CBPF-NF-028/02

Lightfront holography and area density of entropy associated with quantum localization on a wedge-horizon

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to be published in Int. J. Phys.A

August 2002

Abstract

The lightfront quantization of the 70s is reviewed in the more rigorous setting of lightfront (LF) restriction of free fields in which the lightfront is considered to be the linear extension of the upper causal horizon of a wedge region. Particular attention is given to the change of localization structure in passing from the wedge to its horizon which results in the emergence of a transverse quantum mechanical substructure of the QFT on the horizon and its lightfront extension. The vacuum fluctuations of QFT on the LF are compressed into the direction of the lightray (where they become associated with a chiral QFT) and lead to the notion of area density of a "split localization" entropy.

To overcome the limitation of this restriction approach and include interacting theories with non-canonical short distance behavior, we introduce a new concept of algebraic lightfront holography (LFH) which uses ideas of algebraic QFT, in particular the modular structure of its associated local operator algebras. In this way the localization properties of LF degrees of freedom including the absence of transverse vacuum fluctuations are confirmed to be stable against interactions. The important universality aspect of lightfront holography is emphasized. Only in this way one is able to extract from the "split-localization" entropy a split-independent additive entropy-like measure of the entanglement of the vacuum upon restriction to the horizon algebra.

PACS: 11.10.-z, 11.30.-j, 11.55.-m

1 Constructive Aims of Lightfront Holography

Lightfront quantum field theory and the closely related p$\rightarrow\infty$ frame method have a long history. The large number of articles on this subject (which started to appear at the beginning of the 70ies) may be separated into two groups. On the one hand there are those papers whose aim is to show that such concepts constitute a potentially useful enrichment of standard local quantum physics [1][2][3], but there are also innumerable attempts to use lightfront concepts as a starting point of more free-floating "effective" approximation ideas in high-energy phenomenology (notably for Bjorken scaling) whose relations to causal and local quantum physics remained unclear or was not addressed.

As will become clear in the next section where we will recall some of the old ideas from a modern perspective, the old lightfront approach was severely limited since in d=1+3 spacetime dimensions its prerequisites were met only in the absence of interactions. Nevertheless algebraic lightfront holography[1] (LFH), which is the subject of this paper, may be viewed as a revitalization of the old approach with new concepts. The aim of the old approach (never satisfactorily achieved) was the simplification of dynamics by encoding some of its aspects into more sophisticated kinematics; this is precisely what LFH aims to achieve, but this time without suffering from short distance limitations which eliminate interactions and with the full awareness of the locality issue and the problem of reconstruction of the original theory (or family of original theories) which have the same holographic projection.

Whereas the old approach amounted to the lightfront restriction of pointlike fields (which also caused the mentioned limitation), the LFH reprocesses the original fields first into nets of algebras which by algebraic holography is then converted into a net of operator algebras which is indexed by regions on the lightfront. This net turns out to be a "generalized chiral net" (a chiral net extended by a vacuum-polarization-free transverse quantum mechanics) with a 7-parametric symmetry group which corresponds to a subgroup of the 10-parametric Poincaré group of the ambient theory. It also possesses additional higher symmetries which originate from the conformal covariance of the chiral LF theory. The latter amount to diffuse-acting automorphisms in the ambient theory ("fuzzy symmetries"). This does not only include the rigid rotation which belongs to the Moebius group with L_0 as its infinitesimal generator, but also all higher diffeomorphism of the circle $Diff(S^1)$.

It is important to realize that the absence of a direct relation between the ambient fields A and those generating the lightfront net A_{LF} in the presence of interactions

$$A_{LF}(x) \neq A(x)|_{LF} \tag{1}$$

is the prize to pay for the enormous holographic simplification of interacting quantum field theories. The inverse holography i.e. the classification of ambient theories which belong to one LFH class (including the action of its 7-parametric symmetry group) is the main unsolved problem.

The LFH does not accomplish dynamical miracles, but as already mentioned, it shifts the separating line between kinematical substrate and dynamical actions in a helpful way by placing more structure (as e.g. compared to the canonical formalism) onto the kinematical side which is described by the holographic projection. Last not least it avoids having an "artistic" starting point as e.g. canonical commutation relations or functional integral representations which the physical (renormalized) theory cannot maintain in the presence of interactions. Instead it fits well into the spirit of Wightman QFT or AQFT where the result obtained at the end of a computation do verify the requirements at the start. Among the reasonably easy structural consequences of LFH is the surface proportionality of localization entropy associated with a causal horizon. This will be the subject of the previous to last section.

Although the connection between the local net on the lightfront and that on the full ambient spacetime turns out to be quite nonlocal (in contradistinction to the "AdS-CQFT holography" which still maintains

[1]The term "holography" was coined by 't Hooft [4] in order to describe his intuitive idea about the organization of degrees of freedom in the presence of event horizons for QFT in CST. The present setting of LFH is algebraic QFT (AQFT) in Minkowski spacetime.

many relative local aspects [5]), the modular localization approach succeeds to convert this holographic idea into rigorous mathematics. With these remarks on what is meant by LFH as compared to many other meanings to the word "holography" in the recent literature, we conclude our historical and verbal remarks and pass to the mathematical description[2].

2 Elementary facts on pointlike fields restricted to the lightfront

For some elementary observations we turn to the simple model of a d=1+1 massive free field (used in the second section)

$$A(x) = \frac{1}{\sqrt{2\pi}} \int \left(e^{-ipx}a(\theta) + e^{ipx}a^*(\theta)\right) d\theta \tag{2}$$
$$p = m(ch\theta, sh\theta)$$

where for convenience we use the momentum space rapidity description. In order to get onto the light ray $x_- = t - x = 0$ in such a way that $x_+ = t + x$ remains finite, we approach the $x_+ > 0$ horizon of the right wedge $t^2 - x^2 < 0$, $x > 0$ by taking the $r \to 0$, $\chi = \hat{\chi} - ln\frac{r}{r_0} \to \hat{\chi} + \infty$ limit in the x-space rapidity parametrization

$$\begin{aligned} &x = r(sh\chi, ch\chi),\ x \to (x_- = 0, x_+ \geq 0,\ finite) \\ &A(x_+, x_- \to 0) \equiv A_{LF}(x_+) = \frac{1}{\sqrt{2\pi}} \int \left(e^{-ip_-x_+}a(\theta) + e^{ip_-x_+}a^*(\theta)\right) d\theta \\ &= \frac{1}{\sqrt{2\pi}} \int \left(e^{-ip_-x_+}a(p) + e^{ip_-x_+}a^*(p)\right) \frac{dp}{|p|} \end{aligned} \tag{3}$$

where the last formula exposes the limiting $A_{LF}(x_+)$ field as a chiral conformal (gapless P_- spectrum) field; the mass in the exponent $p_-x_+ = mr_0e^{\theta}e^{-\hat{\chi}}$ is a dimension preserving parameter which (after having taken the limit) has lost its physical significance of a mass gap (the physical mass is the gap in the $P_- \cdot P_+$ spectrum).

Since this limit only effects the numerical factors and not the Fock space operators $a^{\#}(\theta)$, we expect that there will be no problem with the horizontal restriction i.e. that the formal method (the last line in 3) agrees with the more rigorous result using smearing functions. Up to a fine point which is related to the bad infrared behavior of a scalar chiral $dimA = 0$ field, this is indeed the case. Using the limiting χ-parametrization we see that for the smeared field with $supp\tilde{f} \in W$, $\tilde{f}$ real, one has the identity

$$\begin{aligned} &\int A(x_+, x_-)\tilde{f}(x)d^2x = \int_C a(\theta)f(\theta) = \int A_{LF}(x_+)\tilde{g}(x_+)dx_+,\ \tilde{g}\ real \\ &\tilde{f}(x) = \int_C e^{ip(\theta)x}f(\theta)d\theta,\ \tilde{g}(x_+) = \int e^{ipx_+}g(p)\frac{dp}{|p|} = \int_C f(\theta)e^{ip_-(\theta)x_+}d\theta \end{aligned} \tag{4}$$

These formulas, in which a contour C appears, require some explanation. The on-shell character of free fields restricts the Fourier transformed test function to their mass shell values with the backward mass shell corresponding to the rapidity on the real line shifted downward by $-i\pi$

$$f(p)|_{p^2=m^2} = \begin{cases} f(\theta),\ p_0 > 0 \\ f(\theta - i\pi),.\ p_0 < 0 \end{cases}$$

and the wedge support property is equivalent to the analyticity of $f(z)$ in the strip $-i\pi < \mathrm{Im}\, z < 0$. The integration path C consists of the upper and lower rim of this strip and corresponds to the negative/positive frequency part of the Fourier transform. By introducing the test function $\tilde{g}(x_+)$ which is supported on

[2]The present work combines and supersedes previous reports by the author: hep-th/0106284, 0108203, 0111188.

the halfline $x_+ \geq 0$, it becomes manifest that the smeared field on the horizon rewritten in terms of the original Fourier transforms must vanish at $p = 0$ as required by L_1-integrability of

$$f(p)|_{p^2=m^2, p_0>0} \frac{dp}{\sqrt{p^2+m^2}} = f(\theta)d\theta \equiv g(\theta)d\theta = g(p)|_{p^2=0, p_0>0} \frac{dp}{|p|} \tag{5}$$
$$\curvearrowright g(p=0) = 0, \; or \int \tilde{g}(x_+)dx_+ = 0$$

with a similar formula for negative p_0 and the corresponding θ-values at the lower rim. This infrared restriction is typical for spinless free fields with $dimA = 0$. The equality of the f-smeared $A(x)$ fields with the g-smeared $A_+(x_+)$ leads to the vanishing of $g(p)$ at the origin and finally to the equality of the Weyl operators and hence of the generated operator algebras

$$\begin{aligned} \mathcal{A}(W) &= \mathcal{A}(R_+) = \mathcal{A}(R_-) \\ \mathcal{A}(W) &= alg\left\{ e^{iA(f)} |\, supp\tilde{f} \subset W \right\} \\ \mathcal{A}(R_+) &= alg\left\{ e^{iA_{LF}(g)} |\, supp\tilde{g} \subset R_+, \int \tilde{g}dx_+ = 0 \right\} \end{aligned} \tag{6}$$

Here the equality with $\mathcal{A}(R_-)$ expresses the fact that the lower horizon of the wedge would have led to the same global algebra. This equality is the quantum version of the classical propagation property of characteristic data on the upper or lower lightfront of a wedge. With the exception of $d = 1+1$, $m = 0$, the classical amplitudes inside the causal shadow W of R_+ are uniquely determined by the upper or lower lightfront data.

Although the above argument shows that this classical aspect prevails, it would be incorrect to think that the global identity of the algebras persists on the local level i.e. that there exists a region in W whose associated operator algebra corresponds to a $\mathcal{A}(I)$ algebra when $I \in R_+$ is a finite interval. The net structure on W and R_+ is very different; the subalgebras localized in intervals $\mathcal{A}(I_\pm)$ on the two half-lines $R_\pm$ (the upper/lower horizons W) have no local relation to subalgebras $\mathcal{A}(\mathcal{O})$ of $\mathcal{A}(W)$ and vice versa. As a result of the global identity (6), the position of compactly localized algebras in one net is diffuse ("fuzzy") relative to the net structure of the others. That a finite interval on R_+ does not cast a 2-dimensional causal shadow does not come as surprise since even in the classical setting a causal shadow is only generated by characteristic data which have at least a semi-infinite extension. Related to this is the fact that the opposite lightray translation

$$\begin{aligned} &AdU_-(a)\mathcal{A}(R_+) \subset A(R_+) \\ &U_-(a) = e^{-iP_- a} \end{aligned} \tag{7}$$

is a totally "fuzzy" endomorphism of the $A(R_+)$ net; whereas in the setting of the $\mathcal{A}(W)$ net spacetime indexing $AdU_-(a)$ is a geometric map.

It is very important to notice that even in the free case the horizontal limit is conceptually different from the scale invariant massless limit. The latter cannot be performed in the same Hilbert space since the $m \to 0$ limit needs a compensating $\ln m$ term in the momentum space rapidity θ in the argument of the operators $a^\#(\theta)$ whereas in the horizon limit (3) was only effecting the c-number factors.

There is however no problem of taking this massless limit in correlation functions if one uses spacetime smearing functions whose integral vanishes (or $f(p=0) = 0$). The limiting correlation functions define via the GNS construction a new Hilbert space which contains two chiral copies of the conformal $dimA = 0$ field corresponding to the right/left movers. This difference between the scaling limit and the lightray holography limit for free fields is easily overlooked since the conformal dimensions of the resulting fields are in this case the same and the only difference is that the scaling limit leads to a 2-dimensional conformal theory which decomposes into two independent chiral algebras. We will see that in the case of interacting theories the appropriately defined holographic projection onto the lightfront is different from the scaling limit by much more than just a doubling.

There arises the question whether the chiral theories originating from the lightray restriction are intrinsically different from those which are obtained by factorizing d=1+1 conformal theories into its two chiral components. At least in the present example this is not the case; we can always extend a free chiral field independent of its origin to a d=1+1 massive field by defining an additional action $U_-(a)$ which creates a phase factor $e^{ip_+x_-}$ on the $a^\#(p)$ appearing in (3) without enlarging the Hilbert space. The situation is reminiscent of Wigner's finite helicity massless representations which allow an extension from the Poincaré- to the conformal- symmetry without enlargement of the one-particle space.

Passing now to the higher dimensional case we notice, that by introducing an effective mass which incorporates the transverse degrees of freedom on the upper lightfront horizon of W, the previous arguments continue to hold

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ip_+x_- - ip_-x_+ + ip_\perp x_\perp} a(p) + h.c.\right) \frac{d^3p}{2\omega},\ x_\pm = x^0 \pm x^1 \tag{8}$$
$$p = (m_{eff}ch\theta, m_{eff}sh\theta, p_\perp),\ m_{eff} = \sqrt{m^2 + p_\perp^2},\ p_\pm = \frac{p^0 \pm p^1}{2}$$

The limiting field can again be written in terms of the same Fock space creation/annihilation operators and as before the (effective) mass looses its spectral role and becomes a pure engineering scale parameter

$$A_{LF}(x_+, x_\perp) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ip_-x_+ + ip_\perp x_\perp} a(p) + h.c.\right) \frac{dp_-}{2\left|p_-\right|} d^2p_\perp \tag{9}$$
$$A_{LF}(gf_\perp) = \int a^*(p_-, p_\perp) g(p_-) f_\perp(x_\perp) \frac{dp_-}{2\left|p_-\right|} d^2p_\perp + h.c.$$

where in the second line the resulting operator is written in terms of a dense set of test functions which factorize in a longitudinal (along the lightray) and a transverse part and where the longitudinal part may be again brought into the rapidity space form involving a path C. The dependence of the longitudinal part on the transverse momenta is concentrated in the effective longitudinal mass which in turn only enters via a scale-setting factor in $p_-x_+ = m_{eff}e^\theta e^{-\chi}$ and is therefore extremely simple, even simpler than the mass dependence of a generalized free field. In fact the best way to formulate the resulting structure on the lightfront is to say that the longitudinal structure is that of a chiral QFT (with the typical vacuum polarization leading to long range correlations) whereas transversely it is quantum mechanical i.e. free of vacuum polarization and the ensuing correlations, as evidenced by the form of the inner product

$$\langle A_{LF}(gf_\perp) A_{LF}(g'f'_\perp)\rangle = \int \bar{g}(p) g'(p) \frac{dp}{2\left|p\right|} \int \bar{f}_\perp(p_\perp) f'_\perp(p_\perp) d^2p_\perp$$
$$\left[A_{LF}(x_+, x_\perp), A_{LF}(x'_+, x'_\perp)\right] = i\Delta(x_+ - x'_+)_{m=0}\delta(x_\perp - x'_\perp) \tag{10}$$

i.e. the commutation of the transverse part is like that of Schrödinger field. In fact the analogy to QM is much stronger since the vacuum does not carry any transverse correlation at all, a fact which can be best seen in the Weyl generators

$$\langle W(g, f_\perp) W(g', f'_\perp)\rangle = \langle W(g, f_\perp)\rangle \langle W(g', f'_\perp)\rangle\ \ if\ suppf \cap suppf' = \emptyset \tag{11}$$
$$W(g, f_\perp) = e^{iA_{LF}(gf_\perp)}$$

i.e. the vacuum behaves like a quantum mechanical vacuum with no correlations in the transverse direction[3]. To make this relation with transverse QM complete, we will now show that, as a result of loss of vacuum correlations, there is also a Galilei group acting on these transverse degrees of freedom.

[3]The reader should remind himself that LF-restriction refers to the full free field operators and does not maintain the original spacelike correlations between spacelike separated points.

For this it is helpful to understand the symmetry group of the lightfront restriction. It is not difficult to see that it consists of a 7-parametric subgroup of the 10-parametric Poincaré group; besides the longitudinal lightray translation and the W-preserving L-boost (which becomes a dilation in lightray direction on the light front) there are two transverse translation and one transverse rotation. The remaining two transformations are harder to see; they are the two "translations" of the Wigner little group of the lightray of the lightfront. The little group is isomorphic to the 3-parametric Euclidean group in two dimensions. As a subgroup of the 6-parametric Lorentz group it consists of a rotation around the spatial projection of the lightray and two "translations" (in the Euclidean setting) which turn out to be specially tuned combinations of L-boosts and rotations which tilt the edge of the wedge in such a way that it stays inside the lightfront but changes its angle with the lightray. This two-parametric abelian subgroup corresponds in the covering $SL(2,C)$ description of the Lorentzgroup to the matrix

$$\begin{pmatrix} 1 & a_1 + ia_2 \\ 0 & 1 \end{pmatrix} \tag{12}$$
$$G_i = \frac{1}{\sqrt{2}}(M_{it} + M_{iz}),\ i = x, y$$

where in the second line we have written the generators in terms of the Lorentz-generators $M_{\mu\nu}$ so that the above interpretation in terms of a combination of boosts and rotations is obvious. The velocity parameter of the Galilei transformations in the $x_\perp$-x_+ variables in terms of the $a_i, i = 1, 2$ can be obtained from the $SL(2,C)$ formalism.

The important role of this Galilei group in the partial return to quantum mechanics as *the* simplifying aspect of "lightfront holography" cannot be overestimated. In the algebraic LFH the lightray translation and dilation set the longitudinal net structure whereas the Galilei transformations are indispensable for creating the transverse localization structure of the LFH algebra.

Note that as in the 2-dimensional example the physical particle spectrum is not yet determined by these 7 generators; one rather needs the action of the x_- lightlike generator normal to the lightfront in order to obtain the physical mass operator of the ambient theory.

For free Bose fields (as for certain more general chiral fields) there is no problem to cast the above pointlike formalism into the setting of bounded operator algebras by either using the spectral theory of selfadjoint unbounded operators or by Weyl-like exponentiation (see below). For Fermi fields the test function smearing suffices to convert them into bounded operators; in this case one can elevate the above observations on smeared fields directly into properties of spacetime-indexed nets of operator algebras.

There exists however a disappointing limitation for this lightfront restriction formalism for pointlike interacting fields which forces one to adopt the operator-algebraic method. Namely the pointlike restriction method suffers from the same shortcomings which already affected the canonical equal time formalism: with the exception of some superrenormalizable interactions in low dimensional spacetime there are no interacting theories which permit a restriction to the lightfront or to equal times. Fields of strictly renormalizable type (to which all Lagrangian fields used in d=1+3 particle physics belong) are outside the range of the above restriction formalism. In fact a necessary condition for a restriction can be abstracted from the two-point function and consists in the finiteness of the wave function renormalization constants Z which in the non-perturbative setting amounts to the convergence of the following integral over the Kallen-Lehmann spectral function

$$Z \simeq \int \rho(\kappa^2) d\kappa^2 < 0 \tag{13}$$
$$\langle A(x)A(0)\rangle = \int i\Delta^{(+)}(x,\kappa^2)\rho(\kappa^2)d\kappa^2$$

The operator algebra approach overcomes this restriction by bypassing the problem of "bad pointlike coordinatization". It uses the "causal shadow property" i.e. the requirement that the (weakly closed) operator algebra associated with a simply connected convex spacetime region is equal to the algebra of its causal completion[4] (causal shadow). In this way nontrivial operator algebras also become associated with certain lower dimensional regions. For example a semiinfinite strip on the lightfront in lightray direction casts a causal shadow which is simply the 4-dim. causal completion i.e. the 4-dim. slab which this 3-dim. strip cuts into the ambient space. A special case of this is the characteristic initial value problem (Cauchy problem with lightfront data) for which the causal (upper) horizon H_W and the wedge W which have identical algebras. Of course the net on the lightfront also contains regions with a finite longitudinal extension which do not cast causal shadows (independent of whether their transverse direction is compact or extends to infinity). The algebras of those regions do not correspond to algebras in the ambient net; they have to be constructed by modular methods (modular inclusions, modular intersections) which will be presented in the next section.

Despite the totally different nature of the modular LFH method in the formulation of an algebraic lightfront holography of the next section, most of the structural statements about algebraic nets obtained by the method of LF-restriction of pointlike fields remain valid. Considered as a QFT in its own right the LFH has some unusual properties which are a consequence of the fact that the lightfront is not belonging to the family of globally hyperbolic spacetime manifolds to which one usually restricts field theoretic considerations (but precisely this makes them such useful auxiliary QFTs). Not only do longitudinal compact regions not cast any causal shadows into the ambient Minkowski spacetime, but there are also no such shadows (and related Cauchy propagation) inside the lightfront. The related transverse quantum mechanical behavior without vacuum polarization has the consequence that algebras whose transverse localization does not overlap admit a tensor factorization, just like the inside/outside tensor factorization in Schroedinger theory in the multiplicative second quantization formulation[5]. As a consequence, the application of subalgebras with finite transverse extension to the vacuum does not lead to dense subspaces of the Fock space; in fact this is the only known case where the Reeh-Schlieder properties are violated in a nontrivial way (the cyclically generated space is a genuine subspace). Only for algebras with arbitrary longitudinal extension whose transverse extension remains unlimited maintain the cyclicity and separating property of the vacuum. As far as I am aware, this is the only known case of a partial return to QM within QFT.

The crucial operator-algebraic property which is responsible for this somewhat unexpected state of affairs is the existence of positive lightlike translations[6] $U_{e_+}(a)$ in lightlike strip (whose causal shadows are lightlike slabs) algebra $\mathcal{A}(l\text{-}strip)$. Let $A \in \mathcal{A}(l\text{-}strip)$ and $A' \in \mathcal{A}(l\text{-}strip)'$. Since $U_{e_+}(a)$ acts on both the algebra and its commutant (associated with the complement of the strip within the lightfront) as an automorphism, we obtain for the vacuum expectation values

$$\left\langle 0 \left| A U_{e_+}(a) A' \right| 0 \right\rangle = \left\langle 0 \left| A' U_{e_+}(a)^* A \right| 0 \right\rangle \tag{14}$$

But according to the positivity of the translations this requires the function to be a boundary value of an analytic function which is holomorphic in the upper as well as the lower halfplane. Due to the boundedness the application of Liouville's theorem yields the constancy in a. The cluster property, which in the weak form as it is needed here also applies to infinite lightlike separations, leads to

$$\curvearrowright \langle 0 \left| A A' \right| 0 \rangle = \langle 0 \left| A \right| 0 \rangle \langle 0 \left| A' \right| 0 \rangle \tag{15}$$

which is the desired tensor factorization. By successive application of this argument to a strip-subalgebra of the commutant, the lightfront algebra can be made to factorize into an arbitrary number of nonoverlapping strip algebras. In fact for those lightfront algebras which are associated with the restriction of free field,

[4] The extension by spacelike "caps" of a timelike interval is already a consequence of the spectrum condition and spacelike commutativity.

[5] In the standard quantum mechanical formulation the perfect statistical independence between inside/outside corresponds to the additive decomposition of the Hilbert space.

[6] This argument is analogous to the proof that certain algebras have translational invariant centers [6][2] as in the case of the proof of the factorial property of wedge algebras [8]. The loss of correlations results from the fact that lightlike translation act in a two-sided way on strips.

the two-sided strip algebras can easily be seen to be type I_∞ factors i.e. the full lightfront algebra tensor-factorizes into full strip algebras

$$\begin{aligned}\mathcal{A}(LF) &= \mathcal{A}(M^{(3,1)}) = B(\mathcal{H}) \\ &= \bigotimes_i \mathcal{A}(LF_i) = \bigotimes_i B(H_i) \\ \mathcal{H} &= \bigotimes_i \mathcal{H}_i\end{aligned} \tag{16}$$

Algebras with smaller longitudinal localizations are imbedded as unique hyperfinite type III_1 factors in suitable strip algebras $B(H_i)$. Subalgebras with a semi-infinite or finite longitudinal extension which are associated with non-overlapping strip algebras inherit this factorization; in fact they fulfill a Reeh-Schlieder theorem within the factor spaces associated with the respective strip algebras. The transverse correlation-free factorization for arbitrarily small strips in lightray direction written in terms of pointlike fields has the form (10) where in the interacting case the chiral fields A_{LF}^{ch} may be any conformal fields with (half)integer scale dimensions. These arguments are not affected by interactions and therefore it is helpful to collect them in form of a structural theorem (all algebras are weakly closed i.e. von Neumann algebras)

Theorem 1 *A subalgebra $\mathcal{A}$ of $B(H)$ admitting a two-sided lightlike translation with positive generator is of type I, i.e. splits $B(H)$ as $B(H) = \mathcal{A} \otimes \mathcal{A}'$*

The two-sidedness of the lightlike positive energy translations is important; if they act only one-sided (i.e. as an algebraic endomorphism) as e.g. in the case of the wedge algebras, one can only conclude that the center is translation-invariant and agrees with the center of the full algebra; in this case there is no tensor-factorisation and the algebras turn out to be of the same kind as typical sharply localized algebras, namely hyperfinite type III_1 factors [8].

With these remarks which (as shown in the next section) continue to apply in the presence of interactions, we prepared at least intuitively the ground in favor of area laws for transverse additive quantities as e.g. the localization-entropy of a horizon. Later this problem will be considered in more detail.

The restriction for free fields can also be carried out for the more interesting case of the lower rotational symmetric causal horizon of a symmetric double cone. Again the analogous restriction limit (with the lower apex placed at the origin, $r_+ = t + r$ plays the role of x_+) maintains the creation and annihilation operator structure and the Hilbert space and leads to a conformal invariant limit from which the original massive field may be reconstructed via the application of suitable symmetries which lead away from the horizon. In this case there is no geometric modular theory (no Killing vector) for the massive free theory, however the massless restriction to the horizon acquires a geometric modular group in form of a subgroup of a double cone-preserving conformal transformation. Similar to the LF situation, the algebra on the (lower) horizon $H(\mathcal{C})$ is equal to the double cone algebra (but with different subnet structure)

$$\mathcal{A}(H(\mathcal{C})) = \mathcal{A}(\mathcal{C}) \tag{17}$$

This phenomenon which leads to an enlarged symmetry in the same Hilbert space has been termed "symmetry-enhancement" [7]. As a result of the equality two algebras and the shared Hilbert space one could also say that a diffuse acting modular symmetry becomes geometric (a diffeomorphism) on the horizon. Unfortunately it is presently not known how to derive this result for double cones in the presence of interactions when the method of restricting pointlike fields to the horizon breaks down [7].

[7]There is presently no formulation of modular inclusions (see next section) for a diffuse acting modular group which restricts to the smaller algebra as a diffuse compression of the smaller region.

3 Algebraic holography and modular localization

The algebraic construction for interacting theories with trans-canonical Kallen-Lehmann spectral functions starts from the position of a wedge algebra $\mathcal{A}(W)$ within the algebra of all operators in Fock space $B(\mathcal{H})$ and the action of the Poincaré group on $\mathcal{A}(W)$. It is based on the modular theory of operator algebras and adds two new concepts: *modular inclusions* and *modular intersections.* Since both concepts have already received attention in the recent literature on algebraic QFT, we will limit ourselves to remind the reader of the relevant definitions and theorems (with a formulation which suits our purpose) before commenting on them and applying them to the lightfront holography.

Definition 2 *(Wiesbrock, Borchers [8]) An inclusion of operator algebras ($\mathcal{A} \subset \mathcal{B}, \Omega$) is "modular" if ($\mathcal{A}$,$\Omega$), ($\mathcal{B}$,$\Omega$) are standard and $\Delta_{\mathcal{B}}^{it}$ acts for (t<0) as a compression on $\mathcal{A}$*

$$Ad\Delta_{\mathcal{B}}^{it}\mathcal{A} \subset \mathcal{A} \tag{18}$$

A modular inclusion is standard if the relative commutant ($\mathcal{A}' \cap \mathcal{B}, \Omega$) is standard. If the sign of t for the compression is opposite it is advisable to add this sign and talk about a $\pm$modular inclusion.

Modular inclusions are different from the better known inclusions which arise in the DHR superselection theory associated with internal symmetries in quantum field theory. The latter are characterized by the fact that they possess conditional expectations [9]. The prototype of a conditional expectation in the conventional formulation of QFT in terms of charge carrying fields is the projection in terms of averaging over the compact internal symmetry group with its normalized Haar measure. If $U(g)$ denotes the representation of the internal symmetry group we have

$$\mathcal{A} = \int AdU(g)\mathcal{F}d\mu(g) \tag{19}$$
$$E : \mathcal{F} \xrightarrow{\mu} \mathcal{A}$$

i.e. the conditional expectation E projects the (charged) field algebra $\mathcal{F}$ onto the (neutral) observable algebra $\mathcal{A}$.

Modular inclusions have no conditional expectations. This is the consequence of a theorem of Takesaki [10] which states that the existence of a conditional expectation for an inclusion between two noncommutative algebras (in standard position with respect to the same vector) is equivalent to the modular group of the smaller being the restriction of that of the bigger algebra. Since a genuine modular compression (endomorphism) excludes the modular group of the smaller being the restriction of that of the bigger algebra there can be no projection E. Nevertheless the modular inclusion situation may be considered as a generalization of the situation covered by the Takesaki theorem.

The main aim of modular inclusion is to generate spacetime symmetry and nets of spacetime indexed algebras which are covariant under these symmetries. ¿From the two modular groups $\Delta_{\mathcal{B}}^{it}, \Delta_{\mathcal{A}}^{it}$ one can form the translation-dilation group with the commutation relation $\Delta_{\mathcal{B}}^{it}U(a) = U(e^{-2\pi t}a)\Delta_{\mathcal{B}}^{it}$ and a system of local algebras obtained by applying these "symmetries" to the relative commutant $\mathcal{A}' \cap \mathcal{B}$ which may be combined to a possibly new algebra $\mathcal{C}$

$$\mathcal{C} \equiv \overline{\bigcup Ad\Delta_{\mathcal{B}}^{it}(\mathcal{A}' \cap \mathcal{B})} \tag{20}$$

In general $H_{\mathcal{C}} = \overline{\mathcal{C}\Omega\mathcal{C}} \subset H_{\mathcal{B}} = \mathcal{B}\otimes \equiv H$, and whereas the modular groups $\Delta_{\mathcal{B}}^{it}, \Delta_{\mathcal{A}}^{it}$ of the inclusion $\mathcal{A} \subset \mathcal{B}$ are different, the $\mathcal{C} \subset \mathcal{B}$ inclusion leads to a Takesaki situation with $\Delta_{\mathcal{C}}^{it} = \Delta_{\mathcal{B}}^{it}|_{H_{\mathcal{C}}}$ with the conditional expectation being $E : \mathcal{B} \to \mathcal{C} = P\mathcal{B}P$, $H_{\mathcal{C}} = H_{\mathcal{B}}$. If the inclusion is however standard (which means $H_{\mathcal{C}} = H$), the equality $\mathcal{C} = \mathcal{B}$ follows. In that case a modular inclusion gives rise to a chiral AQFT.

Theorem 3 *(Guido,Longo and Wiesbrock [11]) Standard modular inclusions are in correspondence with strongly additive chiral AQFT*

Here chiral AQFT is any net of local algebras indexed by the intervals on a line with a Moebius-invariant vacuum vector and strongly additive refers to the fact that the removal of a point from an interval does not "damage" the algebra i.e. the von Neumann algebra generated by the two pieces is still the original algebra. This begs the question of whether the present use of the word chiral is the same as that in the standard literature where chiral refers the the left/right moving component of a d=1+1 conformal observable algebra with a diffeomorphism generating energy-momentum tensor. If one does not restrict the observable fields by requirements of "rationality", the two notions probably coalesce. There are two reasons for this. On the one hand it has been shown that every local chiral net is generated by pointlike covariant fields. On the other hand it has been shown that a Moebius-invariant theory in terms of Wightman functions has states different from the vacuum which are invariant under higher diffeomorphisms of the circle. Hence this Witt-Virasoro covariance which is the hallmark of standard chiral models appears to be shared by the more general looking algebraic definition. This is of course the prerequisite for the property "chiral" in the higher dimensional holographic projection to which we now return.

First we adapt the abstract theorem to our concrete case of a wedge algebra in a massive interacting QFT in d=1+1 spacetime dimensions.

$$\begin{aligned} \mathcal{B} &= \mathcal{A}(W) \\ \mathcal{A} &= U(e_+)\mathcal{A}(W)U^*(e_+),\, e_+ = (1,1) \\ &\equiv \mathcal{A}(W_{e_+}) \end{aligned} \tag{21}$$

This is the inclusion of the algebra translated via a lightlike translation into itself so the geometrically the relative commutator

$$\mathcal{A}(W_{e_+})' \cap \mathcal{A}(W) \equiv \mathcal{A}(I(0,1)) \tag{22}$$

is by causality localized in the upper horizontal interval $(0,1)$. The standardness of this inclusion then leads to a chiral conformal AQFT i.e. a net (more precisely a pre-cosheaf)

$$\begin{aligned} I &\rightarrow \mathcal{A}(I),\ I \subset S^1 \\ \mathcal{A}(R_+) &= \overline{\cup_t Ad\Delta^{it}\mathcal{A}(I(0,1))} \\ \mathcal{A}(R) &= \mathcal{A}(R_+) \vee J\mathcal{A}(R_+)J \end{aligned} \tag{23}$$

on which the Moebius group (which preserves the vacuum vector) acts. With the help of an external (i.e. non-Moebius) automorphism on $\mathcal{A}(R)$ implemented by the opposite lightray translation $U_-(a)$, we are able to return from the chiral net on the right upper horizon to the original 2-dim. net. If we call the transition from the d=1+1 original net via the modular inclusion of wedge algebras to the $\mathcal{A}(R)$ net the "holographic projection", then the reconstruction of the d=1+1 theory from its holographic projection together with the opposite lightray translation $U_-(a)$ (which acts as a kind of positive spectrum Hamiltonian) should be called the "holographic construction". Since chiral theories are simpler than massive d=1+1 models, the gain by looking first at the holographic projection should be obvious. In fact the kind of chiral theory which is a candidate for such a start is restricted to models with generating fields with (half)integer scaling dimension which are closed under commutation i.e. where the delta function terms and their derivatives are multiplied with field from the generating set. Such models are "Lie-fields" or in modern parlance "W-algebras" in the wider sense (not necessary rational).

In analogy to the free case of the previous section one should expect that the $U_-(a)$ translation does not enlarge the Hilbert space but rather leads to the identification of the physical mass via the mass operator P_+P_- in that Hilbert space. Although the first step in the holographic construction is quite simple, the second step, namely the construction of the massive theory by the use of the opposite lightlike momentum (in the spirit of an Hamiltonian), may turn out to be more demanding. The important point however which should be emphasized here is that the holographic approach is free of short distance specters; unlike the canonical or functional integral method which is not compatible with infinite Z-factors (and which in the process of renormalization is automatically abandoned in favor of spacelike (anti)commutation), it is not endangered by large scaling dimensions. A second not less important related advantage is that the lightray is better adapted to the Poincaré extension of the holographic construction. An interesting family of models, for which these holographic aspects may be studied on the level of sesquilinear forms (instead of operators), are the factorizing models of the previous section.

For the extension of holographic projection to higher dimensional theory one needs one more mathematical definition and theorem about "modular intersections"

Definition 4 *([12]) A (±) modular intersection is defined in terms of two standard pairs $(\mathcal{N},\Omega),(\mathcal{M},\Omega)$ whose intersection is also standard $(\mathcal{N}\cap\mathcal{M},\Omega)$ and which in addition fulfill*

$$((\mathcal{N}\cap\mathcal{M})\subset\mathcal{N},\otimes)\ \ and\ \ ((\mathcal{N}\cap\mathcal{M})\subset\mathcal{M},\otimes)\ \ is\ \ \pm\ \mathrm{mod}\,ular \tag{24}$$
$$J_{\mathcal{N}}(\lim_{t\to\mp\infty}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})J_{\mathcal{N}}=\lim_{t\to\mp\infty}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}=J_{\mathcal{M}}\lim_{t\to\mp\infty}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})J_{\mathcal{M}}$$

All limits are in the modular setting are to be understood in the sense of strong convergence on Hilbert space vectors.

In the geometric setting of local quantum physics the modular intersection property is realized by the pair of intersecting wedge algebras which share the same lightray on their upper horizons $\mathcal{M}=\mathcal{A}(W),\mathcal{N}=AdU(\Lambda_{e_+}(a))\mathcal{M}$ together with the Ω =vacuum. Here $\Lambda_{e_+}(a)$ denotes a "translation" (transverse Galilei transformation) in the Wigner little group which fixes the lightray vector e_+, i.e. the Lorentz transformation which tilts $\mathcal{W}$ around this lightray. In fact the limit in the second line in (24) is geometrically nothing else but $\Lambda_{e_+}(a=1)$ and the commutation relation with $J_{\mathcal{N},\mathcal{M}}$ is easily checked as a geometric relation in the extended Lorentz group. In d=3+1 there are two independent "translations" which maintain the given lightray. Whereas it appears relatively straightforward to extend a given bosonic/fermionic chiral theory by transverse quantum mechanical degrees of freedom in order to form the lightfront substrate with the seven-dimensional symmetry subgroup of the Poincaré group, the remaining three symmetries, notably the Hamiltonian-like lightray translation away from the lightfront which installs the physical interpretation of particle physics, pose a more challenging task. It is presently not clear whether all chiral theories with (half)integer scaling spectrum allow such a holographic inversion. From the experience with the canonical Hamiltonian formulation one would expect that this step is highly non-unique, or to phrase it the other way around that the holographic projection is a many-to-one universality class relation. Up to now one only has met such a class projection in the scale invariant short distance limit which is the basis of critical phenomena. The chiral holographic projection classes constitute a quite different universality class relation between (massive) theories in any spacetime dimensions $d\geq 1+1$ and "one-dimensional" chiral theories. Even if the higher dimensional interacting models do possess conventional pointlike field generators, there will be no direct relation between these fields and the chiral fields. The holographic relation rather involves a radical spacetime reprocessing which can only be formulated in terms of changing the subalgebra affiliation while maintaining the relation between certain semi-finite algebras as that between the wedge-localized algebra and its horizon algebra (which lives on half the lightfront).

Since the theorem about the transverse tensor factorization of lightlike strips and their causal shadows (lightlike slabs) of the previous section followed from a general theorem about strip algebras with a two-sided action of a positive lightlike translation, and therefore did not use the free field structure, it remains valid in the interacting situation.

It is reasonable to ask whether these factorizing chiral strip algebras have pointlike generating fields. The results in ([13][14]) on chiral theories suggest strongly that this is the case and that these fields should be of the form of generalized W-algebras (Lie-fields) i.e. their exists a collection of generating fields $A_{LF}^{(i)}(x_+,x_\perp), i=1,2,....$with the following (anti)commutation relation

$$\left[A_{LF}^{(i)}(x_+,x_\perp),A_{LF}^{(j)}(x'_+,x'_\perp)\right]=\left\{\sum_{k=0}^{n(i,j)}B_{LF,k}^{(i,j)}(x_+,x_\perp)\delta^{(k)}(x_+-x'_+)\right\}\delta(x_\perp-x'_\perp) \tag{25}$$

$$B_{LF,k}^{(i,j)}=lin.comb.of\,A_{LF}^{(l)}\ with\,same\,dimension$$

where the sum extends over chiral δ-functions and their derivatives multiplied with B-fields which consist of linear combinations of generating A-fields with the same dimension (equal to $dimB_{LF,k}^{(i,j)}$) such that the dimension of $B's$ together with the dimension arising from the δ-functions match the balance of scale dimension of the left hand side (which leaves only a finite number of terms). For readers who are familiar with chiral conformal QFT these formula are straightforward extensions of W-commutation relation to the generalized chiral algebras which generate the lightfront net. The quantum mechanical behavior in the transverse direction is described by the common transverse δ-function. Besides the covariance under the seven-parametric subgroup of the Poincaré group, the expectation values fulfill the strong factorization

$$\langle CD\rangle=\langle C\rangle\langle D\rangle \tag{26}$$

where C,D are products of generating fields so that the transverse coordinates in the C-cluster are disjoint from those of the D-cluster. A symbolic way for describing these properties would be to say that the generalized chiral lightfront fields are of the form of a product of a chiral field with a transverse second quantized Schroedinger field.

$$A(x_+,x_\perp)=\text{“}A_{chir}(x_+)A_{QM}(x_\perp)\text{”} \tag{27}$$

where the first line is a symbolic association of a (W-algebra) chiral field of (half)integer dimension and the second line expresses that the transverse part of the correlation functions consists of products of transverse δ-function.

Modular intersections also play an analogous role in the construction of 3- and higher- dimensional AQFT starting from a finite set of wedge algebras [15] and the related holographic isomorphisms as the modular inclusions used in section 3 for the 2-dimensional case.

4 Area density of horizon-associated entropy

The use of the LFH for a coarse classification of higher dimensional theories and the aim of their construction through their net of wedge algebras (inverse LFH) is an ambitious new program which is very much in its initial stages and presently without concrete results. There are however qualitative consequences concerning the thermal behavior of localized degrees of freedom which are more accessible. This includes the area proportionality of “localization entropy” associated with the horizon of a wedge.

It is important to be reminded that the issue of quantum entropy is not something which should be affiliated with individual fields, but it rather constitutes a measure of the relative size of those degrees of freedom which contribute to certain distinguished impure quantum states which originate through spacetime restrictions from a pure state. The transverse factorization of the vacuum state and the transverse translational symmetry suggests to define a localization entropy in the chiral theory along the lightray (corresponding to localization on half the lightray) and interpret it as a transverse area density of localization entropy. The hope that this area density has a universal behavior is based on the already mentioned fact that holographic images loose the memory on the details of the original quantum matter and forms rather large holographic chiral universality classes.

Whereas it is easy to associate a temperature with the vacuum state $\omega(\cdot)\equiv(\Omega,\cdot\Omega)$ restricted to a local algebra $\mathcal{A}(\mathcal{O})$ via the resulting KMS property of the correlation functions, the definition of an entropy for the pair $(\mathcal{A}(\mathcal{O}),\Omega)$ is a more subtle matter.

To see the entropy problem in physical terms it is helpful to go back to the beginnings of QFT when Heisenberg discovered vacuum polarization and Weisskopf and others elaborated these findings. In modern parlance [16] it was connected with the behavior of "partial" Noether charges (which have the correct algebraic properties in commutation relations with observables localized in double cones of size $< R$)

$$Q(f_{R,\varepsilon}f_T) = \int j_0(x) f_{R,\varepsilon}f_T(x_0)d^4x \tag{28}$$

$$f_{R,\varepsilon}|_{t=0} = \begin{cases} 1, \ r < R \\ 0, \ r > R + \varepsilon \end{cases}$$

in their dependence on the localization radius R and the "fuzziness" (or roughening) of the boundary with thickness ε. The gist of Heisenberg's observation in modern terminology is that for $\varepsilon = 0$ the two-point function of $Q(f_{R,\varepsilon}f_T)$ diverges as a result of uncontrollable vacuum fluctuations at the boundary. A closer look at the form of the two point function of a free current reveals that the norm square for large R behaves as

$$\left\| Q(f_{R,\varepsilon}f_T)\Omega \right\|^2 \sim R^2 \tag{29}$$

with a proportionality factor which has an inverse power law divergence for $\varepsilon \to 0$ (with the power depending on the dimension of the charge density). This effect of vacuum polarization near boundaries requires some caution in viewing the global charges as limits of partial charges. The omnipresence of vacuum polarization and the inexorable correlation they create between spacetime regions and their causally separated outside is the reason behind the fact that contrary to (zero temperature-) QM the operators localized in a finite region $\mathcal{O}$ applied to the vacuum are able to approximate any state in the Hilbert space (even those with far away localizations). It also allows a unique identification of the dense set of vectors obtained in this way with operators affiliated with the local algebra $\mathcal{A}(\mathcal{O})$ (the Reeh-Schlieder theorem, also known in conformal field theory under the more folkloristic heading of "the state-field relation").

The mathematical consequences of vacuum polarization in conjunction with causality are most dramatically taken care of in the algebraic setting by emphasizing that the local algebras in the algebraic net are significantly different from quantum mechanical type I algebras (minimal projectors) and belong to the unique hyperfinite type III_1(e.g. without pure states). Such properties are not expressible in terms of generating pointlike field coordinates. It is extremely interesting to note that as a consequence of the previous theorem besides the global algebra also lightlike slabs (which are causal shadows of longitudinal strips with a fixed finite transverse extension on the lightfront) remain type I; in fact they are quantum mechanical type I subalgebras and probably even type I subfactors (true for free fields). This is the cause of the previously observed very strong form of transverse statistical independence in LFH and the surface proportionality of horizon associated localization entropy. It distinguishes localization entropy from normal heat bath entropy is proportional to the volume.

In special circumstances, notably free systems in a nonrelativistic quantization box coupled to a heat bath, it is possible to use the oscillator modes of a free field in the entropy counting. In conformal field theory the presence of a compact operator representing rotations through the compactified Dirac-Weyl world (the analog of L_0 in the chiral case) which has a discrete oscillator-like spectrum is another example. The associated Gibbs formulas have been quite useful in structural studies of chiral conformal models. But the entropy which accompanies the Hawking-Unruh temperature aspects of Rindler-wedge localization is more abstract and the relevant modular "Hamiltonians" are not present among the geometrical symmetries (in particular they are not given by L_0) but need to be constructed through the process of "splitting" which approximates hyperfinite type III_1 algebras by quantum mechanical type I factors. The reader should think of splitting as the algebraic version of the above test function control of surface vacuum polarization in Heisenberg's partial charges. Splitting kills the correlations between a region and its spacelike disjoint at an expenditure in entropy which increases with an inverse power of ε. This is the prize to pay for mimicking in AQFT a quantum mechanical separation into inside/outside tensor factorization. The relevant mathematical theorem is as follows.

Definition 5 *An inclusion $\mathcal{A} \subset \mathcal{B}$ is called split if there exists an intermediate type I factor $\mathcal{N} : \mathcal{A} \subset \mathcal{N} \subset \mathcal{B}$.*

We remind the reader that in theories with reasonable phase space properties there exists a canonical way of constructing for a standard inclusion (i.e. $(\mathcal{A}' \cap \mathcal{B}, \Omega)$ is also standard) a type I factor which is explicitly given by the formula

$$\mathcal{N} = \mathcal{A}J\mathcal{A}J = \mathcal{B}J\mathcal{B}J \qquad (30)$$
$$with\ J \equiv J_{(\mathcal{A}' \cap \mathcal{B}, \Omega)}$$

The verification of these structural properties is not difficult [21], but the explicit computation of a modular conjugation J which belongs to not simply connected or disconnected localization region is presently difficult[8]. The fact that $\mathcal{N}$ is type I implies that the modular group of $(\mathcal{N}, \Omega)$ is implemented by the unitary group $h^{it} = e^{it\mathbf{H}_{\mathcal{N}}}$ with a generator $\mathbf{H}_{\mathcal{N}}$ which is affiliated with the algebra $\mathcal{N}$

$$Ad\Delta^{it}_{(\mathcal{N},\Omega)}\mathcal{N} = Adh^{it}\mathcal{N} \qquad (31)$$
$$\Delta^{it}_{(\mathcal{N},\Omega)}H = h^{it}H_{\mathcal{N}} \otimes h^{-it}H_{\mathcal{N}'},\ H = H_{\mathcal{N}} \otimes H_{\mathcal{N}'}$$

where the positive operator $h = h^{it}|_{t=-i}$ represents the Connes-Radon-Nikodym derivative of the vacuum state $\omega_\Omega(\cdot)$ restricted to $\mathcal{N}$ with respect to the trace (tracial weight) on $\mathcal{N}$. By construction this operator has discrete spectrum and finite trace $tr\, h < \infty$ (which by normalization $\omega_\Omega(\cdot) = tr(h\cdot)$ can be set equal to one). Assuming that h is also Hilbert Schmidt, the finiteness of the entropy

$$E_{(\mathcal{N},\Omega)} \equiv -tr|_{H_{\mathcal{N}}} hlnh \qquad (32)$$

requires in addition the absence of an infrared accumulation of too many eigenvalues near zero.

For the case at hand (the chiral algebra in 27) the split property is a consequence of the nuclearity property of the chiral algebra which in turn would be guarantied by the trace condition [22]

$$tre^{-\beta L_0} < \infty \qquad (33)$$

For the families of chiral theories (loop-groups, W-algebras) which originate from $d = 1 + 1$ conformal models which have been studied up to now, this property is satisfied; however even though L_0 is a Moebius group generator, we are not aware of any argument that the discreteness of its spectrum and the existence of the trace (33) follows from the defining properties of chiral theories. In fact it is certainly violated in the presence of a transverse symmetry; in this case one can of course see directly that there can be no intermediate canonical $\mathcal{N}$-factor since it would inherit the transverse symmetry which leads to an infinite degeneracy of eigenvalues of h and lead to a violation of the trace condition..

We now adjust the previous abstract situation to the calculation of the area density of the split horizon localization entropy. It is convenient to use the circular compactification. of the lightray. The split inclusion of interest is $\mathcal{A} \subset \mathcal{B}$ with $\mathcal{A} = \mathcal{A}(I(0 + \varepsilon, \pi - \varepsilon))$, $\mathcal{B} = \mathcal{A}(I(0, \pi))$ in the limit $\varepsilon \to 0$; i.e. the bigger algebra is the quarter circle algebra (the image of the $(0, \infty)$ interval under the Cayley transformation) and its shortened version with the distance ε from both ends. Our aim is to understand the restriction of the vacuum state $\omega(\cdot) = \langle \Omega |\cdot| \Omega \rangle$ to the intermediate fuzzy quantum mechanical algebra $\mathcal{N} \subset \mathcal{B}$. In fact our aim is more ambitious: we want to take the minimal entropy over all intermediate type I factors which comply with the above definition. There is a very elegant way to do this which is due to Kosaki [20] which in principle avoids explicit constructions in terms of an intermediate $\mathcal{N}$ and tensor factorization of vectors.

[8] There has been some recent progress in the understanding of modular objects associated with double interval chiral algebras [18][19].

It requires only the knowledge of the original and the "split" vacuum ω_p

$$\begin{aligned} &\omega_p(AB') = \omega(A)\omega(B'),\ A \in \mathcal{A}, B' \in \mathcal{B}' \\ &E(\omega_p, \omega) = sup_{y(t)} \int_0^\infty \left[\frac{\omega^2(1)}{1+t} - \omega_p(y^*(t)y(t)) - \frac{1}{t}\omega(x(t)x^*(t)) \right] \frac{dt}{t} \\ &with\ x(t) \equiv 1 - y(t)\ \ a\ path\ in\ \mathcal{A} \vee \mathcal{B}' \end{aligned} \tag{34}$$

We have not been able to use this beautiful variational formula for the Araki relative entanglement entropy of two states (adjusted to the situation at hand of one state being the product state of the other). For previous attempts to use this formula for localization entropy without using LFH see [23].

The desire to make some headway forced us to use the (nonunique) vector implementation of such states[9] and to restrict our calculation to free fields. This is related to find an implementation of the isomorphism Φ of the double localized algebra with its tensor factorized form

$$\mathcal{A} \vee \mathcal{B}' \overset{\Phi}{\simeq} \mathcal{A} \otimes \mathcal{B}'$$

We then can follow [24] and use the so-called "flip trick" in the "duplicated" representation of Φ i.e. the representation in $H \otimes H$ with the duplicated vacuum $\Omega \otimes \Omega$. The generating fields of the duplicated operator algebra are

$$\psi_1 = \psi \otimes \mathbf{1},\ \psi_2 = \mathbf{1} \otimes \psi \tag{35}$$

where ψ is a generating Bose free field (the construction also works with a free Fermi field) of the original algebra and the algebra $\mathcal{N}$ is that of the first tensor factor. The "flip trick" which is an implementation of Φ in terms of concrete unitary operators. For the case at hand we notice that the fields in the different tensor product factors can be interpreted as a doublet i.e. $\psi_1 = \psi \bar{\otimes} 1$, $\psi_2 = 1 \bar{\otimes}\, \psi$. In the spirit of a SO(2) Noether symmetry the implementation of the unitary flip operation can then be done in terms of a Noether current [24] formalism (so that the computations are reminiscent of the surface vacuum fluctuations in the above historical remark about partial charges)

$$\begin{aligned} \Phi(\psi(x)) &= e^{ij(f)}\psi_1(x)e^{-ij(f)} = \left\{ \begin{array}{c} \psi_1(x),\ x \in I(0+\varepsilon, \pi - \varepsilon) \\ \psi_2(x),\ x \in I(\pi, 2\pi) \end{array} \right. \\ j(x) &= \psi_1^*(x)\psi_2(x), \quad f = \left\{ \begin{array}{c} 0,\ \ x \in I(0+\varepsilon, \pi-\varepsilon) \\ 1,\ \ x \in I(\pi, 2\pi) \end{array} \right. \end{aligned} \tag{36}$$

Clearly $U(f) = e^{ij(f)}$ is an implementation of the split isomorphism Φ acting on $H \bar{\otimes} H$ and restricted to the doubled vacuum $\Omega_{vac} = \Omega \bar{\otimes} \Omega$ yields a vector which implements the product state (34). As expected the state vector $\eta' = U(f)(\Omega \bar{\otimes} \Omega)$ becomes orthogonal on all vectors in $H \bar{\otimes} H$ for $\varepsilon \to 0$. Let us check this for the vacuum Ω_{vac}

$$\begin{aligned} &\langle \Omega_{vac} | \eta' \rangle = \langle \Omega_{vac} | U(f) | \Omega_{vac} \rangle \\ &= e^{-\frac{1}{2}\langle j(f), j(f) \rangle_0} \sim \varepsilon,\ \varepsilon \to 0 \\ &\langle \eta' | AB' | \eta' \rangle = \langle \Omega_{vac} | A | \Omega_{vac} \rangle \langle \Omega_{vac} | B' | \Omega_{vac} \rangle \\ &A \in \mathcal{A}(I(0+\varepsilon, \pi-\varepsilon)),\ B' \in \mathcal{A}(I(\pi, 2\pi)) \end{aligned} \tag{37}$$

As stated before we have arbitrarily chosen one implementing vector η' and therefore we do not know whether this is the one closest to Ω (which would be the one with the smallest localization entropy). The following inequality has general validity

[9] We of course hope that in the limit $\varepsilon \to 0$ the differences in the vector-implementations will not matter.

$$\begin{aligned} \|\eta - \Omega\|^2 &= 2\,|1 - (\eta, \Omega)| \\ &= inf\left\{ \|\eta' - \Omega\|^2, \eta' \, is \; split \right\} \\ &= \|\omega_{split} - \omega_\Omega\| \leq \|\eta' - \Omega\|^2 \end{aligned} \tag{38}$$

where η is the implementing vector with the smallest distance and the last line uses the so-called canonical Bures distance in the convex space of states[10]. It is generally believed that in the limit $\varepsilon \to 0$ all implementing state vectors show the same behavior. We will assume that this is true, and that there is no physically preferred implementation.

The norm square of the smeared current $j(f)$ can be explicitly computed from the known current two-point function

$$\langle j(f), j(f) \rangle_0 \overset{\varepsilon \to 0}{\sim} -\log \varepsilon$$

i.e. the resulting logarithmic dependence on the collar size ε leads to the above positive power law after exponentiation. This is a quantitative expression for Wald's qualitative discussion of the inequivalence (orthogonality) of the two Hilbert spaces in the limit [25].

In the same vein the inner product with all basis vectors converges with a power law to zero for $\varepsilon \to 0$

$$\langle \Omega_{vac} \,|U(f)|\, a_1^*(p_1)...a_n^*(p_n) \otimes a_2^*(k_1)...a_2^*(k_m)\Omega_{vac} \rangle \to 0 \tag{39}$$

i.e. the original vacuum becomes a highly entangled state on the split algebra which in the limit $\varepsilon \to 0$ even leaves the Hilbert space i.e. belongs to an inequivalent representation of the algebra. By tracing out the second tensor factor after squaring (39) one obtains a density matrix ρ_ε in the first factor space which represents the vacuum Ω_{vac} as a mixed state on the factor space $\overline{\mathcal{A}(I(0+\varepsilon, \pi - \varepsilon))\eta'}$. A convenient basis for this partial tracing is the discrete basis of the rotational generator L_0. The actual computations will be presented elsewhere; here we will only pay attention to the qualitative consequences of the logarithmic divergence of the entropy area density

$$s_\varepsilon = -\rho_\varepsilon ln \rho_\varepsilon = -ln\varepsilon \cdot s \tag{40}$$

As a consequence of the universality of the LFH we expect a universal behavior of the entropy area density in that the leading singularity of s_ε is universally logarithmic with a possible dependence of the "reduced" entropy on holographic matter classes. We want to stress that in contrast to our discussions in previous sections this is only a working hypothesis subject to future verification. It is this hypothesis which makes it possible to associate a entropy-like additive measure with the horizon. Since for $\varepsilon \to 0$ the split type I factor $\mathcal{N}$ converges against the horizon- or wedge- algebra and the modular group coalesces with the dilation respectively the Lorentz boost, the reduced area density s is really the localization entropy aspect of the Hawking-Unruh localization temperature (which was the KMS temperature associated with the Lorentz boost automorphism).

It needs to be stressed that the surface parameter ε is not what is usually in quantum field theory called a cut-off. In contrast to the latter which modifies the theory (including its Hilbert space, causality etc.) in an uncontrollable way and for this reason has to be removed at the end in order to be able to recover a theory in the original setting defined by the physical principles, the splitting procedure takes place in the original causal/local positive energy theory and amounts to a temporary spacetime resettling of degrees of freedom and not to their liquidation. In particular it is quite different from removing a region and its material content for all times (i.e. a spatial collar between a nonrelativistic quantization box and its outside).

[10] The maximum value 2 of the Bures distance is an indication that the state ω_{split} belongs to an inequivalent folium.

5 Outlook

We succeeded to demonstrate that various properties, which previously were expected to require the setting of curved spacetime or even that of QG, are generic properties of the principles of quantum field theory. Among those properties are the presence of diffuse (fuzzy) acting diffeomorphisms of the circle (with Witt-Virasoro genererators) and an absence of transverse vacuum correlation in LFH leading among other things to an area proportionality of localization an additive entropy-like measure of the entanglement of the global vacuum relative to the local vacuum on the horizon.

Since only the ratios of area densities of entropy are fixed by the principles of QFT there is still the open problem of a Bekenstein normalization. In the context of this paper the open problem is the question of whether the understanding of the quantum version of Bekenstein's law is more than the fixing one free constant for all types of quantum matter by extending the thermodynamic basic laws from the realm of the heat bath setting to that of thermality from localization. Although the present results do not rule out a fundamental connection with QG, I do not expect that the entropy issue of black holes reveals more about fundamental properties than the better known temperature and radiation aspects. In any case a serious attempt in that direction should not ignore the fact that curvature is not the cause for the area behavior of entropy of quantum matter associated with localization on a horizon, although it certainly plays an important role to attribute classical/geometric aspects to this pure quantum phenomenon.

The localization entropy issue as a special aspect of LFH is one step in a long range program which I pursued over many years. It consists in a revitalization of some of the ideas of the S-matrix approach of the 60s. The idea to avoid singular field coordinatization (which at least partially are considered to be the origin of ultraviolet problems) by using on-shell coordinatization-independent quantities as the S-matrix always appeared to me very reasonable.

The most mysterious property of on-shell objects is their crossing symmetry observed in the S-matrix and generalized formfactors (matrix elements of operators between bra outgoing and ket incoming particle states)[11]. Although the idea that crossing is the on-shell manifestation of off-shell causality seems to be very sound, there has never been a sufficiently general derivation of crossing from the standard principles. Outside perturbation theory there is the famous proposal by Veneziano of an elastic crossing symmetric S-matrix in terms of Gamma function which after several "revolutions" gave rise to the mathematically successful string theory in 26 or 10 spacetime dimensions. Unfortunately in the heat of these mathematical discoveries the solution of the deep physical questions about the on-shell to off-shell relations, the role of crossing in connection with locality and a possible different view about ultraviolet problems in a local QFT fell into historical oblivion. But these are just the properties which a particle theory needs for its physical interpretation; their lack in my view is responsible why string theory remained (ever since its inception 30 years ago) a collection of calculational recipes without the possibility to communicate its content in terms of a realization of (new?) physical principles. It would not be worthwhile to stress this point in the context of the present paper if it would not be for the fact that string theory has its cradle exactly there where also my motivation for the modular construction of QFT (which among other things led me to the present LFH) is coming from.

Algebraic QFT strengthened the de-emphasis on field-coordinatizations (this time in terms of nets of algebras). A synthesis of both strain of ideas was possible after the discovery of the modular theory of operators and the recognition of its physical relevance. One encouraging result is that the successful bootstrap-formfactor program (for a recent presentation with a complete list of references [26]) of constructing special families of d=1+1 field theories became incorporated into the setting of AQFT i.e. all its construction recipes have been understood in terms of the basic principles [17][27]. The basic role in this new constructive approach is played by the (Rindler) wedge algebras; smaller localizations have to be constructed by intersections. In general the direct construction of wedge algebras is very difficult and

[11] For a formal derivation (which ignores the on-shell analytic properties) within the LSZ framework we refer to the appendix in [26].

the LFH is an attempt to simplify this construction. There are encouraging signs that certain d=1+2 dimensional free anyons [28] will be the next family of models constructed by modular methods.

Progress in this area will depend somewhat on whether sufficient interest in conceptual problems in local quantum physics and particle theory can be passed on and whether young people can be motivated to take the risk of long term intellectual investments away from the fast moving globalized fashions.

Perhaps one can learn something from a past when a similar speculative mood of "everything goes" was governing the Zeitgeist; I am referring to the wild speculations which started immediately after the discovery of the ultraviolet divergencies and only came to a halt around the beginning of the 50s after the 1948 discovery of renormalization theory (a totally conservative success which vindicated all the prior principles of QFT). The most surprising sociological aspect of that conservative discovery is that it was made to a large part by very young people as Feynman, Schwinger and Dyson (see the introductory remarks in [29]). Of course that surprise disappears somewhat once one reminds oneself that research in particle physics was much more difficult and demanding in those under the constant threat of experimental falsification and the very restrictive imposition of established physical principles than in a free-floating setting of some of the sociologically accepted presently fashionable theories of which their proponents even after 30 years could not even explain their possible underlying principles.

Acknowledgements: I am indebted to Jakob Yngvason for a invitation for a visit of the ESI and I thank the organizers of the ESI workshop "Quantum field theory on curved space time" for the opportunity to present my results.

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