty) = t^{(q)} \qquad \forall q \qquad \forall t^{(q)} \tag{4}$$

(We shall see later on that this restriction enables exact results concerning Bethe lattices and the asymptotically exact behaviour on regular lattices). These hypothesis and the demand to contain expression (1) as a particular case can be simultaneously satisfied through the definition

$$s\left[q,h(d);\ t^{(q)}\right] \equiv \frac{\ln\left[1+(q-1)h(d)\ t^{(q)}\right]}{\ln\left[1+(q-1)h(d)\right]} \quad (5)$$

where h(d) is a number which depends on the particular d-dimensional (d > 1) lattice we are dealing with; restriction (4) implies that

$$\lim_{d\to\infty} h(d) = 0 \quad (6)$$

Remark that the previous results obtained for planar lattices[2] can be recovered by making h(2) = 1 for any planar lattice. By the way let us notice that

$$s\left[q,h(d);\ \frac{1-t^{(q)}}{1+(q-1)h(d)\ t^{(q)}}\right] = 1 - s\left[q,h(d);\ t^{(q)}\right] \quad (7)$$

which, for d = 2 and h(2) = 1, expresses a duality transformation[1,2,17].

At this level we may state our conjecture: *for any d-dimensional lattice (d > 1) exists a number* (namely h(d)) *such that the* (ferromagnetic) *critical point* $s_c \equiv s(q,h(d);t_c)$ *depends on the lattice but practically not on q* ($1 \leq q \leq q_c(d)$; q is not necessarily integer). By imposing, for example, $s(1,\ h;\ t_c^{(1)}) = s(2,\ h;\ t_c^{(2)})$ we obtain, for a given lattice, the corresponding value of h. The use of Tables 1a and 1b and refs (2,18) provides the values indicated in Table 2 (see also Fig. 1), where we remark that, for a given dimensionality d, h does not vary very much ($h(2) \simeq 1$, $h(3) \simeq 0.38$ within about 10% error). Therefore, for lattices where variations of independent coupling constants J_1, $J_2,\ldots,J_n$ do not change d, we can extend this conjecture to the entire critical frontier (CF) similarly to our previous work[2], namely:

if $\phi\left\{s\left(q_0,h;\ t_1^{(q_0)}\right),\ldots,\ s\left(q_0,h;\ t_n^{(q_0)}\right)\right\} = 0$ is the CF for a fixed value q_0 $(1 \leq q_0 \leq q_c(d))$ then

$$\phi\left\{s\left(q,h;\ t_1^{(q)}\right),\ldots,\ s\left(q,h;\ t_n^{(q)}\right)\right\} = 0 \tag{8}$$

represents, within good accuracy, the CF for other values of q as well $\left(1 \leq q \leq q_c(d)\right)$; $t_i^{(q)}$ is related to J_i through eq. (2).

3. APPLICATION TO CRITICAL POINTS OF PURE FERROMAGNETS

3.1 General Remarks and Finite Dimensional Regular Lattices

This conjecture has been verified[2] on a large set of planar lattices using the value h(2) = 1 which leads to the exact frontier only for the square lattice. However, if we use the values of h indicated in Table 2, the accuracy of the conjecture is expected to increase. For example, for the anisotropic triangular lattice the maximal error in the s-variable (which occurs in the isotropic limit[2]) introduced by this conjecture reduces from 2%[2] (with h = 1) to 0.17% (with h = 1.08448) for q = 3 and from 2.4%[2] (with h = 1) to 0.32% (with h = 1.08448) for q = 4. Similarly, for the isotropic diced lattice[18], the error in the s-variable introduced by this conjecture reduces from 1% (with h=1) to 0.22% (with h = 0.953) for q = 3 and from 1.3% (with h = 1) to 0.43% (with h = 0.953) for q = 4. For their dual lattices (honeycomb and kagomé respectively) their percentual errors are even smaller. For the SC and FCC lattices our estimates for the q = 3 critical points agree quite well (the discrepancy in s being about 0.4% and 1% respectively) with previous ones (see Table 1a).

Tables 1a and 1b have been calculated by using:

i) the values of h indicated in Table 2 whenever the critical points are known for two or more values of q;

ii) the value of h(d) corresponding to the d- dimensional hypercubic lattice for all the first-neighbour lattices where only one critical point is known ;

iii) the first-neighbour lattice h value for associated first- and higher-neighbour lattices (this choice has proved to be a good one for the first and second-neighbour square lattice: the discrepancy in h is about 1%; see Table 2).

Through this procedure, the present conjecture has provided 26 new independent critical points which are presented in Tables 1a and 1b (region delimited by heavy lines). The central values of the s- variable indicated in these tables have been obtained by using the central values of $t^{(2)}$ and/or $t^{(1)} \equiv p_c$ reproduced in the same line (top value). The error bars were estimated by taking into account the error and/or dispersion existent in the literature, the error coming from the uncertainty in the determination of h and finally the intrinsic error of the conjecture itself (the latter has proved to be neglectable whenever check was possible). We may remark in these tables that the overall uncertainty in the values of $t_c^{(q)}/q$ and k_BT_c/qJ is about 1% for all the cases but the hcp lattice (for which no recent estimation of p_c is available as far as we know).

For the three dimensional lattices (Table 1a) we have restricted ourselves to $1 \leqslant q \leqslant 3$ in order to be sure that the transition is either a continuous or a slightly first order one. For higher finite dimensionalities (Table 1b) the transition is a second order one if $q \leqslant 2$; it is however interesting to remark that for the d=4 HSC lattice and q=3 the present conjecture still holds ($3J/k_BT_c = 0.3875 \pm 0.0010$[6] hence $s_c = 0.158 \pm 0.004$ to be compared with 0.161 ± 0.002 from Table 1b).

3.2 Regular Lattices in the $d \to \infty$ Limit

Gaunt and Ruskin[19] have derived the following (probably asymptotic) expansion for the critical bond probability p_c on d- dimensional simple hypercubic lattice:

$$p_c \equiv s\left(1, h;\ t_c^{(1)}\right) = \frac{1}{\sigma} + \frac{5}{2\sigma^3} + \frac{15}{2\sigma^4} + \frac{57}{\sigma^5} + \ldots \qquad (9)$$

where $\sigma \equiv z - 1 = 2d - 1$, z being the coordination number.

Furthermore Fisher and Gaunt[20] have obtained a (probably asymptotic) expansion for the Ising critical point. From their results it immediately follows

$$t_c^{(2)} = \frac{1}{\sigma} + \frac{1}{\sigma^3} + \frac{3}{\sigma^4} + \frac{15}{\sigma^5} + \frac{256}{3\sigma^6} + \ldots \qquad (10)$$

It is more convenient to work herein with expansions in σ rather than in z because of the better numerical results which are obtained for d varying from 2 to 7 (possibly for $d > 7$ as well); consequently we have chosen σ^{-1} for abcissa in Fig. 1.

Through the imposition $s\left(1,h;\ t_c^{(1)}\right) = s\left(2,h;\ t_c^{(2)}\right)$, we get the following (probably asymptotic) expansion for h

$$h\ (\sigma) = \frac{3}{\sigma^2} + \frac{12}{\sigma^3} + \ldots \qquad (11)$$

Observe that the term σ^{-1} appearing in expansions (9) and (10) corresponds to the Bethe approximation (see refs. (19, 20)) which is expected to be asymptotically exact in the limit $d \to \infty$ for *all* regular lattices. We immediately verify condition (6), which ensures the validity of the conjecture within the dominant asymptotic term for *all* regular lattices. The cases $q > 2$ have not been included in Table 1b, however, for all values of q and $d \gg 1$ for which the transition is only slightly first order, we expect

$$s\left[q, h(d \to \infty);\ t_c^{(q)}\right] \sim t_c^{(q)} \sim \frac{1}{\sigma} \qquad (q \geqslant 1) \qquad (12)$$

3.3 Bethe Lattice $\left(d_{ef}^{-1} = 0\right)$

The critical point of a z- coordinated Bethe lattice (whose effective dimensionality can be considered as infinite[16]) is given (through notation changements) for *all* values of q by[15,16,21]

$$s\left(q, 0;\ t^{(q)}\right) = t^{(q)} = \frac{1}{z-1} \qquad (\forall\, q) \qquad (13)$$

Therefore the conjecture is strictly verified for *all* values of z.

3.4 Husimi Trees

We consider here the z- coordinated triangular cactus (z/2 triangles coming together at each site) and square Husimi tree (z/2 squares coming together at each site) whose exact critical points are known for $q = 1,2$. Through notation changements, the $q=1$ critical value p_c of the triangular cactus is given by[16,22]

$$p_c + p_c^2 - p_c^3 = \frac{1}{\sigma-1} \qquad (14)$$

where $\sigma \equiv z-1$; see Table 1b for the values corresponding to $z=4,6,8,12$; $\sigma\to\infty$ leads to $p_c \sim 1/\sigma$, i.e., the Bethe lattice limit (13). The $q=2$ critical value leads to[22]

$$t_c^{(2)} = \frac{1}{2}\left[\sigma - \sqrt{\sigma^2-4}\right] \qquad (15)$$

see Table 1b for the values corresponding to $z=4,6,8,12$; $\sigma\to\infty$ leads to $t_c^{(2)} \sim 1/\sigma$, i.e. the Bethe lattice approximation[20]. Analogously we have, for the $z=4$ square Husimi tree, that $q=1$ leads to[16]

$$1 - 2p_c - 2p_c^2 - 2p_c^3 + 3p_c^4 = 0 \qquad (16)$$

(see Table 1b for the value p_c)

and $q=2$ leads to[23]

$$1 - 2t_c^{(2)} - 2\left(t_c^{(2)}\right)^2 - 2\left(t_c^{(2)}\right)^3 + \left(t_c^{(2)}\right)^4 = 0 \qquad (17)$$

(see Table 1b for the value $t_c^{(2)}$).

In the present situation, no check can be performed in what concerns our conjecture, but we can use it $\left[p_c = s\left(2,h(d_{ef}); t_c^{(2)}\right)\right]$ to calculate the associated values of h (see Table 2) and, through Fig.1, to estimate the effective dimensionalities d_{ef} (Table 2). Since these values are finite and bigger than 4, it seems plausible that in the Husimi trees (including the cactus) the transitions, for $q \geqslant 3$, are of the first order. We have seen before (d = 4 HSC and q = 3) that the conjecture provides reasonable estimates even for first order phase transitions. If this is still true for the Husimi trees then we should expect, for $q = 3$, $t_c^{(3)} \simeq 0.339$ for the $z = 4$ square Husimi tree and $t_c^{(3)} \simeq 0.364$, 0.204, 0.144, 0.0911 for the triangular cacti with coordination numbers 4,6,8 and 12 respectively.

4. APPLICATION TO CRITICAL FRONTIERS OF PURE FERROMAGNETS

Let us now apply our conjecture to calculate entire critical frontiers as formulated in eq.(8).

4.1 First and Second-Neighbour Cubic Lattice

Let us note J_1 and J_2 the independent ferromagnetic coupling constants respectively associated to the first and second neighbour interactions in the cubic lattice. Since in both extreme cases $J_2/J_1 = 0$ (first-neighbour SC) and $J_1/J_2 = 0$ (two independent first-neighbour FCC lattices) there is no change of dimensionality, we can apply our conjecture.

As far as we know, no results concerning the entire critical frontier have been proposed for $q \neq 2$. The approximate $q = 2$ critical temperatures have been calculated, through series expansions, for $0 \leqslant J_2/J_1 \leqslant 1$, by Dalton and Wood[24]. Their results lead (by using $h = h^{SC} = 0.377$), in the $s_1^{(1)} - s_2^{(2)}$ space, to the curve in heavy line shown in Fig. 2. The uncertainty in the h^{SC} value leads to the two broken lines; the superior one coincides, within the scale of Fig. 2, with the Ising frontier obtained by using $h = h^{FCC} = 0.412$. We conjecture that the critical frontier, *for every q* (at least for $1 \leqslant q \leqslant 3$), is well approximated, for

$0 \leqslant J_2/J_1 \leqslant 1$, by the curve in heavy line whose extrapolation (dotted line) for higher values of J_2/J_1 must intersect the $s_2^{(q)}$ axis at the value $s_2^{(q)} = p_c^{FCC} = 0.119 \pm 0.001$ (see Table 1a).

4.2 First and Second-Neighbour BCC Lattice

Let us consider a BCC lattice where J_1 and J_2 are respectively associated to the first and second neighbour ferromagnetic interactions. Analogously to the previous case, we can apply our conjecture since in both limit cases $J_2/J_1 = 0$ (first-neighbour BCC lattice) and $J_1/J_2 = 0$ (two independent first-neighbour SC lattices) we have three-dimensionality as well.

As far as we know, the entire critical frontier has been calculated only for q = 2. By using Dalton and Wood's[24] values for the Ising critical temperatures for $0 \leq J_2/J_1 \leq 1$ and $h = h^{BCC} = 0.372 \pm 0.037$ we obtain the q = 2 critical frontier (heavy line of Fig. 3), within some error (broken lines). Remark that h^{BCC} and h^{SC} differ so little (see Table 2) that if we had used h^{SC} we would have obtained a curve indistinguishable, within the scale of Fig. 3, from the previous one. According to our conjecture, the referred frontier holds as well for *other* values of q (at least for $1 \leq q \leq 3$). The extrapolation for $J_2/J_1 > 1$ (dotted line in Fig. 3) must intersect the $s_2^{(q)}$ axis at the value $p_c^{SC} =$ $= 0.247 \pm 0.003$ (see Table 1a).

4.3 First, Second and Third-Neighbour FCC Lattice

Let us respectively note J_1, J_2 and J_3 the first, second and third neighbour ferromagnetic interactions in the FCC lattice. Once more the limit cases $J_2/J_1 = J_3/J_1 = 0$ (first-neighbour FCC lattice), $J_1/J_2 = J_3/J_2 = 0$ (four independent first-neighbour SC lattices) and $J_1/J_3 = J_2/J_3 = 0$ (third-neighbour FCC lattice; $z = 24$) have all the same dimensionality $d=3$, thus allowing the application of our conjecture. As far as we know the critical surface for this lattice has been discussed only for $q=2$. Philhours[25] has calculated the Ising transition temperatures for $0 \leq J_2/J_1 \leq 2$ and $0 \leq J_3/J_1 \leq 2$, while Dalton and Wood[24] have considered only the particular case of first- and second-neighbour FCC lattice where $0 \leq J_2/J_1 \leq 1$ $(J_3/J_1 = 0)$. These results lead (for $h = h^{FCC} = 0.412 \pm 0.026$) to the critical surface shown in Fig. 4. The error introduced by the uncertainty on h can not be seen within this scale. The same happens with the discrepancy (inferior to 0.3% in the $s_i^{(2)}$ variables) between the results of refs. [24] (crosses in Fig. 4) and [25]. Once again we conjecture that this critical surface is valid for, at least, $1 \leq q \leq 3$; its extrapolation must intersect the $s_2^{(q)}$ axis in $p_c^{SC} = 0.247 \pm 0.003$ (Table 1a) and the $s_3^{(q)}$ axis in $p_c^{FCC}(3) = 0.054 \pm 0.004$ (we have estimated this value by comparing extrapolations, in the $s_1^{(2)} - s_2^{(2)} - s_3^{(2)}$ space, of different iso - J_2/J_1 critical lines).

4.4 Anisotropic Bethe Lattice

Turban[26] has obtained the exact bond percolation critical frontier for a z- coordinated Bethe lattice with n_1 and $n_2 = z - n_1$ bonds with respective occupancy probabilities p_1 and p_2 (see eq.(5.4) of ref. [26]). Following his procedure[26], we generalize it for the case of $n_1, n_2, \ldots$ and n_N bonds $\left(\sum_{i=1}^{N} n_i = z \right)$ with probabilities $p_1, p_2, \ldots$ and p_N respectively, and obtain the following critical equation

$$\begin{vmatrix} 1-p_1(n_1-1) & -p_1 n_2 & \cdots & -p_1 n_N \\ -p_2 n_1 & 1-p_2(n_2-1) & \cdots & -p_2 n_N \\ \vdots & \vdots & & \vdots \\ -p_N n_1 & -p_N n_2 & \cdots & 1-p_N(n_N-1) \end{vmatrix} = 0 \qquad (18)$$

As we have seen that for the Bethe lattice $s\left\{q,0;t_i^{(q)}\right\} = t_i^{(q)}$ we conjecture (according to eq. (8)) that the *exact* critical frontier *for every* q is given by

$$\begin{vmatrix} 1-t_1^{(q)}(n_1-1) & -t_1^{(q)} n_2 & \cdots & -t_1^{(q)} n_N \\ -t_2^{(q)} n_1 & 1-t_2^{(q)}(n_2-1) & \cdots & -t_2^{(q)} n_N \\ \vdots & \vdots & & \vdots \\ -t_N^{(q)} n_1 & -t_N^{(q)} n_2 & & 1-t_N^{(q)}(n_N-1) \end{vmatrix} = 0 \qquad (19)$$

5. GENERAL ISOTROPIC BOND-MIXED POTTS FERROMAGNET ON BETHE LATTICE

Up to this point we have considered only *pure* Potts ferromagnets, but we can also apply our conjecture for the quenched random-bond Potts model on a z- coordinated Bethe lattice. The q=2 critical frontier for the general bond mixed problem (where each bond strength J is an independent random variable with an *arbitrary* probability distribution P(J), $J \geqslant 0$) is given, through notation changements, by[27]

$$\langle t_c^{(2)} \rangle_{P(J)} = \frac{1}{z-1} \qquad (20)$$

We conjecture that the *exact* critical frontier for *any* q is

$$\langle t_c^{(q)} \rangle_{P(J)} = \frac{1}{z-1} \qquad (\forall q) \tag{21}$$

hence

$$\left\langle \frac{1 - e^{-qJ/k_BT_c}}{1 + (q-1)\, e^{-qJ/k_BT_c}} \right\rangle_{P(J)} = \frac{1}{z-1} \qquad (\forall q) \tag{21'}$$

where $<\cdots>$ stands for the average associated to $P(J)$.
We observe that in the dilute case ($P(J) = (1-p)\delta(J)+p\delta(J-J_0)$; $J_0 > 0$) we recover Southern and Thorpe's result[15] $pt_0^{(q)} = (z-1)^{-1}$.

6. CONCLUSION

We formulate a conjecture concerning the critical frontier of q- state Potts ferromagnets. This conjecture essentially states that, through a convenient variable, quasi- universality with respect to q can be exhibited. We have recently[2] provided a large number of verifications for planar lattices, to which we can presently add two more, namely, the anisotropic Kagomé and diced lattices whose exact critical frontiers for any q have been conjectured by Wu[18]. We add herein a certain amount of verifications for non planar lattices: SC, FCC, $d \to \infty$ regular lattices, isotropic Bethe lattice for the quenched bond-dilute ferromagnetic problem.

The present conjecture enables us to state a certain amount of predictions, namely

i) 26 new approximate critical points (see regions of Tables 1a and 1b within heavy lines) for pure Potts ferromagnets on regular lattices;

ii) the possibly exact critical point of the q- state Potts ferromagnet on *any* z- coordinated d- dimensional regular lattice is, in the limit $d \to \infty$, asymptotically given, for all $q \geqslant 1$, by Eq.12, i.e. $\frac{K_B T_C}{J} \sim q / \ln (\frac{z+q-2}{z-2})$

iii) the approximate critical points for pure Potts ferromagnets on a certain amount of Husimi trees (we expect the transition to be of the first order for $q > 2$);

iv) the approximate critical lines (or surface) for pure Potts ferromagnets on the first and second-neighbour cubic and BCC as well as on the first, second and third-neighbour FCC lattices;

v) the possibly exact critical frontier for the pure Potts ferromagnet on fully anisotropic Bethe lattices (Eq.(19));

vi) the possibly exact critical frontier for the general isotropic quenched random- bond Potts ferromagnet on Bethe lattices (Eq.(21')).

Cross-checking, by other procedures, of the present predictions would be very wellcome.

REFERENCES

1) Tsallis, C., J. Phys. C14 (1981) L85

2) Tsallis, C. and de Magalhães, A.C.N., to appear in J.Physique Lett. 42 (1981)

3) Stephen, M.J., Phys. Lett. 56A (1976) 149

4) Lubensky, T.C., "La Matière Mal Condensée", Les Houches, ed. by Balian, R., Maynard, R. and Toulouse, G. (North Holland, 1978)

5) Kasteleyn, P.W. and Fortuin, C.M., J. Phys. Soc. Jap.(suppl.) 26 (1969) 11

6) Blöte, H.W.J. and Swendsen, R.H., Phys. Rev. Lett. 43 (1979) 799

7) Domany, E., Schick, M. and Walker, J.S., Phys. Rev. Lett. 38 (1977) 1148

8) Pearce, P.A. and Griffiths, R.B., J. Phys. A13 (1980) 2143

9) Mittag, L. and Stephen, M.J., J. Phys. A7 (1974) L109

10) Baxter, R.J., J. Phys. C6 (1973) L445

11) Straley, J.P. and Fisher, M.E., J. Phys. A6 (1973) 1310

12) Kim, D. and Joseph, R.I., J. Phys. A8 (1975) 891

13) Jensen, S.J.K. and Mouritsen, O.G., Phys. Rev. Lett. 43 (1979) 1736

14) Pytte, E., Phys. Rev. B22 (1980) 4450 and references therein

15) Southern, B.W. and Thorpe, M.F., J. Phys. C12 (1979) 5351

16) Fisher, M.E. and Essam, J.W., J. Math. Phys. 2 (1961) 609

17) Southern, B.W., J. Phys. C13 (1980) L 285

18) Wu, F.Y., J. Phys C12 (1979) L 645

19) Gaunt, D.S. and Ruskin, H., J. Phys. A11 (1978) 1369

20) Fisher, M.E. and Gaunt, D.S., Phys. Rev. 133 (1964) A224

21) Eggarter, T.P., Phys. Rev. B9 (1974) 2989

22) Thorpe, M.F., J. Phys. A11(1978) 955

23) Ono, I., J. Phys. Soc. Jap/Lett. 44 (1978) 1033

24) Dalton, N.W. and Wood, D.W., J. Math. Phys. 10 (1969) 1271

25) Philhours, J., Phys. Rev. B4 (1971) 929

26) Turban, L., J. Phys. C12 (1979) 1479

27) Horiguchi, T., and Morita, T., J. Phys. A13 (1980) L71

28) Fisch, R., PhD Thesis, University of Pennsylvania, USA (1977)

29) Kirkpatrick, S., "La Matière Mal Condensée", pg. 339, Les Houches, ed. by Balian, R., Maynard, R. and Toulouse, G. (North Holland, 1978)

30) Magalhães, A.C.N. de, Tsallis, C. and Schwachheim, G., J. Phys. C13 (1980) 321

31) Zinn-Justin, J., J. Physique 40 (1979) 969

32) Gaunt, D.S. and Guttman, A.J., "Phase Transitions and Critical Phenomena" vol. 3, ed. Domb, C. and Green, M.S. (New York Academic) pg. 181-243 (1974)

33) Sykes, M.F., Gaunt, D.S., Roberts, P.D. and Wyles, J.A., J. Phys. A5 (1972) 640

34) Sykes, M.F., Gaunt, D.S. and Glen, M., J. Phys. A9 (1976)1705

35) Dunn, A.G., Essam, J.W. and Ritchie, D.S., J. Phys. C8 (1975) 4219

36) McKenzie, S., J. Phys. A12 (1979) L185

37) Guttmann, A.J. and Joyce, G.S., J. Phys. A5 (1972) L81

38) Levy, P.M. and Sudano, J.J., Phys. Rev. B18 (1978) 5087

39) Gaunt, D.S. and Sykes, M.F., J. Phys. A6 (1973) 1517

40) Vyssotsky, V.A., Gordon, S.B., Frisch, H.L. and Hammersley, J.M., Phys. Rev. 123 (1961) 1566

41) Ho-Ting-Hun, J. and Oitmaa, J., J. Phys. A8 (1975) 1920

42) Leu, J.A., Betts, D.D. and Elliott, C.J., Can. J. Phys. 47 (1969) 1671

43) Domb, C. and Dalton, N.W., Proc. Phys. Soc. 89 (1966) 859

44) Gaunt, D.S., Sykes, M.F. and McKenzie, S., J. Phys. A12 (1979) 871

45) Moore, M.A., Phys. Rev. B1 (1970) 2238

46) McKenzie, S. and Gaunt, D.S., J. Phys. A13 (1980) 1015

47) Baker, G.A., Phys. Rev. B9 (1974) 4908

48) Ono, I., J. Phys. Soc. Jap. 48 (1980) 24.

CAPTIONS FOR FIGURES AND TABLES

FIG. 1 Parameter h as a function of σ^{-1} ($\sigma = 2d - 1$) for the hypercubic lattices. The broken line is a guide-to-eye curve which has the correct asymptotic behaviour in the $\sigma \to \infty$ limit. Values of h corresponding to other $d = 2,3$ lattices are also indicated. (a) Square (b) 4-8 (c) Non-crossing diagonal square lattice (d) Kagomé (e) Diced (f) 3-12 (g) Asanoha (h) First and second-neighbour square lattice (i) Triangular (j) Honeycomb. See Table 2 for the references that have been used.

FIG. 2 The approximate para (P) -ferro (F) magnetic critical frontier of the first and second-neighbour SC lattice Ising model for $0 \leq J_2/J_1 \leq 1$ obtained by using Dalton and Wood's results[24] and $h = 0.377 \pm 0.044$. The dotted line is a guide-to-eye extrapolation which contains the correct limit for $J_2/J_1 \to \infty$ ($p_c^{FCC} = 0.119 \pm 0.001$).

FIG. 3 The approximate para (P) -ferro (F) -magnetic critical frontier of the first and second-neighbour BCC lattice Ising model for $0 \leq J_2/J_1 \leq 1$ obtained by using Dalton and Wood's results[24] and $h = 0.372 \pm 0.037$. The dotted line is a guide-to-eye extrapolation which contains the correct limit for $J_2/J_1 \to \infty$ ($p_c^{SC} = 0.247 \pm 0.003$).

FIG. 4 The approximate para (P) -ferro (F) -magnetic critical surface (heavy line) of the first, second and third-neighbour FCC lattice Ising model for

$0 \leqslant J_2/J_1 \leqslant 2$ and $0 \leqslant J_3/J_1 \leqslant 2$ which was obtained using Philhours' results[25] and $h = 0.412 \pm 0.026$ (we have also indicated Dalton and Wood's values[24] (x)). The dotted line is a guide-to-eye extrapolation which contains the correct limits for $J_2/J_1 \to \infty$, $J_2/J_3 \to \infty$ ($p_c^{SC} = 0.247 \pm 0.003$) and for $J_3/J_1 \to \infty$, $J_3/J_2 \to \infty$ (we estimate $p_c^{FCC}(3) = 0.054 \pm 0.004$).

TABLE 1 Critical points for isotropic and homogeneous q-state Potts ferromagnets in a set of d- dimensional lattices. In the indicated references appear either the numbers we have quoted or others which immediately imply them. The region delimited by a heavy line contains results that, as far as we know, have not yet been calculated by any other procedure; (...) denotes an exact value. (a) d = 3 lattices (b) $d \geq 4$ lattices.

TABLE 2 Estimates for the values of h in a set of d- dimensional lattices ($d \geq 2$). For $d \geq 3$ we have used the q=1 and q=2 results appearing in Tables 1a and 1b. For the cacti and Bethe lattices an effective dimensionality is indicated; (...) denotes an exact value. Although not strictly two - dimensional the first- and second-neighbour square lattice clearly belongs to the d = 2 class[2].

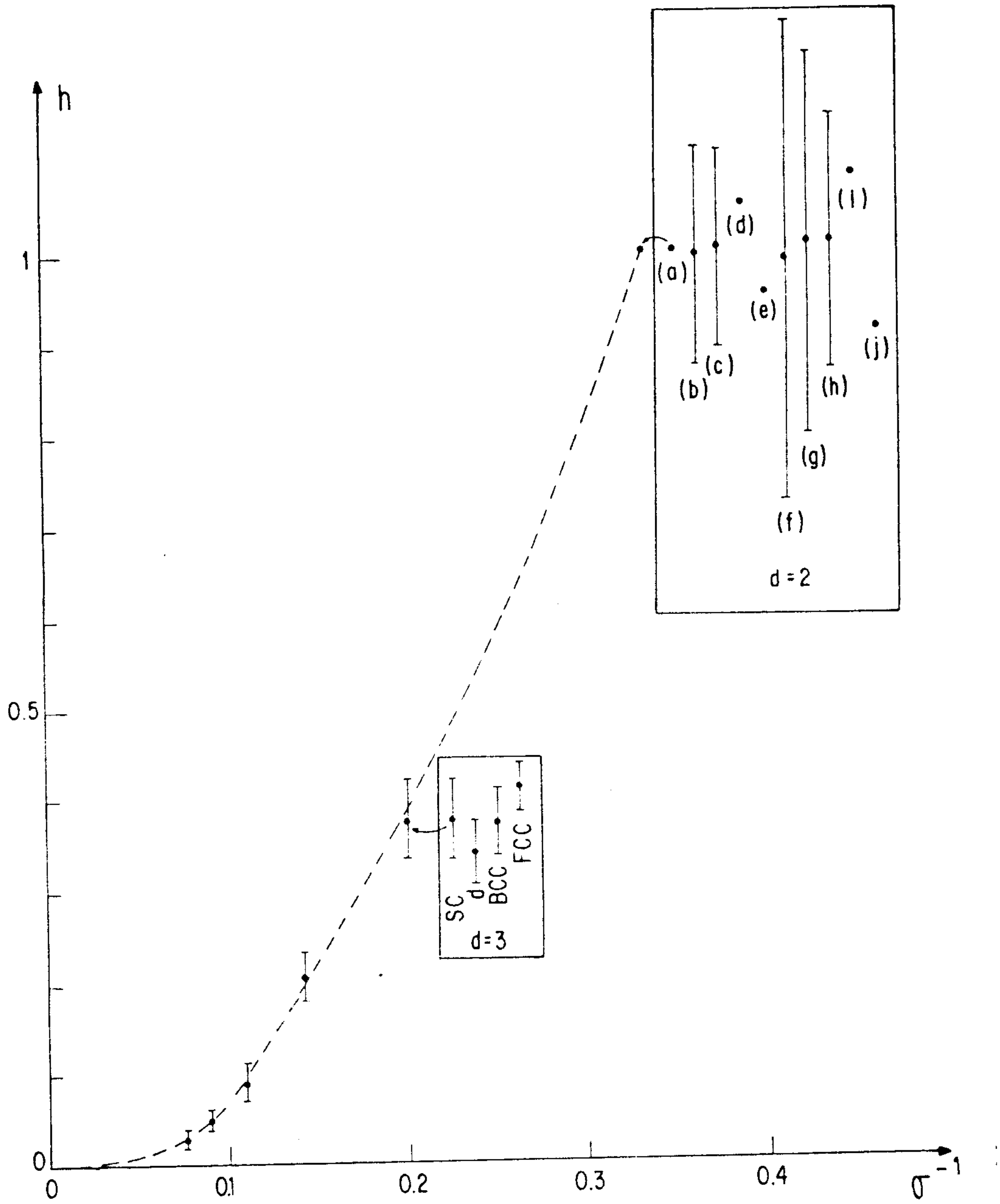
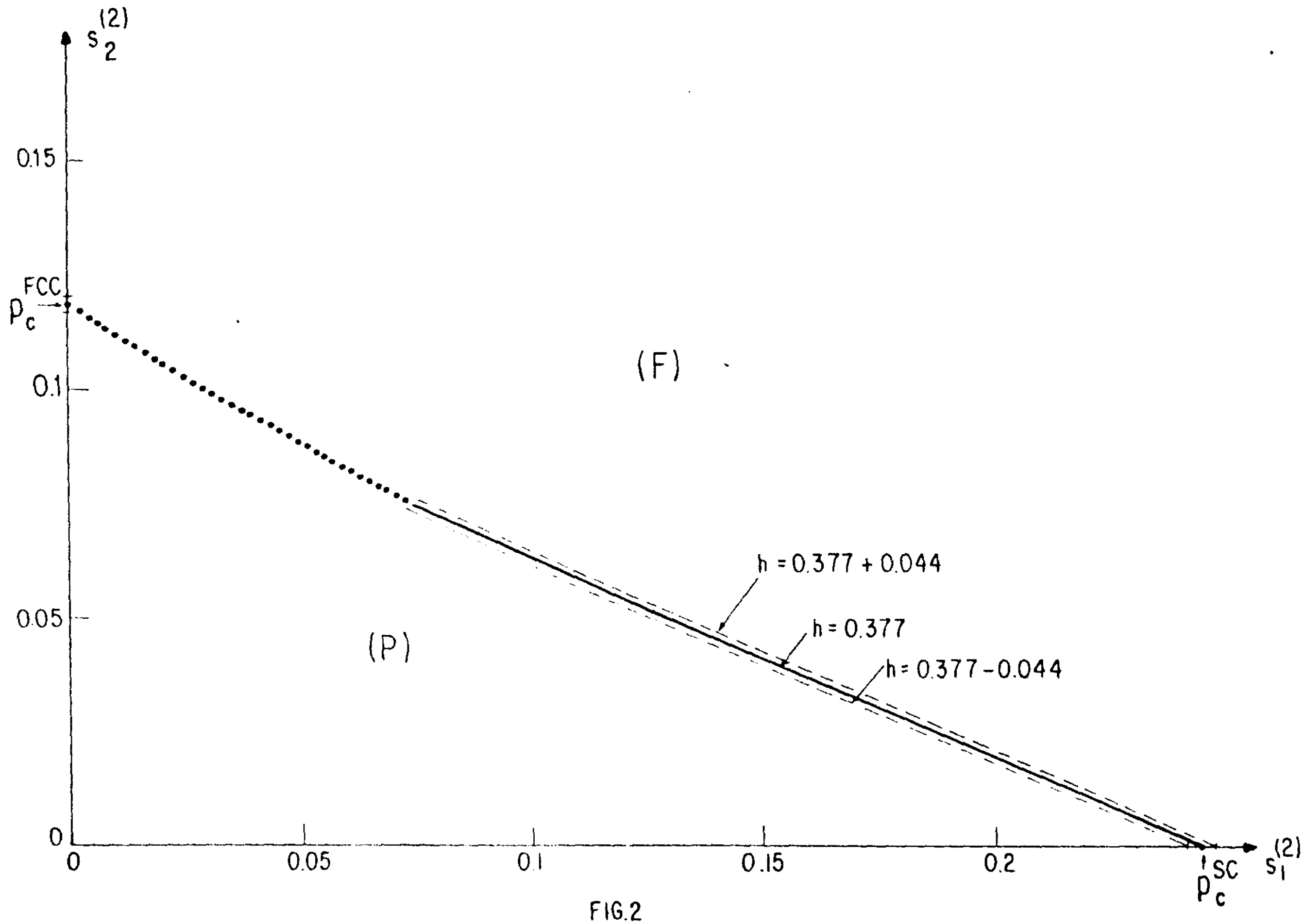
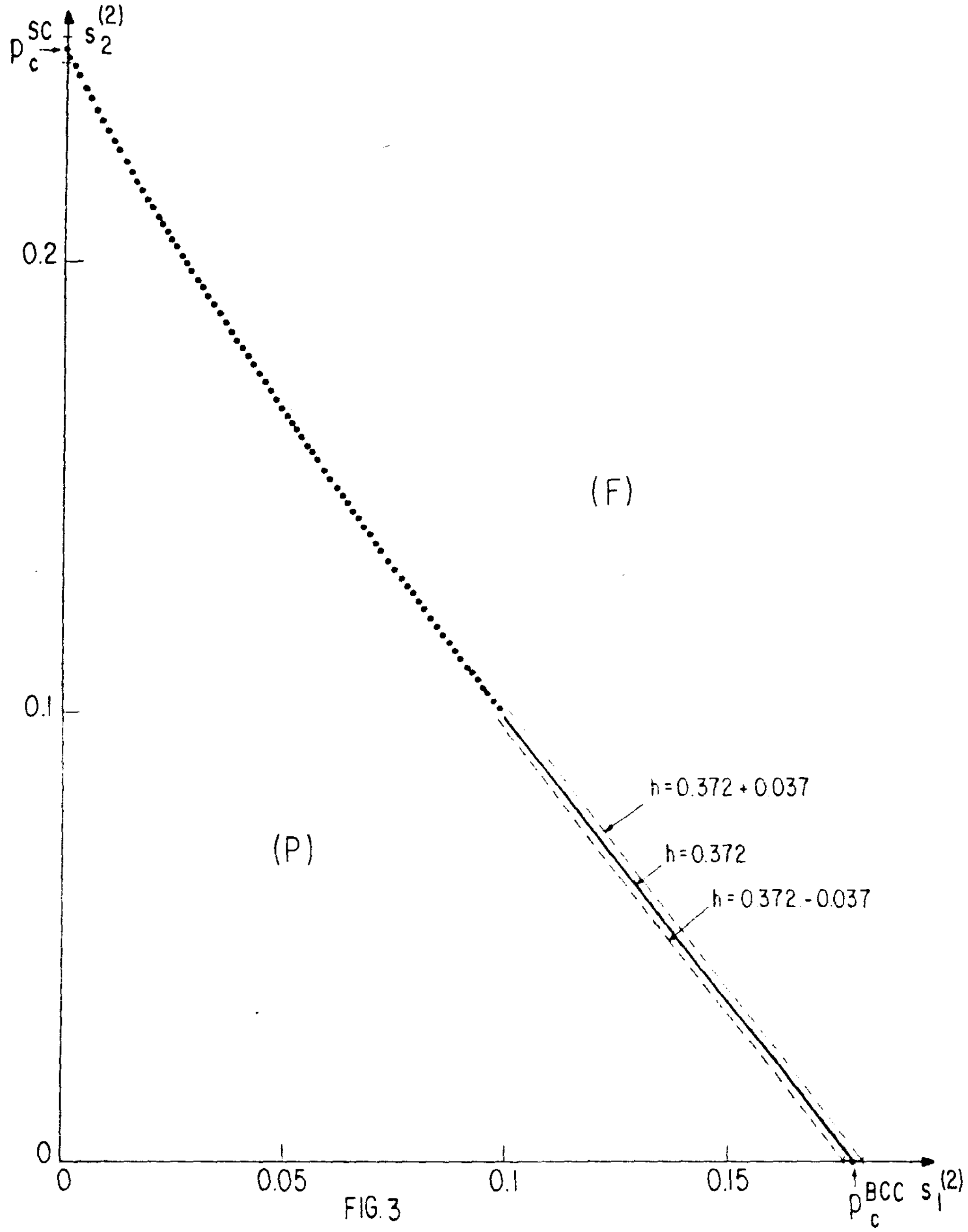


FIG. 1



$s_2^{(2)}$
0.15
p_c^{FCC}
0.1
(F)
0.05
(P)
h = 0.377 + 0.044
h = 0.377
h = 0.377 − 0.044
0
0
0.05
0.1
0.15
0.2
p_c^{SC}
$s_1^{(2)}$

FIG.2



p_c^{SC}
$s_2^{(2)}$
0.2
0.1
0
(F)
(P)
h=0.372+0.037
h=0.372
h=0.372-0.037
0
0.05
0.1
0.15
p_c^{BCC}
$s_1^{(2)}$

FIG. 3

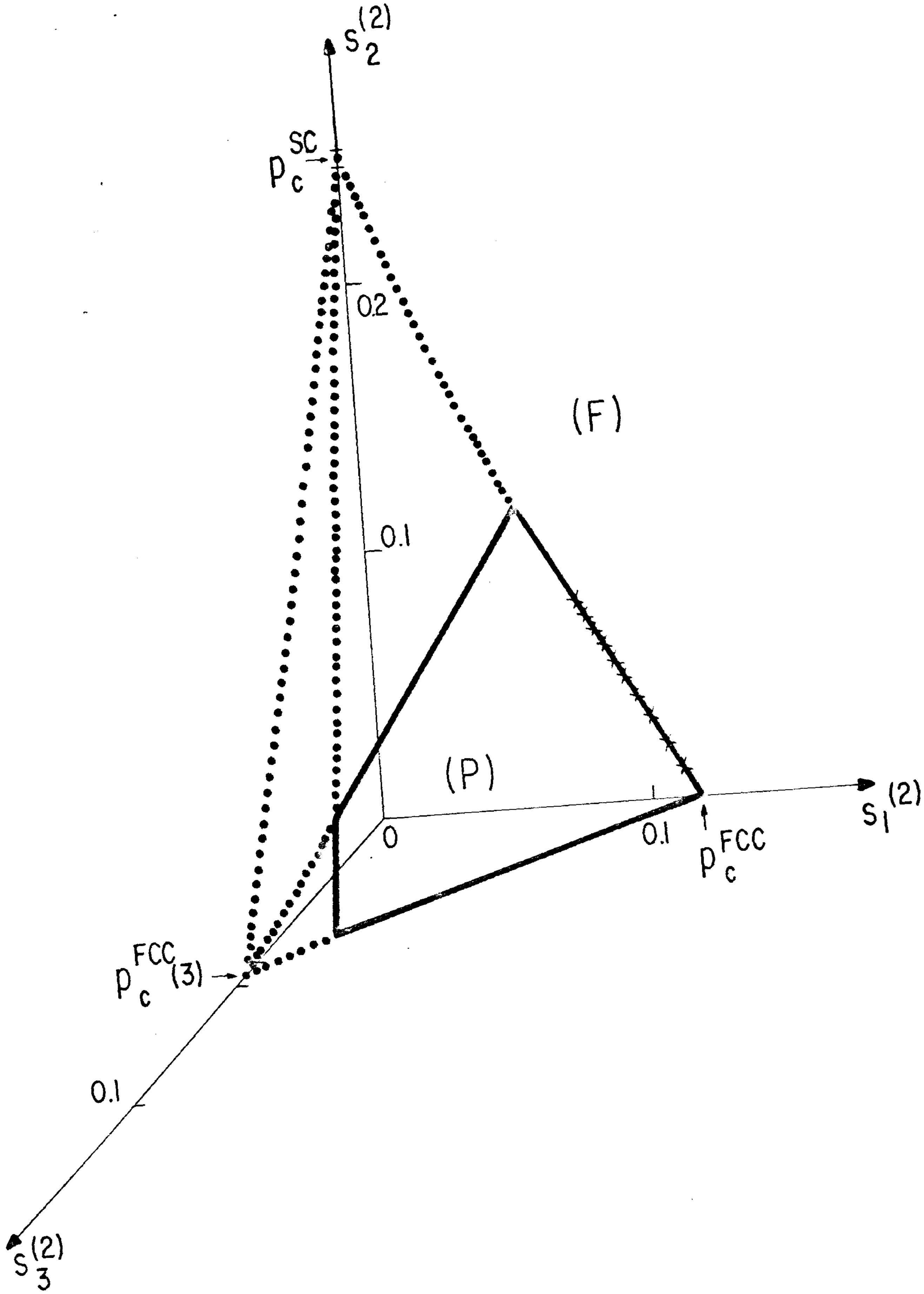


FIG. 4

TABLE 1.a

LATTICE $d=3$	q = 1		q = 2			q = 3		
	$s_c=t_c=p_c$	$\frac{k_BT_c}{J}$	s_c	t_c	$\frac{k_BT_c}{J}$	s_c	t_c	$\frac{k_BT_c}{J}$
1st neighbour SC	0.247±0.003 [19,34] 0.2465±0.0002 [28] 0.2495±0.0005 [29] 0.2526±0.0013 [30]	3.52 ± 0.05 3.533 ± 0.003 3.484 ± 0.008 3.435±0.021	0.247±0.003 0.247±0.003 0.247±0.003 0.247±0.003	0.21811 [31] 0.21814±0.00001 [32] 0.21815 [20] 0.21813 [33]	4.5112 4.5106±0.0002 4.5103 4.5108	0.247±0.003 0.246±0.005 0.246±0.005	0.197±0.007 0.1964 0.19656	5.44±0.18 5.45 [6] 5.4506 [13]
1st neighbour FCC	0.119±0.001 [34] 0.1190±0.0005 [35]	7.89 ± 0.07 7.893 ± 0.035	0.119±0.0015 0.119±0.001 0.119±0.001 0.119±0.001	0.101727 [31] 0.101722±0.000001 0.101716 [37] 0.101740 [33]	9.7962 9.7967±0.0001 [36] 9.7973 9.7950	0.119±0.0015 0.1178±0.0015 0.1175±0.0015	0.090±0.002 0.0890 0.0888	11.55±0.25 11.67 [12] 11.70 [38]
1st neighbour BCC	0.1785±0.0020 [34]	5.086±0.063	0.1785±0.0020 0.1785±0.0020	0.156099 [31] 0.156116 [33]	6.3538 6.3531	0.1785±0.0020	0.140±0.005	7.54±0.26
diamond (d)	0.388±0.005 [34]	2.037±0.034	0.388±0.005 0.388±0.003	0.35374 [31] 0.353806±0.000013 [39]	2.7048 2.70425±0.00010	0.388±0.005	0.328±0.009	3.33±0.09
hcp	0.124±0.005 [40]	7.55±0.33	0.124±0.005	0.107±0.006	9.31±0.53	0.124±0.005	0.096±0.006	10.8±0.7
tetrahedron (cristobalite)	0.263±0.003	3.28±0.04	0.263±0.003	0.23300±0.00001 [41]	4.2130±0.0002	0.263±0.003	0.211±0.007	5.09±0.16
hidrogen peroxide	0.558±0.007	1.225±0.024	0.558±0.007	0.518140 [42]	1.74279	0.558±0.007	0.488±0.013	2.22±0.06
hypertriangular	0.251±0.003	3.46±0.05	0.251±0.003	0.222087 [42]	4.42771	0.251±0.003	0.201±0.007	5.34±0.18
hyperKagomé	0.433±0.006	1.76±0.03	0.433±0.006	0.394384 [42]	2.39819	0.433±0.006	0.365±0.012	2.99±0.10
1st and 2nd neighbour SC	0.075±0.001	12.83±0.18	0.075±0.001 0.075± 0.001	0.06443 [24] 0.06441 [43]	15.499 15.504	0.075±0.001	0.057±0.002	18.0±0.6
1st and 2nd neighbour FCC	0.076±0.001	12.65±0.17	0.076±0.001 0.076±0.001 0.076±0.001	0.06444 [24] 0.06436 [43] 0.0643	15.497 15.516 15.53 [25]	0.076±0.001	0.057±0.002	18.0±0.6
1st and 2nd neighbour BCC	0.099±0.001	9.59±0.10	0.099±0.001 0.099±0.001	0.08578 [24] 0.08571 [43]	11.629 11.639	0.099±0.001	0.076±0.002	13.6±0.4
1st, 2nd and 3rd neighbour SC	0.0505±0.0008	19.30±0.31	0.0505±0.0008	0.0432 [43]	23.1	0.0505±0.0008	0.0382±0.0016	26.6±1.1
1st, 2nd and 3rd neighbour FCC	0.0305±0.0004	32.28±0.43	0.0305±0.0004 0.0302±0.0003	0.0257 [43] 0.02539	38.9 39.37 [25]	0.0305±0.0004	0.0225±0.0008	44.9±1.6
1st, 2nd and 3rd neighbour BCC	0.0501±0.0007	19.46±0.28	0.0501±0.0007	0.0429 [43]	23.3	0.0501±0.0007	0.0380±0.0014	26.8±1.0

TABLE 1b

LATTICE		q = 1		q = 2		
		$s_c = t_c = p_c$	$\frac{k_B T_c}{J}$	s_c	t_c	$\frac{k_B T_c}{J}$
d=4	HSC	0.161 ± 0.0015 [19] 0.1600 ± 0.0002 [28]	5.697 ± 0.058 5.735 ± 0.008	0.161 ± 0.002 0.161 ± 0.002 0.161 ± 0.002	0.14856 ± 0.00003 [44] 0.1487 [45] 0.14877 ± 0.00003 [20]	6.6815±0.0014 6.675 6.6719±0.0014
d=4	HFCC	0.0498 ± 0.0006	19.58 ± 0.24	0.0498±0.0006 0.0498±0.0006	0.04548 ± 0.00002 [46] 0.04549 [45]	21.97 ±0.01 21.97
d=4	HBCC	0.075 ± 0.001	12.83 ± 0.18	0.075 ±0.001 0.075 ± 0.001	0.06889 ± 0.00005 [46] 0.06892 [45]	14.49 ± 0.01 14.487
d=5	HSC	0.118 ± 0.001 [19] 0.1181 ± 0.0002 [28]	7.96 ± 0.07 7.957 ± 0.014	0.118 ± 0.0015 0.118± 0.001	0.11354 ± 0.00002 [47] 0.11354 ± 0.00001 [20]	8.7695±0.0016 8.7695±0.0008
d=6	HSC	0.0941 ± 0.0005 [19] 0.0943 ± 0.0002 [28]	10.12 ± 0.06 10.10 ± 0.02	0.0941±0.001	0.09210 ± 0.00001 [20]	10.827 ± 0.001
d=7	HSC	0.0786 ± 0.0002 [19] 0.0788 ± 0.0002 [28]	12.22 ± 0.03 12.18 ± 0.03	0.0786±0.001	0.0776 [20]	12.86
$d \to \infty$	$(\sigma+1)$-coordinated regular lattice	$\sim \frac{1}{\sigma}$ [19]	$\sim\sigma$	$\sim \frac{1}{\sigma}$	$\sim \frac{1}{\sigma}$ [20]	$\sim\sigma$
$d_{ef}^{-1}=0$	z-coordinated Bethe Lattice	$\frac{1}{z-1}$ [15,16]	$1/\ln\left(\frac{z-1}{z-2}\right)$	$\frac{1}{z-1}$	$\frac{1}{z-1}$ [15,21]	$2/\ln\left(\frac{z}{z-2}\right)$
z-coordinated Square Husimi tree	z=4	0.353933... [16]	2.28910...	0.353933...	0.3460... [23]	2.771...
z-coordinated triangular cactus	z=4	0.4030... [16,22,48]	1.938...	0.4030...	0.3820...	2.4853... [22,48]
z-coordinated triangular cactus	z=6	0.2140... [22]	4.152...	0.2140...	0.2087...	4.7209... [22]
z-coordinated triangular cactus	z=8	0.1480... [22]	6.243...	0.1480...	0.1459...	6.8052... [22]
z-coordinated triangular cactus	z=12	0.0923... [22]	10.32...	0.0923...	0.09167...	10.8777.. [22]

T A B L E 2

LATTICE		h
d = 2	Square [2]	1
	Triangular [2]	1.084...
	Honeycomb [2]	0.913...
	Kagomé [18]	1.049...
	Diced [18]	0.953...
	4-8 [2]	0.996±0.121
	Non-crossing diagonal square lattice [2]	1.003±0.109
	3-12 [2]	0.990±0.264
	Asanoha [2]	1.009±0.211
	1st and 2nd neighbour square lattice [2]	1.010±0.140
d = 3	SC	0.377 ± 0.044
	FCC	0.412 ± 0.026
	BCC	0.372 ± 0.037
	Diamond (d)	0.339 ± 0.034
d = 4	HSC	0.207 ± 0.029
d = 5	HSC	0.0905 ± 0.022
d = 6	HSC	0.0484 ± 0.0127
d = 7	HSC	0.0281 ± 0.0114
$\sigma = 2d-1 \to \infty$	HSC	$\sim \frac{3}{\sigma^2} + \frac{12}{\sigma^3}$
$d_{ef} \to \infty$	Bethe lattice	0
$d_{ef} = 5.5 \pm 0.3$	z=4 square Husimi tree	0.0721 ...
$d_{ef} = 4.0 \pm 0.2$	z=4 triangular cactus	0.193 ...
$d_{ef} = 5.6 \pm 0.4$	z=6 triangular cactus	0.0653 ...
$d_{ef} = 6.7 \pm 0.7$	z=8 triangular cactus	0.0340 ...
$d_{ef} = 8.5 \pm 1.1$	z=12 triangular cactus	0.0151 ...