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ABSTRACT

Within an appropriate renormalization framework, we discuss $a^*(b)$ and δ associated with the bifurcation road to chaos of a Hénon-like map generalized as follows: $(x_{t+1}, y_{t+1}) = (1-a|x_t|^z + y_t, -bx_t)$; $(b \geq 0, z \geq 1)$. For fixed z, we obtain (i) *only two* universality classes, namely the *conservative* (b=1) and *non conservative* $(b\neq 1)$ ones and (ii) $a^*(1/b) = a^*(b)/b^z$. For b=1, $\delta(z)$ presents a minimum, and diverges for $z \to 1$ and $z \to \infty$ (this contrasts with the $b\neq 1$ case).

The Hénon map has attracted a lot of attention in the last few years, because it presents interesting two-dimensional Feigenbaum-like bifurcation schemes, Birkhoff chains and strange attractors[1-4]; as particular cases, it contains, in the extreme dissipative limit (non invertible map), the one-dimensional map studied by Feigenbaum, as well as a typical (area-preserving) conservative two-dimensional mapping. Finally it seems to be deeply related to experimental situations[5].

Herein we focus on the bifurcation road to chaos of the following $(x,y) \to T(x,y)$ map:

$$T(x,y) = (1 - a|x|^z + y, - bx) \tag{1}$$

with $z \geq 1$ and $b \geq 0$ ($b > 0$ corresponds to orientation preserving maps, hence more physical in principle; $b < 0$ will be focused elsewhere). The Hénon and Lozi maps are respectively recovered for $z = 2$ and $z = 1$. The associated Jacobian equals b ($b=1$, $b < 1$ and $b > 1$ respectively correspond to conservative, dissipative and "expansive" maps; $b = 0$ recovers the standard one-dimensional map[6]).

Following along the lines of Derrida and Pomeau[3] we linearize the map $T^{(n)}$ in the neighborhood of the elements of the n-cycle, and obtain the matrix

$$M_n = \prod_{i=1}^{n} \begin{pmatrix} - z\, a \,\mathrm{sign}\left(x_i^{(n)}\right) |x_i^{(n)}|^{z-1} & 1 \\ -b & 0 \end{pmatrix} \tag{2}$$

where $x_i^{(n)}$ are the abcissa of the elements of the n-cycle.

To treat the criticality of the bifurcation road to chaos, we construct a renormalization group (RG) as follows: we renormalize a n-cycle into a n'-cycle ($n = 2,4,8,\ldots; n' = 1,2,\ldots,n/2$) while *preserving both eigenvalues of* M_n *and* $M_{n'}$. This procedure recovers, for b = 1, that of Ref. [3], and *differs*, for arbitrary b, from that introduced by Zisook[7] and that introduced by Hu[8]. For both the extremely dissipative (b = 0) and conservative ($|b| = 1$) cases, all these procedures provide the same bifurcation rates $\delta(b = 0)$ and $\delta(|b| = 1)$. The present procedure yields, for n = 2 and n' = 1, the following recursive relations (in the (a,b)-space):

$$-za'\,\mathrm{sign}[x_1^{(1)}(a',b')]\,|x_1^{(1)}(a',b')|^{z-1}$$

$$= z^2 a^2\,\mathrm{sign}\left[x_1^{(2)}(a,b)\,x_2^{(2)}(a,b)\right]|x_1^{(2)}(a,b)\,x_2^{(2)}(a,b)|^{z-1} - 2b \tag{3}$$

$$b' = b^2 \tag{4}$$

where $x_1^{(1)}$ satisfies $T(x,y) = (x,y)$ (we recall that $y_1^{(1)} = -bx_1^{(1)}$), and $x_1^{(2)}$ and $x_2^{(2)}$ are solutions (different from $x_1^{(1)}$) of $T(T(x,y)) = (x,y)$.

Eq. (3) can be made explicit for z = 2, thus yielding

$$a' = 4a^2 - 6(b+1)^2 a + 2b^4 + 9b^3 + 13b^2 + 9b + 2 \tag{3'}$$

Eqs. (3') and (4) present a fully unstable fixed point $(a_c, 1)$ with $a_c \equiv (25 + \sqrt{65})/8 \simeq 4.1328$ (the numerically exact value is[4] 4.1362), as well as two semi-stable fixed points, namely $(a_d, 0)$ with $a_d \equiv (7 + \sqrt{17})/8 \simeq 1.3904$ (the numerically exact value is 1.4014) and (∞, ∞). The asymptotic behaviors of the critical line $a^*(b)$ are

$$a^* \sim a_d + \frac{3}{2}b \qquad (b \to 0) \tag{5}$$

$$a^* \sim a_c b \qquad (b \to 1) \tag{6}$$

and

$$a^* \sim a_d b^2 \qquad (b \to \infty) \tag{7}$$

The RG flow diagram is indicated in Fig. 1. The bifurcation rate δ equals the greater eigenvalue of the Jacobian $\partial(a',b')/\partial(a,b)$ calculated at the corresponding fixed point; we obtain, for $b = 1$, $\delta_c \simeq 9.0623$ (the numerically exact value is[4] 8.7211), and, for $b = 0$, $\delta_d \simeq 5.1231$ (the numerically exact value is 4.6692).

The whole picture illustrated above for $z = 2$, remains completely similar for arbitrary z (see Fig. 2). In particular, Eqs. (5)-(7) are generalized into

$$a^* \sim a_d(z) + a_d'(z)\, b \qquad (b \to 0) \tag{8}$$

$$a^* \sim a_c(z)\left[1 + \frac{z}{2}(b-1)\right] \qquad (b \to 1) \tag{9}$$

and

$$a^* \sim a_d(z) b^z \qquad (b \to \infty) \qquad (10)$$

where $a_d'(z)$ varies smoothly with z.

Note also an interesting property: Eqs. (3) and (4) remain *invariant* under the transformation $(a,b) \to (a/b^z, 1/b)$, hence the $b > 1$ region can be mapped into the $b < 1$ region. Consequently the critical line $a^*(b)$ satisfies the following (possibly *exact*) property:

$$a^*(1/b) = a^*(b)/b^z \qquad (11)$$

This property is consistent with Eqs. (8)-(10). Furthermore *the expansive* $(b > 1)$ *and dissipative* $(b < 1)$ *regions belong to one and the same universality class* (this can be intuitively understood from the following standpoint: *both* cases asymptotically become *one-dimensional*, the dissipative systems ultimately retaining the *less dissipating* direction in phase space, and the *expansive* systems primarily retaining the *more expansing* direction in phase space). Let us comment on this last point which is an important one: the present picture implies that a *crossover* occurs at $b = 1$. This is to say, while the critical point a^* varies smoothly with b, the bifurcation rate takes *only two* values (*for fixed z*), namely δ_c for $b = 1$ (conservative case), and δ_d for $b \neq 1$ (non conservative case). Although we have not constructed a RG suited for $b < 0$, the picture which emerges, for the first time, is the following: δ equals δ_c for $b = \pm 1$ and equals δ_d for $b \neq \pm 1$, while a^* smoothly varies for b varying from

$-\infty$ to $+\infty$. This picture satisfactorily covers the numerical results by Derrida et al[2] who obtained $a^*(b=-0.3) \simeq 1.0580$ and $\delta(b=-0.3) \simeq 4.6694$, by Bountis[4] who obtained $a^*(b=1) \simeq 4.1362$, $a^*(b=-1) \simeq 1.1536$ and $\delta(b=\pm 1) \simeq 8.721$; it is in agreement with numerical results obtained by Zisook[7] and with comments by Cvitanovic[9], as well as with analytical asymptotic results obtained by Collet et al[10] in the $z \to 1$ limit; on the other hand, it disagrees with results obtained by Hu[8] which suggest that δ varies *smoothly* with b. Furthermore the present picture provides a natural framework for understanding the fact that many experimental systems[9] belong to the Feigenbaum universality class, *in spite of their obviously different degrees of dissipation*. Eq. (11) possibly holds, for *all values* of b, as follows:

$$a^*(1/b) = a^*(b)/|b|^z \qquad (12)$$

Let us go back to the map (1). Its inversion yields

$$x_t = -y_{t+1}/b \qquad (13.a)$$

$$y_t = -1 + x_{t+1} + a|y_{t+1}|^z/|b|^z \qquad (13.b)$$

where the subscript t denotes the recurrence order. If we perform now the transformation $x \rightleftarrows -y$ we obtain

$$x_t = 1 - \frac{a}{|b|^z}|x_{t+1}|^z + y_{t+1} \qquad (14.a)$$

$$y_t = -x_{t+1}/b \qquad (14.b)$$

which coincides with the map (1) if we run "backwards" (time reversal) and perform the transformation $(a,b) \to (a/|b|^z, 1/b)$. No doubt this fact is directly related to property (12), although we have not succeded in putting this in transparent terms.

Let us conclude by stressing the interesting difference which was detected in the z-dependence of the non-conservative and conservative bifurcation rates associated with the 2^k road to chaos. More precisely, in the non-conservative case, $\delta(z)$ monotonously increases when z increases from 1 to infinity, whereas for the conservative case, $\delta(z)$ presents a minimum (near z=2) and diverges in the $z \to 1$ limit as well as (possibly) in the $z \to \infty$ one (thus presenting a shape similar to that found[2,6] in the non-conservative p^k road to chaos *with* $p > 2$).

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CAPTION FOR FIGURES

FIG. 1 - RG flow, in the (a,b) parameter space, corresponding to the bifurcation road to chaos of the $z = 2$ (Hénon) map; ● (0) is a semi-stable (fully unstable) fixed point. The critical lines $a^*(b)$ corresponding to the dissipative ($b < 1$) and creative ($b > 1$) regions belong to the Feigenbaum universality class (one-dimensional, ideally dissipative), and are transformed, one into the other, through $(a,b) \leftrightarrow (a/b^2, 1/b)$; the conservative case ($b = 1$) constitutes a different universality class.

FIG. 2 - RG z-dependence of the critical points a_c and a_d and the corresponding bifurcation rates δ_c and δ_d associated with the bifurcation road to chaos of the present Hénon-like orientation preserving ($b > 0$) map: the subscripts c and d respectively refer to the conservative ($b = 1$) and ideally dissipative ($b = 0$) cases (a_d and δ_d are reproduced from Ref. [6]).

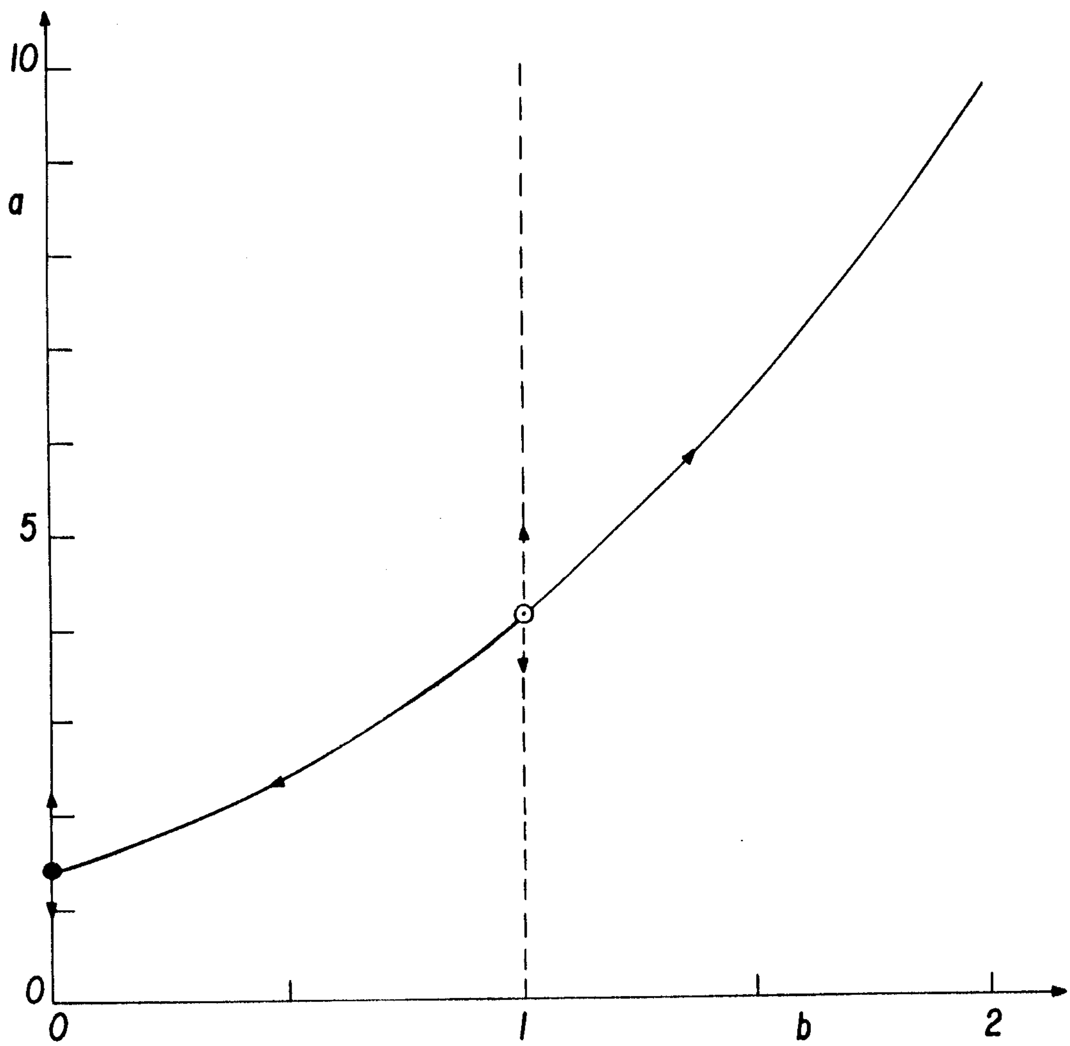

FIG. 1

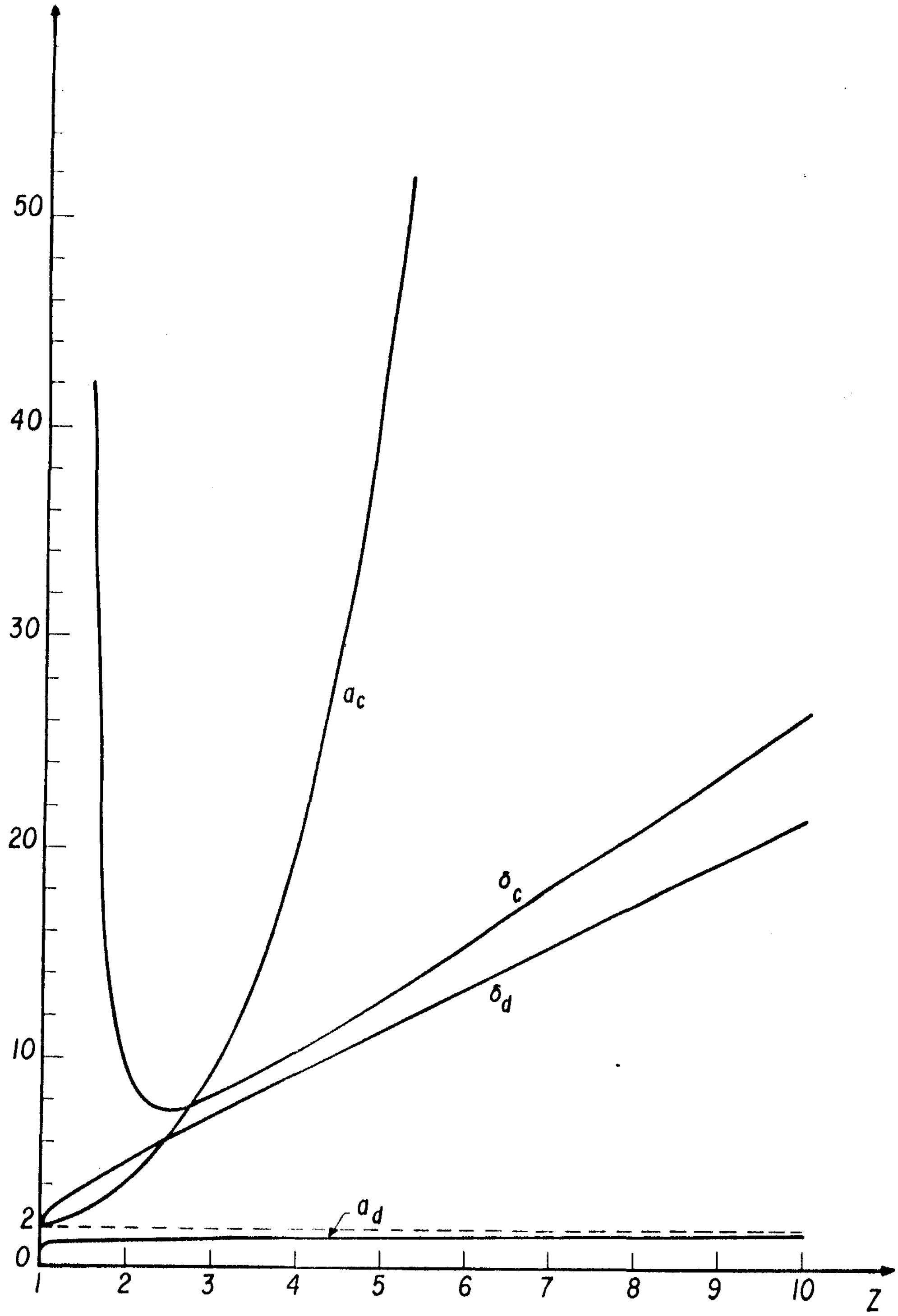

FIG. 2